Wiener's lemma

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Abstract

This note is about an elementary proof of the classical Wiener's lemma, which asserts that the pointwise inverse of an absolutely convergent Fourier series without zeros is again an absolutely convergent Fourier series.

Let $\mathscr{A}[0,1]$ denote the space of functions defined on $[0,1]$ having a absolutely convergent Fourier expansion. That is, every $f(t)$ belonging to $\mathscr{A}[0,1]$ is given by

$$f(t)=\sum_{k\in\mathbf{Z}} c_k e^{j2\pi kt}$$

where

$$\sum_{k\in\mathbf{Z}} |c_k| < \infty.$$

The Wiener's lemma states that the inverse of each non-zero element in $\mathscr{A}[0,1]$ is again in $\mathscr{A}[0,1]$.

Theorem 1 (Wiener's lemma)**.** *Suppose that $f(t)$ belongs to $\mathscr{A}[0,1]$ and is non-zero on $[0,1]$. Then $1/f(t)$ belongs to $\mathscr{A}[0,1]$.*

In other words, if $f(t)$ is an non-zero element of $\mathscr{A}[0,1]$, then we can write $1/f(t)$ as

$$1/f(t)=\sum_{k\in\mathbf{Z}} b_k e^{j2\pi kt}$$

where $\sum_{k\in\mathbf{Z}} |b_k| < \infty$.

To demonstrate the utility of the lemma, we state without proof the following result on the invertibility of certain convolution operators. Let $h\in\ell^1(\mathbf{Z})$ and define C_h by

$$(C_h f)(n)=(h*f)(n)=\sum_{k\in\mathbf{Z}} h(k)f(n-k).$$

Then C_h maps every f in $\ell^1(\mathbf{Z})$ into a sequence $C_h f\in\ell^1(\mathbf{Z})$. We call $h(n)$ the kernel or filter associated with C_h. Using the Wiener's lemma, one can readily obtain the following characterization of the inverse operator C_h^{-1}, if it exists.

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Theorem 2 (Inverse convolution operator)**.** *If C_h is invertible on $\ell^1(\mathbf{Z})$, then the inverse operator is again a convolution operator C_g where $g \in \ell^1(\mathbf{Z})$.*

To prove theorem 1, we need some structure on $\mathscr{A}[0,1]$. It is clear that every $f(t)$ in $\mathscr{A}[0,1]$ is continuous. Moreover, the Fourier coefficients (c_k) corresponding to a given $f(t)$ are unique. This allows one to impose a norm on $\mathscr{A}[0,1]$ by setting

$$||f||_{\mathscr{A}} = \sum_{k\in\mathbf{Z}} |c_k| \qquad (f \in \mathscr{A}[0,1]).$$

One can show that $\mathscr{A}[0,1]$ is complete with respect to this norm, making this a Banach space. Moreover, if $f(t)$ and $g(t)$ belong to $\mathscr{A}[0,1]$, then so do $f(t)g(t)$, and that

$$||fg||_{\mathscr{A}} \leq ||f||_{\mathscr{A}} ||g||_{\mathscr{A}}. \tag{1}$$

This multiplicative structure, along with its usual interaction with additions and scalar multiplications, makes $\mathscr{A}[0,1]$ a Banach algebra.

Proof of Theorem 1. Since $1/f = (f\bar{f})/|f|^2$, it suffices to prove the result for positive functions in $\mathscr{A}[0,1]$. Let $g(t)$ be such a positive function. We need to show that $1/g(t) \in \mathscr{A}[0,1]$.

Since $g(t)$ is bounded on $[0,1]$, we normalize it to $0 < g(t) \leq 1$. Consider the function $u(t) = 1 - g(t)$. Note that

$$\sum_{n=0}^{\infty} u(t)^n = \frac{1}{1-u(t)} = \frac{1}{g(t)} \qquad (t \in [0,1]). \tag{2}$$

Our goal would be to show that the above series converges in $\mathscr{A}[0,1]$. A particular way of establishing this is to show that the series

$$\sum_{n=0}^{\infty} ||u^n||_{\mathscr{A}}$$

is convergent. This would imply that the partial sums of (2) form a Cauchy sequence in $\mathscr{A}[0,1]$. Then, by the completeness of $\mathscr{A}[0,1]$, we could conclude that $1/g(t)$ is an element of $\mathscr{A}[0,1]$.

We proceed by approximating $u(t)$ by a trigonometric polynomial. Let

$$u(t) = \sum_{k\in\mathbf{Z}} c_k e^{j2\pi kt},$$

and select N to be so large that $\sum_{|k|>N} |c_k| < \varepsilon$. Setting

$$p(t) = \sum_{|k|\leq N} c_k e^{j2\pi kt}$$

we have $||u-p||_{\mathscr{A}} < \varepsilon$. In other words, we have the approximation $u(t) = p(t) + r(t)$, where $||r||_{\mathscr{A}} < \varepsilon$ (the error ε will be fixed as per requirement).

We can write

$$u^n = (p+r)^n = \sum_{i=0}^{n} \binom{n}{i} p^i r^{n-i}.$$

Then, by (1),

$$||u^n||_{\mathscr{A}} \leq \sum_{i=0}^{n} \binom{n}{j} ||p^i||_{\mathscr{A}} ||r||_{\mathscr{A}}^{n-i}.$$

A straightforward computation, fundamentally based on the trigonometric polynomial nature of $p(t)$, shows that

$$||u^n||_{\mathscr{A}} \leq (1-\delta+2\varepsilon)^n \sqrt{2Nn+1}$$

where $\delta > 0$ is the minimum of $g(t)$ over $[0,1]$. Setting $\varepsilon < \delta/2$, we have $q = 1-\delta+2\varepsilon < 1$, so that

$$\sum_{n=0}^{\infty} ||u^n||_{\mathscr{A}} \leq \sum_{n=0}^{\infty} q^n \sqrt{2Nn+1} < \infty.$$

This completes the proof. $\square$