Convergence Analysis of an Iterative Method for the Reconstruction of Multi-Band Signals from their Uniform and Periodic Nonuniform Samples

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Abstract

One of the proposed methods for recovery of a band-limited signal from its samples, whether uniform or nonuniform, is the so called frame Method[1]. In this method the original signal is reconstructed by iterative use of sampling-filtering blocks. Convergence of this method for linear invertible operators has been previously proved. In this paper we show that this method for non-invertible periodic nonuniform samplings as well as non-invertible uniform samples of bandpass (or multi-band) signals will lead to the pseudo-inverse solution. Convergence conditions in case of additive noise will also be discussed.

Key words and phrases :Iterative Method, Periodic Nonuniform Sampling, Pseudo-Inverse Solution, Bandpass and Multiband sampling.

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[1]It is also called marvasti's method by Feichtinger and Gröchenig [1] and Gröchenig and Strohmer in the sixth chapter of [2].

1 Introduction

Although uniform sampling at the Nyquist rate for low pass signals is quite straitforward, the extensions to bandpass and multi-band signals are not trivial. The bandwidth for such signals consists of separate positive frequency intervals with the overall length of B (thus the Nyquist rate is $2B$). Therefore, uniform sampling at the rate $2B$ is likely to lead aliasing. The minimum alias-free uniform sampling rate for bandpass and multiband signals are not necessarily $2B$ [3].

One of the old proposed methods for reducing the sampling rate is the second order sampling [4]. In this method the classical uniform sampling points are substituted by two interlaced uniform sampling sets. The average sampling rate of $2B$ can be reached by proper selection of the interlacing parameter. Generalization to N^{th} order samplings (or *periodic nonuniform sampling*) for bandpass and multi-band signals have been studied by [5, 6, 7, 8]. Periodic nonuniform sampling is also generalized to sampling sets which are unions of shifted lattices ([9, 10]); these sets are not necessarily periodic. Implementations of periodic nonuniform samplings are usually fulfilled by passing the signal through different delaying filters and then uniformly sampling each filter output. The idea of using a general filter bank (instead of delay filters) has been developed by [11, 12].

The proposed reconstruction methods are mainly based on interpolating functions. Since these functions are bandlimited, they cannot be timelimited. For practical implementations these functions should be truncated. In many cases these truncations result in considerable errors. The alternative methods are iterative approaches. In these methods, by repeated use of a simple but not a perfect reconstruction method, the output gradually converges to the perfect solution [2]. For optimizing the iterative method with respect to the convergence rate, accelerated methods has been proposed [13]. One of the advantages of the iterative methods is that when the sampling scheme is non-invertible, it still converges while the interpolating methods diverge. We will show that the converging signal is the pseudo-inverse solution. In the next section, we first introduce our specific iterative method and then in section 3 we will consider its convergence analysis for the case of Hermitian matrices; in this section we consider the effect of additive noise. In section 4 we will check the results for a special non-invertible case of bandpass sampling

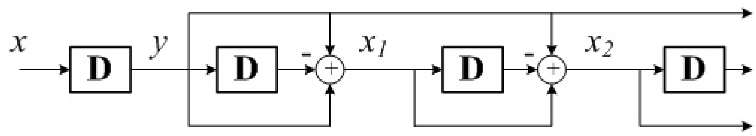

Figure 1: Block diagram of the iterative method for the distorting operator D and the relaxation parameter $\lambda = 1$.

where we show how the sampling-filtering block can be modeled with a Hermitian matrix. The extensions to periodic nonuniform sampling for bandpass and muliband signals is given in section 5. Simulation results and conclusion are given in sections 6 and 7 respectively.

2 The Iterative Method

One of the methods for reconstruction of a signal from its samples, is the iterative method [14, 2]. In this method by successive use of a crude reconstruction method, we increase the quality (measured by SNR) of the reconstructed signal and we may reach the original signal without error. The block diagram of this method is presented in Fig. 1. As it is shown, to recover the original signal x from its distorted version y, we repeatedly apply lowpass filtering and sampling denoted by the operator D. The mathematical formulation of this method is as follows:

$$\begin{cases} x_{k+1} = \lambda\Big(y - D[x_k]\Big) + x_k \\ y = D[x] \\ x_0 = 0 \end{cases} \tag{1}$$

In the above equation λ is called the relaxation parameter. Convergence of the iterative method is determined by this parameter. The range of λ which is required for convergence is usually a continuous interval which includes zero; thus to avoid divergence, λ is normally set near zero.

3 Convergence Analysis

We shall discuss convergence conditions for the iterative method in this section. The analysis is general for any linear operator D which could be modeled with a Hermitian matrix and is not limited to sampling problems. Let us assume that the input $\mathbf{x}$ is an $l \times 1$ vector and the output $\mathbf{y}$ is

$$\mathbf{y} = \mathbf{D}.\mathbf{x}, \tag{2}$$

where $\mathbf{D}$ is an $l \times l$ Hermitian matrix (we will show later that for the convergence of the iterative method, the matrix $\mathbf{D}$ should be either Nonnegative Real or Nonpositive Real). We can also include a noise vector by assuming:

$$\mathbf{y} = \mathbf{D}.\mathbf{x} + \mathbf{n} \tag{3}$$

The statement of the problem is that given the output $\mathbf{y}$ vector, we wish to find the input vector $\mathbf{x}$; i.e., to find the inverse of the system. Since the noise vector or even the existence of the noise is not known, for the purpose of the reconstruction we have to assume the distorting operation is defined by (2) while (3) is more likely to hold. Therefore, we implement the iterative method by successive multiplication of the vector by the $\mathbf{D}$ matrix. Thus the reconstructed vector in each iteration is found by:

$$\begin{cases} \mathbf{x}_{k+1} = \lambda(\mathbf{y} - \mathbf{D}.\mathbf{x}_k) + \mathbf{x}_k \\ \mathbf{y} = \mathbf{D}.\mathbf{x} + \mathbf{n} \\ \mathbf{x}_0 = 0 \end{cases} \tag{4}$$

We define the error vector and the error matrix as:

$$\begin{cases} \mathbf{z}_k = \mathbf{x}_k - \mathbf{x} \\ \mathbf{E} = \mathbf{I}_{l\times l} - \lambda\mathbf{D} \end{cases} \tag{5}$$

Thus, we can rewrite (4) as:

$$\begin{cases} \mathbf{z}_{k+1} = \mathbf{E}.\mathbf{z}_k - \lambda\mathbf{n} \\ \mathbf{z}_0 = -\mathbf{x} \end{cases} \tag{6}$$

$$\Rightarrow \mathbf{z}_k = -\mathbf{E}^k.\mathbf{x} + \lambda\big(\mathbf{I}_{l\times l} + \mathbf{E} + \mathbf{E}^2 + \cdots + \mathbf{E}^{k-1}\big)\mathbf{n} \tag{7}$$

$$\Rightarrow \mathbf{x}_k = \big(\mathbf{I}_{l\times l} - \mathbf{E}^k\big).\mathbf{x} + \lambda\big(\mathbf{I}_{l\times l} + \mathbf{E} + \mathbf{E}^2 + \cdots + \mathbf{E}^{k-1}\big)\mathbf{n} \tag{8}$$

We are interested in the convergence and uniqueness of the sequence $\mathbf{x}_k$ when $k \to \infty$. To find the answers of these questions, we need to evaluate powers of $\mathbf{E}$ as shown in (8).

Assuming $\mathbf{D}$ is nonnegative definite (Hermitian) with eigen-values $d_1, d_2, \ldots, d_l$, we have:

$$0 = d_1 = d_2 = \cdots = d_c < d_{c+1} \leq d_{c+2} \leq \cdots \leq d_l \tag{9}$$

where c is the number of zero eigen-values $(0 \leq c \leq n)$. Moreover, Hermitian matrices have orthonormal eigen-vectors; let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_l$ be these orthonormal eigen-vectors:

$$\begin{cases} \mathbf{D}.\mathbf{v}_i = d_i \mathbf{v}_i \\ \mathbf{v}_i^H.\mathbf{v}_j = \delta[j-i] \\ 1 \leq i \leq j \leq l \end{cases} \tag{10}$$

Therefore:

$$\begin{cases} \mathbf{V} = \big[\mathbf{v}_1 | \mathbf{v}_2 | \ldots | \mathbf{v}_l\big] \\ \mathbf{V}.\mathbf{V}^H = \mathbf{I}_{l \times l} \end{cases} \tag{11}$$

and

$$\mathbf{D} = \mathbf{V}.diag\big(d_1, d_2, \ldots, d_l\big).\mathbf{V}^H, \tag{12}$$

where $diag\big(d_1, d_2, \ldots, d_l\big)$ represents a square diagonal matrix with its diagonal elements as $d_1, d_2, \ldots, d_l$. Similarly we can write:

$$\mathbf{E} = \mathbf{I}_{l \times l} - \lambda \mathbf{D} = \mathbf{V}.diag\big(1 - \lambda d_1, 1 - \lambda d_2, \ldots, 1 - \lambda d_l\big).\mathbf{V}^H \tag{13}$$

We will first study the case with no noise (2) and then we will consider the effect of additive noise as shown in (3).

3.1 The Case Without Additive Noise

Without additive noise, (8) becomes:

$$\mathbf{x}_k = \big(\mathbf{I}_{l \times l} - \mathbf{E}^k\big).\mathbf{x} \tag{14}$$

Hence the convergence of the algorithm is equivalent to the convergence of $\lim_{k \to \infty} \mathbf{E}^k$. From (13) we have:

$$\mathbf{E}^k = \mathbf{V}.diag\big((1 - \lambda d_1)^k, (1 - \lambda d_2)^k, \ldots, (1 - \lambda d_l)^k\big).\mathbf{V}^H \tag{15}$$

To assure convergence, it is required that $|1-\lambda d_i| \leq 1$. Moreover, if for any i we have $1-\lambda d_i = -1$, although $\mathbf{E}^k$ elements stay bounded, (14) does not converge. In summary, to assure convergence, we should have:

$$\forall i : -1 < 1-\lambda d_i \leq 1 \Rightarrow \begin{cases} 0 \leq \lambda < \frac{2}{d_i} & d_i > 0 \\ or & \\ 0 \geq \lambda > \frac{2}{d_i} & d_i < 0 \end{cases} \tag{16}$$

Since all eigen-values are positive, there exists a nonempty set of λ values which satisfy all these inequalities. If we had both positive and negative eigen-values, no solution for λ would be achieved (equivalent to the previously stated condition of non-negativity or non-positivity of the real part). In the iterative method (4), $\lambda = 0$ is a trivial case; thus we have:

$$convergence \Leftrightarrow 0 < \lambda < \frac{2}{d_{max}} \tag{17}$$

From now on we assume that the above condition is fulfilled by proper choice of λ. If $\mathbf{D}$ is an invertible matrix, it has no zero eigen-values ($c = 0$ in (9) and thus for all i, $|1-\lambda d_i| < 1$. This means $\mathbf{E}^k \to \mathbf{0}$ as $k \to \infty$ and from (14) we have:

$$\lim_{k\to\infty} \mathbf{x}_k = \mathbf{x}, \tag{18}$$

which means perfect reconstruction (which was predictable due to the invertibility of $\mathbf{D}$). Now let us assume that $\mathbf{D}$ is not invertible and has c zero eigen-values. As a consequence, $\mathbf{E}$ has c eigen-values equal to 1 and the rest have absolute values less than 1:

$$\mathbf{E}^k = \mathbf{V}.diag\Big(\underbrace{1,1,\ldots,1}_{c\ \ times},(1-\lambda d_{c+1})^k,\ldots,(1-\lambda d_l)^k\Big).\mathbf{V}^H \tag{19}$$

$$\Rightarrow \mathbf{E}^\infty = \lim_{k\to\infty}\mathbf{E}^k = \mathbf{V}.diag\Big(\underbrace{1,1,\ldots,1}_{c\ \ times},\underbrace{0,\ldots,0}_{l-c\ \ times}\Big).\mathbf{V}^H \tag{20}$$

$$\begin{aligned}\Rightarrow \mathbf{x}_\infty &= \lim_{k\to\infty}\mathbf{x}_k = \lim_{k\to\infty}\big(\mathbf{I}_{l\times l}-\mathbf{E}^k\big).\mathbf{x} = \big(\mathbf{I}_{l\times l}-\mathbf{E}^\infty\big).\mathbf{x} \\ &= \mathbf{V}.diag\Big(\underbrace{0,0,\ldots,0}_{c\ \ times},\underbrace{1,\ldots,1}_{l-c\ \ times}\Big).\mathbf{V}^H.\mathbf{x}\end{aligned} \tag{21}$$

Now we shall show that this result is also obtained by pseudo-inverse of $\mathbf{D}$.

$$\begin{aligned}
&\mathbf{D} = \mathbf{V}.diag\Big(\underbrace{0,\ldots,0}_{c\ \ times},d_{c+1},\ldots,d_l\Big).\mathbf{V}^H \\
\Rightarrow\quad & Pseudo-Inverse: \mathbf{D}^{+} = \mathbf{V}.diag\big(\underbrace{0,\ldots,0}_{c\ \ times},d_{c+1}^{-1},\ldots,d_l^{-1}\big).\mathbf{V}^H
\end{aligned} \tag{22}$$

$$\begin{aligned}
\Rightarrow\quad & \mathbf{D}^{+}.\mathbf{D} = \mathbf{V}.diag\Big(\underbrace{0,0,\ldots,0}_{c\ \ times},\underbrace{1,\ldots,1}_{l-c\ \ times}\Big).\mathbf{V}^H \\
\Rightarrow\quad & \mathbf{x}^{+} = \mathbf{D}^{+}.\mathbf{y} = \mathbf{D}^{+}.\mathbf{D}.\mathbf{x} \\
& \quad = \mathbf{V}.diag\Big(\underbrace{0,0,\ldots,0}_{c\ \ times},\underbrace{1,\ldots,1}_{l-c\ \ times}\Big).\mathbf{V}^H.\mathbf{x}
\end{aligned} \tag{23}$$

Comparing (21) and (23), we get:

$$\mathbf{x}_{\infty} = \mathbf{x}^{+} \tag{24}$$

Thus we prove the important result that when the matrix D is not invertible, the iterative method converges to the pseudo-inverse solution.

3.2 The Case of Additive Noise

Assuming the additive noise has zero mean and $\mathbf{D}$ is invertible, we can write (3) as:

$$\mathbf{y} = \mathbf{D}.\mathbf{x} + \mathbf{n} = \mathbf{D}.\big(\mathbf{x} + \mathbf{D}^{-1}.\mathbf{n}\big) \tag{25}$$

Thus by the same argument as given for the noiseless case, we will have:

$$\mathbf{x}_{\infty} = \lim_{k\to\infty} \mathbf{x}_k = \mathbf{x} + \mathbf{D}^{-1}.\mathbf{n} \tag{26}$$

Although the iterative method converges to the above equation, yet $\mathbf{D}^{-1}$ can amplify the noise component, and hence the final SNR may not be acceptable.

Now we assume that $\mathbf{D}$ is not invertible and has c zero eigen-values and therefore (19) holds for powers of $\mathbf{E}$. If we define $e_i = 1 - \lambda d_i$

($1 \leq c \leq l$), we will have $e_1 = \cdots = e_c = 1$ and $|e_i| < 1$ for $c+1 \leq i \leq l$. Rewriting (19), we have:

$$\begin{aligned} \mathbf{E}^i &= \mathbf{V}.diag\Big(\underbrace{1,\ldots,1}_{c\ \ times}, e_{c+1}^i, \ldots, e_l^i\Big).\mathbf{V}^H \\ \Rightarrow \sum_{i=0}^{k-1}\mathbf{E}^i &= \mathbf{V}.diag\Big(\underbrace{k,\ldots,k}_{c\ \ times}, \frac{1-e_{c+1}^k}{1-e_{c+1}}, \ldots, \frac{1-e_l^k}{1-e_l}\Big).\mathbf{V}^H \\ &= k\sum_{i=1}^{c}\mathbf{v}_i.\mathbf{v}_i^H + \sum_{i=c+1}^{l}\frac{1-e_i^k}{1-e_i}\mathbf{v}_i.\mathbf{v}_i^H \end{aligned} \tag{27}$$

Combining (8) with the above result, we get:

$$\Rightarrow \mathbf{x}_k = \big(\mathbf{I}_{l\times l} - \mathbf{E}^k\big).\mathbf{x} + \lambda\Big(k\sum_{i=1}^{c}\mathbf{v}_i.\mathbf{v}_i^H + \sum_{i=c+1}^{l}\frac{1-e_i^k}{1-e_i}\mathbf{v}_i.\mathbf{v}_i^H\Big)\mathbf{n} \tag{28}$$

When λ is properly chosen, $\mathbf{E}^k$ converges as $k \to \infty$. Moreover, since $|e_i| < 1$ for $c+1 \leq i \leq l$, we know that $e_i^k \to 0$ as k increases. Hence, the only part in (28) which may diverge is $k\big(\sum_{i=1}^{c}\mathbf{v}_i.\mathbf{v}_i^H\big).\mathbf{n}$:

$$Convergence \Leftrightarrow \big(\sum_{i=1}^{c}\mathbf{v}_i.\mathbf{v}_i^H\big).\mathbf{n} = 0 \tag{29}$$

$$\begin{aligned} &\Rightarrow \quad \mathbf{n}^H.\big(\sum_{i=1}^{c}\mathbf{v}_i.\mathbf{v}_i^H\big).\mathbf{n} = \sum_{i=1}^{c}\big(\mathbf{n}^H.\mathbf{v}_i\big).\big(\mathbf{n}^H.\mathbf{v}_i\big)^H = 0 \\ &\Rightarrow \quad \forall \quad 1 \leq i \leq c: \quad \mathbf{n}^H.\mathbf{v}_i = 0 \Rightarrow \quad \mathbf{n} \perp \mathbf{v}_1, \ldots, \mathbf{v}_c \end{aligned} \tag{30}$$

On the other hand, we know that $\{\mathbf{v}_1, \ldots, \mathbf{v}_l\}$ form an orthonormal basis for our vector space, thus $\mathbf{n} \perp \mathbf{v}_1, \ldots, \mathbf{v}_c$ implies:

$$\begin{aligned} \exists \quad \beta_{c+1}, \ldots, \beta_l : \quad \mathbf{n} &= \sum_{i=c+1}^{l}\beta_i\mathbf{v}_i = \sum_{i=c+1}^{l}\frac{\beta_i}{d_i}\mathbf{D}.\mathbf{v}_i \\ \Rightarrow \mathbf{n} &= \mathbf{D}.\Big(\sum_{i=c+1}^{l}\frac{\beta_i}{d_i}\mathbf{v}_i\Big) = \mathbf{D}.\widetilde{\mathbf{n}}, \end{aligned} \tag{31}$$

where $\widetilde{\mathbf{n}}$ is an $l \times 1$ vector. Thus we have shown that the convergence condition requires that $\mathbf{n} \in Range(\mathbf{D})$. Now if this condition is satisfied, we have convergence of the iterative method:

$$\begin{aligned} \mathbf{n} &= \mathbf{D}.\widetilde{\mathbf{n}} \Rightarrow \mathbf{y} = \mathbf{D}.\mathbf{x} + \mathbf{n} = \mathbf{D}.\big(\mathbf{x} + \widetilde{\mathbf{n}}\big) \\ &\Rightarrow \lim_{k\to\infty} \mathbf{x}_k = \mathbf{D}^{+}\big(\mathbf{x} + \widetilde{\mathbf{n}}\big) \end{aligned} \quad (32)$$

We summarize the results in the following theorem:

Theorem 1 *The iterative method described in (4) (for a Hermitian distorting matrix* $\mathbf{D}$*) converges if and only if:*

$$\begin{cases} 0 < \lambda < \frac{2}{d_{max}} \\ \\ \mathbf{n} = \mathbf{D}.\widetilde{\mathbf{n}}, \end{cases} \quad (33)$$

where $\widetilde{\mathbf{n}}$ *is an* $l \times 1$ *vector and we have:*

$$\lim_{k\to\infty} \mathbf{x}_k = \mathbf{D}^{+}.\mathbf{D}.\big(\mathbf{x} + \widetilde{\mathbf{n}}\big), \quad (34)$$

where d_{max} *is the maximum eigen-value of the matrix* $\mathbf{D}$ *and* $\mathbf{D}^{+}$ *is its pseudo-inverse. For invertible* $\mathbf{D}$ *we have* $\mathbf{D}^{+} = \mathbf{D}^{-1}$ *and* $\widetilde{\mathbf{n}} = \mathbf{D}^{-1}.\mathbf{n}$ *.*

4 Iterative Reconstruction from Uniform Samples of Bandpass Signals

In this section we study a special case which is often non-invertible. Let $x(t)$ be a bandpass signal with its Fourier transform $X(f)$ (Fig. 2). If we simply sample this signal uniformly with a rate close to the Nyquist rate, i.e., $2B$, the occurrence of aliasing effect for some frequency components is highly probable. Let $f_1, \ldots, f_N$ be equally spaced frequencies which cover the interval $[f_L, f_H]$. We define the original vector as:

$$\mathbf{x}^{(f)} = \Big[X(-f_N), \ldots, X(-f_1), X(f_1), \ldots, X(f_N)\Big]^T \quad (35)$$

After uniform sampling of $x(t)$, followed by a bandpass filtering, some frequency components are distorted due to the aliasing. We assume the interference to be as shown in Fig. 3. Thus the k higher frequency

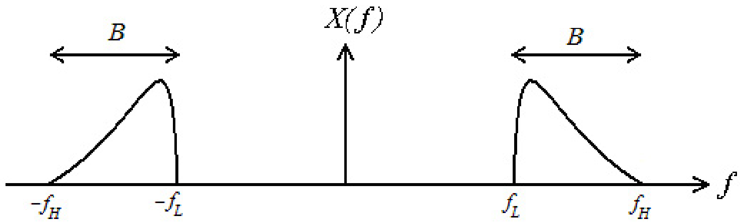

Figure 2: The spectrum of a bandpass signal with bandwidth B.

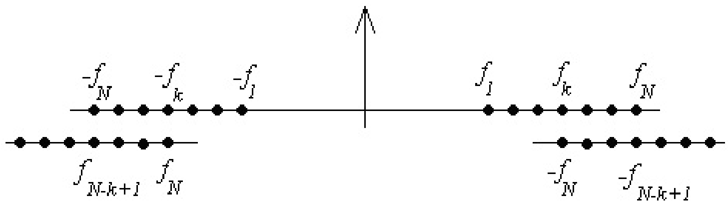

Figure 3: Aliasing effects for uniform sampling. Dots on the axis show nonzero frequency components. Aliasing effects take place between $(f_k, -f_N)$,$(\ f_{k+1}, -f_{N-1})$,$\ldots$,$(f_N\ ,\ -f_{N-k+1})$ and similarly for negative frequencies.

components will interfere with the k lower frequency components and the $2N-k$ middle components remain unchanged, hence the distortion matrix $\big($shown in (2)$\big)$ is given by:

$$\mathbf{D} = \begin{pmatrix} \mathbf{I}_{k\times k} & \mathbf{0}_{k\times(2N-k)} & \mathbf{I}_{k\times k} \\ \mathbf{0}_{(2N-2k)\times k} & \mathbf{I}_{(2N-2k)\times(2N-2k)} & \mathbf{0}_{(2N-2k)\times k} \\ \mathbf{I}_{k\times k} & \mathbf{0}_{k\times(2N-k)} & \mathbf{I}_{k\times k} \end{pmatrix} \tag{36}$$

It is evident that this matrix, only for $k=0$ is invertible and for $k>0$ is non-invertible (since the sampling scheme was irreversible). To find the pseudo-inverse of the matrix, we have to evaluate the eigen-values:

$$Eig\left\{\mathbf{D}\right\} = \left(Eig\left\{\begin{pmatrix} \mathbf{I}_{k\times k} & \mathbf{I}_{k\times k} \\ \mathbf{I}_{k\times k} & \mathbf{I}_{k\times k} \end{pmatrix}\right\}, Eig\left\{\mathbf{I}_{(2N-2k)\times(2N-2k)}\right\}\right) \tag{37}$$

$$\Rightarrow Eig\{\mathbf{D}\} = \Big(\big(\underbrace{0,\ldots,0}_{k\ times},\underbrace{2,\ldots,2}_{k\ times}\big),\big(\underbrace{1,\ldots,1}_{2N-2k\ times}\big)\Big) \tag{38}$$

Eigen-values of $\mathbf{D}$ can be translated to those of $\mathbf{E}$ as:

$$Eig\{\mathbf{E}\} = \Big(\underbrace{1,\ldots,1}_{k\ times},\underbrace{1-\lambda,\ldots,1-\lambda}_{k\ times},\underbrace{1-2\lambda,\ldots,1-2\lambda}_{2N-2k\ times}\Big) \tag{39}$$

$$\begin{aligned}\Rightarrow \mathbf{E}^{\infty} &= \mathbf{V}.diag\big(\underbrace{1,\ldots,1}_{k\ times},\ \underbrace{0,\ldots,0}_{2N-k\ times}\ \big).\mathbf{V}^T\\ &= \Big[\mathbf{v}_1|\mathbf{v}_2|\ldots|\mathbf{v}_k\Big].\Big[\mathbf{v}_1|\mathbf{v}_2|\ldots|\mathbf{v}_k\Big]^T,\end{aligned} \tag{40}$$

where $\mathbf{v}_1,\ldots,\mathbf{v}_k$ are eigen-vectors of $\mathbf{D}$ related to its zero eigen-values. It is easy to check that:

$$\mathbf{D}.\mathbf{v}_i = 0 \Rightarrow \mathbf{v}_i = \frac{1}{\sqrt{2}}\Big[\underbrace{0,\ldots,0}_{i-1},1,\underbrace{0,\ldots,0}_{2N-k-1},-1,\underbrace{0,\ldots,0}_{k-i}\Big]^T,\ 1\le i\le k \tag{41}$$

Now that we have the eigenvectors, we can evaluate the final vector produced by the iterative method. For this purpose, we have to first calculate the $\mathbf{E}^{\infty}$ matrix:

$$\mathbf{E}^{\infty} = \frac{1}{2}\begin{pmatrix} \mathbf{I}_{k\times k} & \mathbf{0}_{k\times(2N-2k)} & -\mathbf{I}_{k\times k}\\ \mathbf{0}_{(2N-2k)\times k} & \mathbf{0}_{(2N-2k)\times(2N-2k)} & \mathbf{I}_{(2N-2k)\times k}\\ -\mathbf{I}_{k\times k} & \mathbf{0}_{k\times(2N-2k)} & \mathbf{I}_{k\times k}\end{pmatrix} \tag{42}$$

$$\Rightarrow\quad \mathbf{x}^{(f)}_{\infty} - \mathbf{x}^{(f)} = -\mathbf{E}^{\infty}.\mathbf{x}^{(f)} = \frac{1}{2}\begin{pmatrix} X(f_N)-X(-f_{N-k+1})\\ \vdots\\ X(f_{N-k+1})-X(-f_N)\\ \left.\begin{matrix}0\\ \vdots\\ 0\end{matrix}\right\}2N-2k\\ X(-f_{N-k+1})-X(f_N)\\ \vdots\\ X(-f_N)-X(f_{N-k+1})\end{pmatrix} \tag{43}$$

$$\Rightarrow \quad \mathbf{x}_{\infty}^{(f)} = \mathbf{x}^{(f)} - \mathbf{E}^{\infty}.\mathbf{x}^{(f)} = \begin{pmatrix} \frac{X(f_N)+X(-f_{N-k+1})}{2} \\ \vdots \\ \frac{X(f_{N-k+1})-X(-f_N)}{2} \\ X(f_{N-k}) \\ \vdots \\ X(f_{-N+k}) \\ \frac{X(-f_{N-k+1})-X(f_N)}{2} \\ \vdots \\ \frac{X(-f_N)-X(f_{N-k+1})}{2} \end{pmatrix} \tag{44}$$

In other words, using the iterative method, we have only divided the aliased frequencies by 2. Although this result could have been achieved without the use of the iterative method, the strength of this method becomes more evident in more complex cases such as multiband signals. In fact this method, without any knowledge about the aliasing parts, accomplishes the desired results.

5 Periodic Nonuniform Sampling for Bandpass and Multi-band Signals

To avoid high sampling rates for perfect reconstruction of bandpass and multi-band signals, Periodic Nonuniform Samplings (PNS) have been suggested. In the N^{th} order PNS, the sampling points $\{t_i\}$ are found by:

$$\begin{aligned} t_{a.N+b} &= a.T + k_b \qquad a \in \mathbb{Z},\ b \in \{0, 1, \ldots, N-1\} \\ & \quad\ 0 \leq k_1 < k_2 < \cdots < k_{N-1} < T \end{aligned} \tag{45}$$

where T is the sampling period and the average sampling rate is $\frac{N}{T}$. We denote the signal and its Fourier transform by $x(t)$ and $X(f)$, respectively. If we apply the N^{th} order PNS to the signal $x(t)$, we get:

$$\begin{aligned}
x_s(t) &= x(t).\sum_{a\in\mathbb{Z}}\sum_{b=0}^{N-1}\delta(t-aT-k_b) \\
\Rightarrow X_s(f) &= X(f)*\sum_{b=0}^{N-1}\sum_{\alpha\in\mathbb{Z}}\delta(f-\frac{\alpha}{T})e^{-j2\pi k_b\frac{\alpha}{T}} \\
&= \sum_{\alpha\in\mathbb{Z}}X(f-\frac{\alpha}{T}).\underbrace{\sum_{b=0}^{N-1}e^{-j2\pi k_b\frac{\alpha}{T}}}_{L(\alpha)} \\
&= \sum_{\alpha\in\mathbb{Z}}X(f-\frac{\alpha}{T}).L(\alpha) \qquad (46)
\end{aligned}$$

As shown in the above equation, each frequency component of the sampled signal may be affected by more than one component of the original signal. Since the original signal is bandlimited, each frequency component of the sampled signal is the combination of finite components. Let f_0,f_1,...,f_m be the initial frequencies which affect the frequency f_0 of the sampled signal. Thus:

$$\begin{aligned}
\forall\, 0\le i,k\le m \quad &: \quad T.(f_i-f_k)\in\mathbb{Z} \\
X_s(f_i) &= \sum_{k=0}^{m}X(f_k).L\big(T.(f_i-f_k)\big) \qquad (47)
\end{aligned}$$

We can represent the above summations as a matrix equation:

$$\begin{aligned}
&\mathbf{x_s}^{(f)} = \mathbf{L}.\mathbf{x}^{(f)} \\
\mathbf{x}^{(f)} &\triangleq [X(f_1),X(f_2),\ldots,X(f_m)]^T \\
\mathbf{x_s}^{(f)} &\triangleq [X_s(f_1),X_s(f_2),\ldots,X_s(f_m)]^T \\
\mathbf{L} &\triangleq \Big(L\big(T.(f_i-f_k)\big)\Big)_{i,k} \quad , \quad 0\le i,k\le m \qquad (48)
\end{aligned}$$

Due to the definition of the L function we know:

$$\begin{aligned}
L_{i,k} &= L\big(T.(f_i-f_k)\big) = \sum_{b=0}^{N-1}e^{-j2\pi k_b(f_i-f_k)} = \left(\sum_{b=0}^{N-1}e^{-j2\pi k_b(f_k-f_i)}\right)^* \\
&= L^*\big(T.(f_k-f_i)\big) = L^*_{k,i} \qquad (49)
\end{aligned}$$

where $*$ denotes the conjugate operation. The above equation shows the fact that the matrix $\mathbf{L}$ is Hermitian ($\mathbf{L}^H = \mathbf{L}$).

Up to now we have only considered $m+1$ single frequency components of the continuous Fourier transform. Let us assume that we have sampled the signal $X(f)$ at the sampling points of $f \in \{\dots, \frac{-2}{r.T}, \frac{-1}{r.T}, 0, \frac{1}{r.T}, \frac{2}{r.T}, \dots\}$, where r is an arbitrary positive integer. Since the frequency step between these sampling points is an integer multiple of $1/(r.T)$, the frequency components which play a role in aliasing are either all included or all excluded. Moreover, we will discard the out of band components and only focus on the components inside the original band (by means of filtering). The original signal was bandlimited and we have only considered the inband frequency components; thus we are dealing with finite number of samples and we can consequently define the original and sampled Fourier transform vectors similar to (48). Similar to the above approach, we can show that these vectors are related to each other within a Hermitian matrix. The vectors are estimates of the continuous Fourier transforms and when the parameter r in the frequency sampling part increases, these estimates represent the continuous functions more precisely. We assume that we have chosen a large enough r so that our estimates are suitable representations. Now we are dealing with discrete signals which could be treated via vectors and as shown here, the sampling-filtering block could be modeled with a Hermitian matrix (distortion matrix).

6 Simulation results

We have verified our theoretical results with MATLAB simulations. Fig. 4 shows a bandpass signals and its reconstructed versions from uniform samples at the Nyquist rate (which are non-invertible) for different iteration numbers. It is evident that for aliased components, we have a division by 2 in the frequency domain (compare the sampled signal and the one at the 20^{th} iteration). Fig. 5 shows that after 20 iterations (for $\lambda = 0.9$) we have almost reached the final signal and the SNR curve is saturated. The case of noisy samples which we expect to diverge, is shown in Fig. 6. Although the initial SNR of noisy samples is 25^{dB}, divergence is clear by the reconstructed signal at the 20^{th} iteration. In each iteration, we improve the signal while amplifying the noise. After a finite number of iterations, the pseudo-inverse solution of the signal is reached (neglecting the noise part) but noise amplification is continued;

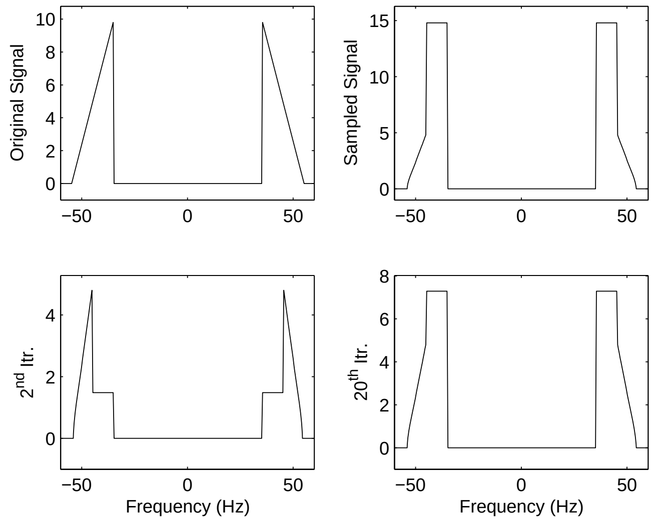

Figure 4: Spectrum of a bandpass signal and its reconstructed versions from noiseless uniform samples at the Nyquist rate with $\lambda = 0.9$.

thus we would expect to have an optimum recovered signal after a finite number of iterations which is given in the 7^{th} iteration with $\lambda = 0.9$ (Fig. 7). To include a more general case of PNS, a multiband signal is sampled at times $\{n.50^{ms}, n.50^{ms} + 37.5^{ms}, n.50^{ms} + 41.7^{ms}\}_{n\in\mathbb{Z}}$ (Fig. 8). Since SNR curve is saturated before the 20^{th} iteration (Fig. 9), the recovered signal at the 20^{th} iteration (Fig. 8) is the pseudo-inverse solution.

7 Conclusion

Uniform sampling at the Nyquist rate for bandpass (or multi-band) signals may not be sufficient to recover the signal exactly. On the other hand, various periodic nonuniform sampling schemes can be used at the Nyquist rate for perfect recovery. In either case, for discrete signals, sampling (uniform or periodic nonuniform) and filtering can be modeled as a Hermitian matrix. For the first case the matrix is usually non-invertible; for the second case the matrix may be invertible. Iterative methods can be used to find the inverse of a Hermitian matrix. Convergence analysis of the iterative methods show that for invertible matrices, the iterative

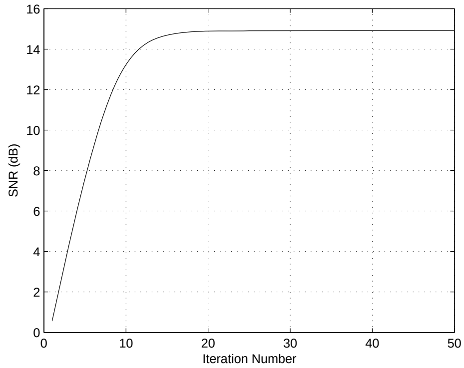

Figure 5: SNR of the reconstructed signals from noiseless uniform samples with $\lambda = 0.9$ for the signal of Fig. 4.

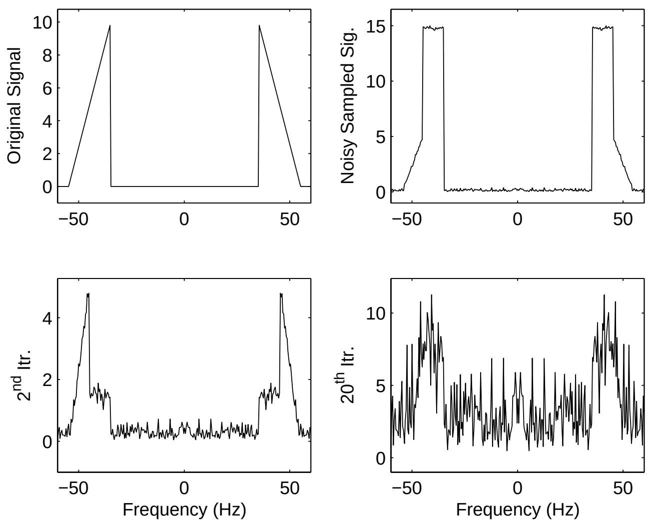

Figure 6: Spectrum of a bandpass signal and its reconstructed versions from noisy uniform samples at the Nyquist rate with $\lambda = 0.9$.

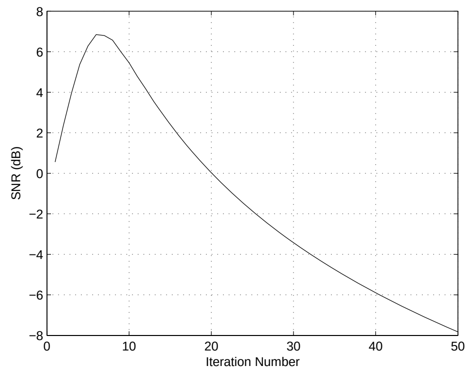

Figure 7: SNR of the reconstructed signals from noisy uniform samples with $\lambda = 0.9$ for the signal of Fig. 6.

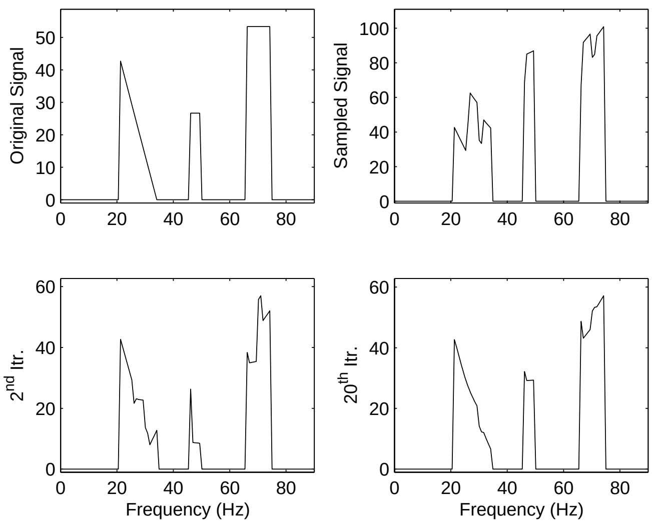

Figure 8: Spectrum of a multiband signal and its reconstructed versions from a non-invertible PNS set at the Nyquist rate with $\lambda = 0.5$.

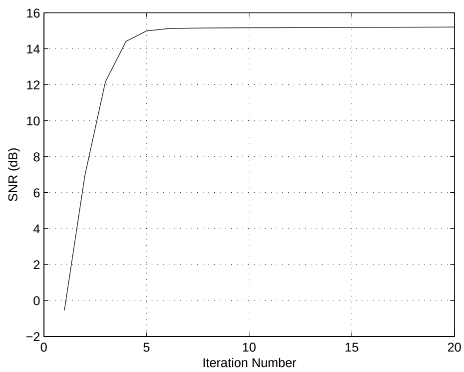

Figure 9: SNR of the reconstructed signals from a non-invertible PNS set with $\lambda = 0.5$ for the signal of Fig. 8.

method converges to the original signal and when the matrix is non-invertible, the iterative result is equivalent to the pseudo-inverse of the matrix. For the case of quantized samples or additive noise in the channel, iterative methods converge to the pseudo inverse if a finite number of iterations are used. On the other hand if infinite number of iterations are used, the recovered signal converges to the inverse of the system; in this case, depending on the noise structure, the inverse system may amplify or attenuate the noise component.

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