Quantitative orientation analysis

The aim is to characterize the orientation and isotropy properties of a region of interest (ROI) in an image. To that end, we first define the weighted inner product

$$\langle f, g\rangle_w = \iint_{R^2} w(x,y) f(x,y) g(x,y) dxdy \qquad (1)$$

where $w(x,y) \geq 0$ is a weighting function that specifies the area of interest. It is typically a normalized square window of size L that is centered on a location of interest (x_0, y_0). The norm associated with this inner-product is $\|f\|_w = \sqrt{\langle f, f\rangle_w}$. Next, we consider the derivative in the direction specified by the unit vector $\mathbf{u}_\theta = (\cos\theta, \sin\theta)$, which is given by

$$D_{\mathbf{u}_\theta} f(x,y) = \mathbf{u}_\theta{}^T \nabla f(x,y)$$

where $\nabla f = (f_x, f_y)$ is the gradient of the image under consideration. We are now interested in finding the direction $\mathbf{u}$ along which the directional derivative is maximized over the ROI. It is given by

$$\mathbf{u}_\theta = \arg \max_{\|\mathbf{u}\|=1} \|D_{\mathbf{u}} f\|_w^2. \qquad (2)$$

A standard inner-product manipulation then yields

$$\|D_{\mathbf{u}} f\|_w^2 = \langle \mathbf{u}^T \nabla f, \nabla f^T \mathbf{u}\rangle_w = \mathbf{u}^T \mathbf{J}\mathbf{u} \qquad (3)$$

where

$$\mathbf{J} = \langle \nabla f, \nabla f^T\rangle_w = \begin{bmatrix} \langle f_x, f_x\rangle_w & \langle f_x, f_y\rangle_w \\ \langle f_x, f_y\rangle_w & \langle f_y, f_y\rangle_w \end{bmatrix}$$

is the so-called structure tensor, which is a 2 x 2 symmetric positive-definite matrix. The solution of the optimization problem (2) is obtained by setting the derivative of $\mathbf{u}^T\mathbf{J}\mathbf{u} + 1 - \frac{\lambda}{2}\mathbf{u}^T\mathbf{u}$ with respect to $\mathbf{u}$ to zero, which yields the eigenvector equation: $\mathbf{J}\mathbf{u} = \lambda\mathbf{u}$.
This implies that the first eigenvector of $\mathbf{J}$ gives the dominant orientation of the ROI; the corresponding eigenvalue is $\lambda_{\max} = \max\|D_{\mathbf{u}} f\|_w^2$. Conversely, the directional derivative is minimized in the orthogonal direction given by the second eigenvector; i.e., $\lambda_{\min} = \min\|D_{\mathbf{u}} f\|_w^2$. This clearly shows that the structure tensor contains all the relevant directional information. The features are:

- Orientation: $\theta = \frac{1}{2}\arctan\left(2\frac{\langle f_x, f_y\rangle_w}{\langle f_y, f_y\rangle_w - \langle f_x, f_x\rangle_w}\right)$
- Coherency: $C = \frac{\lambda_{\max} - \lambda_{\min}}{\lambda_{\max} + \lambda_{\min}} = \frac{\sqrt{\left(\langle f_y, f_y\rangle_w - \langle f_x, f_x\rangle_w\right)^2 + 4\langle f_x, f_y\rangle_w}}{\langle f_x, f_x\rangle_w + \langle f_y, f_y\rangle_w}$, $C \in [0..1]$
- Gradient Energy: $E = \text{Trace}(\mathbf{J})$

The coherency indicates if the local image features are oriented or not: C is 1 when the local structure has one dominant orientation and C is 0 if the image is essentially isotropic in the local neighborhood.