Two fundamental dyadic decompositions

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Abstract

We discuss two classical decompositions in harmonic analysis, namely, the Haar decomposition of square-integrable functions, and the Calderón-Zygmund decomposition of integrable functions, both of which fundamentally rely on dyadic decompositions of the real line.

1 The Haar decomposition

Let

\[ H(x) = \begin{cases} +1 & \text{if } 0 \leq x < 1/2, \\ -1 & \text{if } 1/2 \leq x < 1, \\ 0 & \text{otherwise} \end{cases} \]

Consider the (normalized) dilatations and translations of \( H(x) \), the system of functions

\[ H_{j,k}(x) = 2^{j/2}H(2^j x - k) \quad (j, k \in \mathbb{Z}). \]

Using, among other things, certain unique properties of dyadic intervals, we show that the system \( H_{j,k}(x) \) form an orthonormal basis of \( L^2(\mathbb{R}) \), that is,

- the system \( H_{j,k}(x) \) is orthonormal, and that
- every \( f(x) \) in \( L^2(\mathbb{R}) \) can be represented as

\[ f(x) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} a_{j,k} H_{j,k}(x), \]

where the series on the right converges (unconditionally) in the \( L^2 \) norm.

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Proof. The first part of the assertion follows from the observation that $H_{j,k}(x)$ is supported on a dyadic interval, that is,

$$\text{supp}(H_{j,k}) = [k2^{-j}, (k+1)2^{-j}).$$

Now, the dyadic intervals $[k2^{-j}, (k+1)2^{-j})$, $j, k \in \mathbb{Z}$, have the following structure:

(i) Either two such intervals do not intersect, or one is contained in the other;

(ii) If an interval $I_1$ is contained in $I_2$, then $I_1$ lies in either the left or right half of $I_2$.

Based on these considerations, one can easily verify that the system is indeed orthonormal, so that $\langle H_{j,k}, H_{j',k'} \rangle = 1$ if and only if $j = j'$ and $k = k'$; for all other cases, it equals zero.

The proof of the second assertion is slightly involved. We consider the closed subspace

$$W_n = \text{closure} \left( \text{span} \left\{ H_{j,k} : j < n, k \in \mathbb{Z} \right\} \right),$$

where the closure is in the $L^2(\mathbb{R})$ norm, and the closed subspace $P_n$ of $L^2(\mathbb{R})$ consisting of functions which are constant over intervals of the form $[k2^{-n}, (k+1)2^{-n})$, $k \in \mathbb{Z}$. The desired result follows once we have shown that $W_n = P_n$ for every integer $n$, since it is well-known that $\bigcup_n P_n$ is dense in $L^2(\mathbb{R})$ (simple functions are dense in $L^2(\mathbb{R})$).

Note that the nested spaces $\cdots \subset W_{-1} \subset W_0 \subset W_1 \subset \cdots$ have the property that

- $f(x) \in W_n$ if and only if $f(2x) \in W_{n+1}$, and
- $f(x) \in W_0$ if and only if $f(x + k) \in W_0$ for all $k \in \mathbb{Z}$.

A similar argument also applies to the subspaces $\cdots \subset P_{-1} \subset P_0 \subset P_1 \subset \cdots$. Thus, it will suffice to show that $W_0 = P_0$. The inclusion $W_0 \subset P_0$ is obvious. The reverse inclusion follows from the fact that $\chi_{[0,1)}(x)$, the building block for $P_0$, also belongs to $W_0$. Indeed, a simple computation shows that

$$\chi_{[0,1)}(x) = \sum_{j < 0} \sum_{k \in \mathbb{Z}} 2^{j/2} H_{j,0}(x),$$

where the series on the right (dominated by a geometric series) converges absolutely. This concludes the proof. □

2 The Calderón-Zygmund decomposition

Let $f(x)$ be an arbitrary integrable function defined on the real line, and let $\lambda$ be a positive real number. Let $\text{Avg}_A(f)$ denote

$$\text{Avg}_A(f) = \frac{1}{|A|} \int_A f(x) \, dx,$$
where \(|A|\) is the Lebesgue measure of a (measurable) subset \(A\) of the real line. One can then form a countable system of (almost) disjoint intervals \(\{I_k\}\) such that the following hold:

(i) \(|f(x)| \leq \lambda\) for almost every \(x\) outside the set \(\bigcup_k I_k\);

(ii) The average of \(|f(x)|\) on every \(I_k\) is uniformly bounded from below and above,
\[
\lambda < \operatorname{Avg}_{I_k}(|f|) < 2\lambda. \tag{1}
\]

(iii) The total length of the intervals \(\{I_k\}\) is controlled by the norm of \(f(x)\),
\[
|\bigcup_k I_k| \leq \frac{1}{\lambda} \|f\|_1. \tag{2}
\]

This is the so-called Calderón-Zygmund decomposition of \(f(x)\) at height \(\lambda\). This turns out to be very useful decomposition in harmonic analysis, since it allows one to decompose an integrable function \(f(x)\) into a so-called ‘good’ part \(g(x)\), and a ‘bad’ part \(b(x)\). The part \(g(x)\) is bounded and integrable (and hence belongs to all \(L^p(\mathbb{R})\) spaces), while \(b(x)\) is compactly supported and has a zero mean. To achieve this, one has to set
\[
g(x) = \begin{cases} 
  f(x) & \text{if } x \text{ is outside } \bigcup_k I_k, \\
  \operatorname{Avg}_{I_k}(f) & \text{if } x \in I_k,
\end{cases}
\]
and \(b(x) = f(x) - g(x)\).

**Proof.** We note that it suffices to establish the decomposition for a non-negative integrable function. Then, given an arbitrary integrable function \(f(x)\), one can simply apply the decomposition to the positive and negative components of \(f(x)\), and recombine the estimates to arrive at the desired decomposition of \(f(x)\).

Let \(f(x)\) be an non-negative integrable function. To achieve the above decomposition, we use the following stopping-time strategy: We begin by partitioning the real line into a system of dyadic intervals \(\mathcal{M}\), whose interiors are disjoint and whose common length is so large that
\[
\operatorname{Avg}_{I}(f) \leq \lambda
\]
for every \(I\) belonging to \(\mathcal{M}\). This can clearly be achieved, since \(\operatorname{Avg}_{I}(f)\) approaches zero with the increase in \(|I|\). For instance, one can take \(\mathcal{M}\) to be the system of intervals
\[
[n2^l, (n+1)2^l) \quad (n \in \mathbb{Z})
\]
where \(J\) is chosen to be sufficiently large.
Fix an interval $I_0$ belonging to $\mathcal{M}$, and split it into two halves. If we denote one of the split intervals by $I_1$, then we are faced with two distinct possibilities, either 

$$\text{Avg}_{I_1}(f) > \lambda, \text{ or } \text{Avg}_{I_1}(f) \leq \lambda.$$ 

In the former case, we stop splitting $I_1$, and select it as one of the intervals $I_k$ appearing in the decomposition. We have for it (1), since 

$$\lambda < \frac{1}{|I_1|} \int_{I_1} f(x) \, dx \leq \frac{1}{2^{-1}|I_0|} \int_{I_0} f(x) \, dx \leq 2\lambda.$$ 

In the latter case, we further split $I_1$, and keep repeating the above process until we are forced into the former case (if this happens at all).

We repeat this process starting with every interval belonging to $\mathcal{M}$. Clearly, this results in a system of almost disjoint intervals $I_k$, which are at most countable. This gives us (2), since 

$$|\bigcup_k I_k| = \sum_k |I_k| \leq \frac{1}{\lambda} \sum_k \int_{I_k} f(x) \, dx = \frac{1}{\lambda} \int_{\bigcup_k I_k} f(x) \, dx \leq \frac{1}{\lambda} \|f\|_1.$$ 

Finally, the fact that $f(x) \leq \lambda$ for almost every $x$ outside $\bigcup_k I_k$ can be deduced from the Lebesgue differentiation theorem, which states that the relation

$$f(x) = \lim_{|I| \to 0} \frac{1}{|I|} \int_I f(x - y) \, dy \quad (3)$$

holds for almost every $x \in \mathbb{R}$, provided that $f(x)$ is integrable. Indeed, for every $x$ belonging to the complement $\bigcup_k I_k$ and for sufficiently small $I$, we have by construction

$$\frac{1}{|I|} \int_I f(x - y) \, dy \leq \lambda,$$

so that, letting $|I| \to 0$ and applying (3), we arrive at the desired result. $\square$