Gabor wavelet analysis and the fractional Hilbert transform

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WAVELETS XIII, 2009
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The dual-tree transform

**Hilbert transform (HT) pairs of wavelet bases**

- Dual-tree transform introduced by Kingsbury to improve the shiftability of the decimated discrete wavelet transforms [Kingsbury, 2001].
- HT pairs of wavelet bases \{\psi_\alpha(x)\} and \{\psi'_\alpha(x)\} [Selesnick, 2001].
- Formally, given a **primary** wavelet basis \{\psi_\alpha(x)\}, one constructs a **secondary** wavelet basis \{\psi'_\alpha(x)\} with the correspondence

\[
\psi'_\alpha(x) = \mathcal{H}\psi_\alpha(x) \quad \text{(for every } \alpha)\]

where \(\mathcal{H}\) denotes the HT operator,

\[
\mathcal{H} f(\omega) = -j \text{sign}(\omega) \hat{f}(\omega).
\]

- \(\mathcal{H}\) maps \(\cos(\omega_0 x)\) into \(\sin(\omega_0 x)\), and acts as a orthogonal transform on finite-energy signals.
The dual-tree transform

The invariance trick

- The primary (dyadic) basis generated through dilations and translations:

  \[ \psi_{i,k}(x) = \Xi_{i,k} \psi(x) \quad (i, k \in \mathbb{Z}) \]

  where \( \Xi_{i,k} \) is the discrete dilation-translation operator.

- **Invariances of the HT** [Chaudhury and Unser, 2009]:
  (i) \( \mathcal{H} \) is unitary; maps a basis into a basis,
  (ii) \( \mathcal{H} \) commutes with dilations and translations, particularly with \( \Xi_{i,k} \).

- Setting \( \psi'(x) = \mathcal{H} \psi(x) \) suffices; the functions

  \[ \psi'_{i,k}(x) = \Xi_{i,k} \psi'(x) \]

  constitute the secondary wavelet basis.

- Identical argument holds for the dual bases \( \{ \tilde{\psi}_{i,k}(x) \} \) and \( \{ \tilde{\psi}'_{i,k}(x) \} \).
Signal analysis: local amplitude-phase factors

- Input signal $f(x)$ is analyzed in both the bases:

$$f(x) = \begin{cases} \sum_{(i,k) \in \mathbb{Z}^2} a_i[k] \psi_{i,k}(x), \\ \sum_{(i,k) \in \mathbb{Z}^2} b_i[k] \psi'_{i,k}(x) \end{cases}.$$ 

- Notion of a complex analysis wavelet

$$\tilde{\Psi}(x) = \frac{1}{2} \left( \tilde{\psi}(x) + j \tilde{\psi}'(x) \right).$$

- Complex wavelet coefficients

$$c_i[k] = \langle f, \tilde{\Psi}_{i,k} \rangle = \frac{1}{2} (a_i[k] + j b_i[k]),$$

and the associated **amplitude-phase factors** $|c_i[k]| e^{j \phi_i[k]}$. 
Multiresolution Gabor-like transform realized within the framework of the dual-tree transform [C. and Unser, 2009].

The family of fractional (semi-orthogonal) spline wavelets\(^1\) \(\psi(x; \alpha, \tau)\) is closed w.r.t the HT:

\[
\mathcal{H}\{\psi(x; \alpha, \tau)\} = \psi(x; \alpha, \tau + 1/2).
\]

Moreover, the complex spline wavelet

\[
\Psi(x) = \psi(x; \alpha, \tau) + j\psi(x; \alpha, \tau + 1/2)
\]

asymptotically converges to a Gabor function [Unser et al.]:

\[
\Psi(x; \alpha) \sim \varphi(x) \exp(j\omega_0 x + \xi_0) \quad (\alpha \to +\infty).
\]

(Movie1) Evolution of \(\psi(x; \alpha, \tau)\) with the increase in \(\alpha\).

\(^1\) indexed by the approximation order \(\alpha + 1\) and shift \(\tau\).
The fractional Hilbert transform

- Representation of $f(x)$ in terms of the amplitude-shift factors $\tau$.
- The group of fractional Hilbert transforms $^2$ (fHT)

\[ H_\tau = \cos(\pi \tau) I - \sin(\pi \tau) H \quad (\tau \in \mathbb{R}) \]

comprising the identity ($I$) and the HT ($H$) operator.

- Interpolates the phase-shift property of the HT:

\[ H_\tau \{ \cos(\omega_0 x) \} = \cos(\omega_0 x + \pi \tau). \]

- The fHTs inherit the invariances of the HT:
  (i) Preserves (norm) energy.
  (ii) Invariant to translations and dilations.

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$^2$linear shift $\tau = \phi/\pi$ corresponding to the phase factor $\phi$
Signal reconstruction from the amplitude-phase factors:

\[ f(x) = \frac{1}{2} \sum_{(i,k) \in \mathbb{Z}^2} \left( a_i[k] \psi_{i,k}(x) + b_i[k] \psi'_{i,k}(x) \right) \]

\[ = \sum_{(i,k) \in \mathbb{Z}^2} |c_i[k]| \mathcal{H}_{\phi_i[k]/\pi} \{ \psi_{i,k}(x) \} \]

\[ = \sum_{(i,k) \in \mathbb{Z}^2} |c_i[k]| \Xi_{i,k} \{ \psi(x; \tau_i[k]) \}. \]

The fractionally-shifted wavelets

\[ \psi(x; \tau_i[k]) = \mathcal{H}_{\tau_i[k]} \psi(x) \]

have identical norms.

\[ \Rightarrow |c_i[k]| \text{ indicates the strength of local wavelet correlation.} \]

Local signal displacement encoded in the shift \( \tau_i[k] \).

\[ \Rightarrow \text{ specifies the most "appropriate" wavelet within the family } \{ \mathcal{H}_\tau \psi_{i,k} \}_{\tau \in \mathbb{R}}. \]
fHT of a modulated wavelet

- The case when $\psi(x)$ is a modulated wavelet of the form
  \[ \psi(x) = \varphi(x) \cos(\omega_0 x + \xi_0). \]
  
- The phase-shift action of the fHT is preserved in the presence of the window (under appropriate conditions):

**Proposition (Extension of the Bedrosian theorem)**

Let $\varphi(x)$ be bandlimited to $(-\omega_0, \omega_0)$. Then the following holds

\[ \mathcal{H}_\tau \{ \varphi(x) \cos(\omega_0 x) \} = \varphi(x) \cos(\omega_0 x + \pi \tau). \]

$\implies$ The fHT acts only on the phase of the oscillation while the window remains fixed.
Characterization of the Gabor-like transform

- Mutiresolution windowed-Fourier-like representation,

\[ f(x) \sim \sum_{(i,k) \in \mathbb{Z}^2} \varphi_{i,k}(x) \Xi_{i,k} \left\{ |c_i[k]| \cos (\omega_0 x + \xi_0 + \pi \tau_i[k]) \right\}. \]

- \( \varphi_{i,k}(x) \): fixed Gaussian window at scale \( i \) and translation \( k \).
- \( c_i[k] \): measures the local signal energy.
- \( \tau_i[k] \): shift applied to the modulating sinusoid – the oscillation is shifted to fit the underlying signal singularities/transitions.
Action of the fHT on the Gabor-like wavelet

- **(Movie2)** Visualization of the action of the fHTs on the Gabor-like wavelet.

![Wavelet Visualization](image)

**Figure:** Quadrature pairs of Gabor-like spline wavelets obtained by the action of fHT. Blue: $\mathcal{H}_\tau \psi(x; 8, 0)$, Red: $\mathcal{H}_\tau + \frac{1}{2} \psi(x; 8, 0)$, Black: The fixed Gaussian-like localization window.
Kingsbury constructed direction-selective wavelets by appropriately combining the positive and negative frequency bands of analytic wavelets [Kingsbury, 2001].

Four separable multiresolutions, total of \(3 \times 4 = 12\) separable wavelets.

**Direction-selective** complex wavelets

\[ \Psi_1(x), \ldots, \Psi_6(x) \]

realized through linear combinations of the 12 wavelets.

Similarly, one has the six complex duals \(\tilde{\Psi}_1(x), \ldots, \tilde{\Psi}_6(x)\).
**Directional Gabor-like wavelets**

- Tensor products involving B-spline scaling function and B-spline wavelets [C. and Unser, 2009].
- Dual-tree wavelets resemble Gaussian-windowed plane waves.

**Figure:** Left: Real component of the complex wavelets, Right: Magnitude envelope of the complex wavelets.
Directional HT (dHT)

- Correspondence between the real and imaginary components?
- Directional extension of the HT:
  \[ \hat{\mathcal{H}}_{\theta} f(\omega) = -j \text{sign}(u^T_\theta \omega) \hat{f}(\omega), \]

  where \( u_\theta \) is the unit vector along direction \( \theta \).

- The dHT correspondences for the complex wavelets [C. and Unser, 2009]:
  \[ \Psi_\ell(x) = \psi_\ell(x) + j \mathcal{H}_{\theta_\ell} \psi_\ell(x) \quad (\ell = 1, \ldots, 6) \]

  where \( \theta_1 = \theta_2 = 0; \theta_3 = \theta_4 = \pi/2; \theta_5 = \pi/4; \) and \( \theta_6 = 3\pi/4 \).
Bivariate dual-tree wavelets

**Fractional directional HT**

- (Notion of direction-selective phase shifts) Fractional extensions of the directional HT:

  \[ \mathcal{H}_{\theta, \tau} = \cos(\pi \tau) \mathcal{J} - \sin(\pi \tau) \mathcal{H}_{\theta} \quad (\tau \in \mathbb{R}). \]

- They are unitary, and commute with translations and (uniform) dilations.
- Action on windowed plane waves of the form \( \varphi(x) \cos(\Omega u_{\theta}^T x) \):

  **Proposition**

  *Suppose that \( \varphi(x) \) be bandlimited to the disk \( \{ \omega : ||\omega|| < \Omega \} \). Then*

  \[ \mathcal{H}_{\theta, \tau} \{ \varphi(x) \cos(\Omega u_{\theta}^T x) \} = \varphi(x) \cos(\Omega u_{\theta}^T x + \pi \tau). \]
- Complex wavelet coefficients

\[ c_{\ell,i}^k = \frac{1}{4} \langle f, \tilde{\Psi}_{\ell,i,k} \rangle. \]

- Signal representation for the bivariate dual-tree transform [4]:

\[ f(x) = \sum_{(\ell,i,k)} |c_{\ell,i}^k| \Xi_{\ell,i,k} \{ \psi_{\ell}(x; \tau_{\ell,i}^k) \}. \]

- The explicit form for the Gabor-like transform:

\[ f(x) = \sum_{(\ell,i,k)} \underbrace{\varphi_{\ell,i,k}(x)}_{\text{fixed window}} \Xi_{\ell,i,k} \left\{ \underbrace{|c_{\ell,i}^k| \cos \left( \Omega_{\ell} u_{\theta_{\ell}}^T x + \pi \tau_{\ell,i}^k \right)}_{\text{variable amp–phase directional wave}} \right\}. \]

\[ \implies \text{Superposition of direction-selective plane waves affected with appropriate phase-shifts (locally).} \]
Remarks

1. A mathematical framework linking the reconstructed signal to the processed complex wavelet coefficients.
2. Applicable to generic modulated wavelets, e.g., Shannon wavelet.
4. Potential interest in applications involving the dual-tree transform (e.g., signal denoising, texture analysis/synthesis).
Thank you!

