ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

An Error Analysis for the Sampling of Periodic Signals

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Outline

1. Introduction

2. Sampling of Periodic Signals

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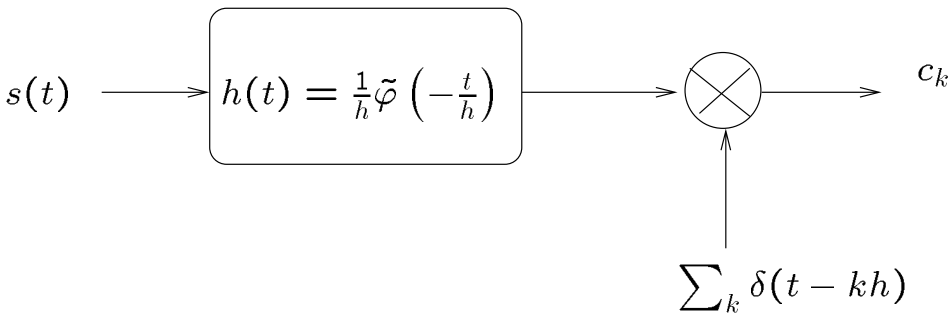
4. Asymptotic Performance

5. Experimental Validation

6. Conclusions

Introduction

Generalized Sampling



$$c_k = \left\langle s(t), \frac{1}{h}\tilde{\varphi}\left(\frac{t}{h} - k\right)\right\rangle$$

$$s_h(t) = \sum_{k=-\infty}^{\infty} c_k \varphi\left(\frac{t}{h} - k\right)$$

- Choice of φ $\Longleftrightarrow$
  - Bandlimited,
  - Spline
  - Wavelet representations
- Well developed theory for $L_2(\mathbb{R})$
- Easily extendable to periodic case by periodization of φ and $\tilde{\varphi}$ [Chuang]

Quantitative Error Analysis

- Sharp Error Estimates [Blu et. al.] → Quantification of Approximation Error

$$MSE(h) \approx \sqrt{\frac{1}{2\pi}\int |\hat{s}\,(\omega)\,|^2 E(h\omega)d\omega}$$

⇒ Optimal choice of sampling step, basis functions and algorithms

- Square modulus of Fourier transform not defined ⇒ Extension

Sampling of Periodic Signals

- Representation of closed curves

- Signal representation with boundary conditions(eg. periodic boundary conditions)

Synthesis and analysis equations

$$s_N(t) = \sum_{k=0}^{N-1} c_k \varphi_p\left(\frac{t}{h} - k\right)$$

$$c_k = \int_0^T s\left(\xi\right) \tilde{\varphi}_p\left(\frac{\xi}{h} - k\right) d\frac{\xi}{h},$$

where

$$\varphi_p(t) = \sum_{l=-\infty}^{\infty} \varphi\left(t - lN\right); \quad T = Nh$$

Approximation Operator

$$\begin{aligned} s_N(t) &= \mathcal{Q}_N s(t) \\ &= \sum_{k=0}^{N-1} \left[\int_0^T s(\xi)\tilde{\varphi}_p\left(\frac{\xi}{h} - k\right) d\frac{\xi}{h}\right] \varphi_p\left(\frac{t}{h} - k\right) \end{aligned}$$

Projection $\leftrightarrow$ Consistent Sampling iff φ_p and $\tilde{\varphi}_p$ are biorthogonal.

Computation of the Square Error

- Reconstruction in V_φ : Not shift invariant in general

Error is dependent on

1. Time shift of the function — τ

$$s(t) \rightarrow s_\tau(t) = s(t-\tau)$$

2. Number of samples — N

Mean Square Approximation Error

$$\begin{aligned}[\gamma_s(\tau, N)]^2 &= \frac{1}{T}\int_0^T |s_\tau(t) - \mathcal{Q}_N s_\tau(t)|^2\, dt \\ &= ||s_\tau - \mathcal{Q}_N s_\tau||^2_{L_2(0,T]}\end{aligned}$$

$T = Nh \quad \Rightarrow \quad \gamma_s(\tau, N)$ is h periodic in τ

Phase is unknown in most applications $\rightarrow$ An average measure of error is desirable

Average Approximation Error

$$\eta_s(N) = \sqrt{\frac{1}{h}\int_0^h [\gamma_s(\tau, N)]^2 \, d\tau}$$

Theorem 1 :

$$\eta_s(N) = \sqrt{\sum_{k=-\infty}^{\infty} |S(k)|^2 E\left(\frac{2\pi k}{N}\right)}$$

$$\begin{aligned} E(\omega) &= \left|1 - \hat{\tilde{\varphi}}(\omega)^* \, \hat{\varphi}(\omega)\right|^2 \\ &\qquad + |\hat{\tilde{\varphi}}(\omega)|^2 \sum_{n \neq 0} |\hat{\varphi}(\omega + 2n\pi)|^2 \\ &= \underbrace{1 - \frac{|\hat{\varphi}(\omega)|^2}{\hat{a}_\varphi(\omega)}}_{E_{\mathrm{min}}(\omega)} + \underbrace{\hat{a}_\varphi(\omega)\left|\hat{\tilde{\varphi}}(\omega) - \hat{\varphi}_d(\omega)\right|^2}_{E_{\mathrm{res}}(\omega)}, \end{aligned}$$

$$S(k) = \frac{1}{M}\int_0^M s(x) e^{j\frac{2\pi k x}{M}} dx$$

Here,

$$\hat{a}_\varphi(\omega) = \sum_{k=-\infty}^{\infty} |\hat{\varphi}(\omega + 2n\pi)|^2$$

and

$$\hat{\varphi}_d(\omega) = \frac{\hat{\varphi}(\omega)}{\hat{a}_\varphi(\omega)}$$

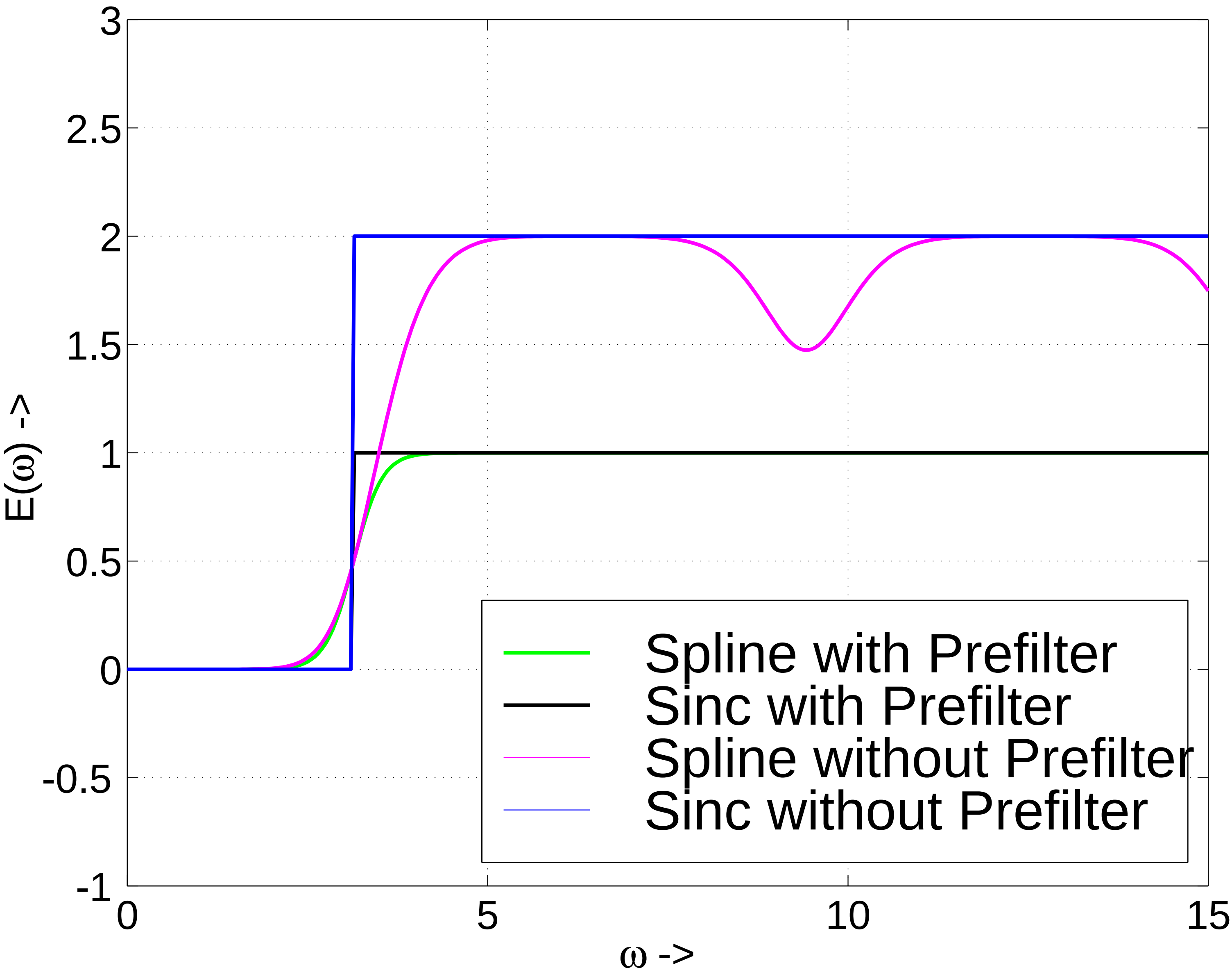
- $\eta_s(N)$ independent of τ $\Rightarrow$ No alias terms

- Kernel identical to $L_2(\mathbb{R})$ case

- Error formula is a discrete sum as compared to an integral in $L_2(\mathbb{R})$

- $\tilde{\varphi} = \varphi_d \Rightarrow E(\omega) = E_{\text{min}}(\omega)$ (orthogonal projection)

$$\eta_s(N) = \eta_{s,\text{min}}(N); \quad \forall N \quad \text{iff} \quad \tilde{\varphi} = \varphi_d$$



Error kernels for cubic B-Spline and Sinc interpolation

Asymptotic Performance

Analysis of $\eta_s(N)$ as $N \to \infty$

Is linked to the behavior of the kernel as $\omega \to 0$

Rate of Decay of Error $\Longleftrightarrow$ Differentiability of the kernel at the origin

Strang Fix Conditions of Order L $\left(E(\omega) = \mathcal{O}\left(\omega^{2L}\right)\right)$

$$\hat{\varphi}(0) \neq 0$$

$$\hat{\varphi}^{(n)}(2k\pi) = 0, \ \forall k \in \mathbb{Z} \setminus \{0\}$$

for $n = 0, 1 \ldots L-1$

$\rightarrow$ φ is called an L^{th} order scaling function

Theorem 2 : *Let φ and $\tilde{\varphi}$ be bi-orthonormal*

φ is an L^{th} order generating function

$$\Updownarrow$$

$$\eta_s(N) \underset{N\to\infty}{=} C_{\varphi,\tilde{\varphi}} \,||\,(2\pi k)^L\, S(k)||_{l_2} \left(\frac{1}{N}\right)^L$$

$$= C_{\varphi,\tilde{\varphi}}\, T^L \,||s^{(L)}||_{L_2[0,T)} \left(\frac{1}{N}\right)^L$$

$$C_{\varphi,\tilde{\varphi}} = \frac{1}{L!}\sqrt{|m^L_{\varphi_d} - m^L_{\tilde{\varphi}}|^2 + \sum_{k\neq 0} |\hat{\varphi}^L(2\pi k)|^2}$$

- Similar expression as [Unser] for functions in $L_2(\mathbb{R})$
- $m^L_{\varphi_d} = m^L_{\tilde{\varphi}} \quad \Rightarrow \quad C_{\varphi,\tilde{\varphi}} = C^-_{\varphi}$

Minimum attainable constant

Independent of the analysis function

Experimental Validation

Reference shape

Map of Switzerland represented as a polygon with 807 edges

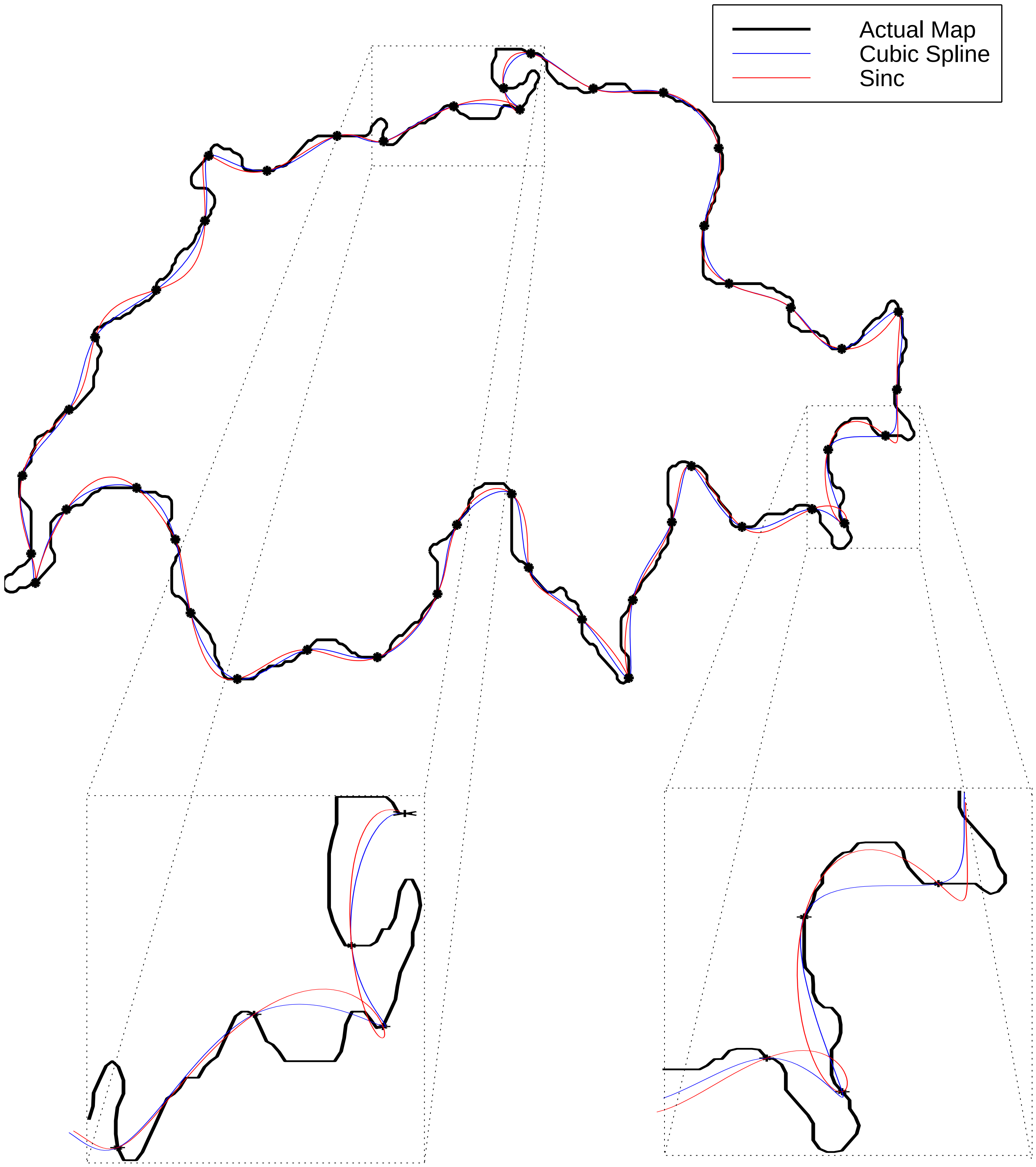
Represented using two periodic functions $x(t)$ and $y(t)$

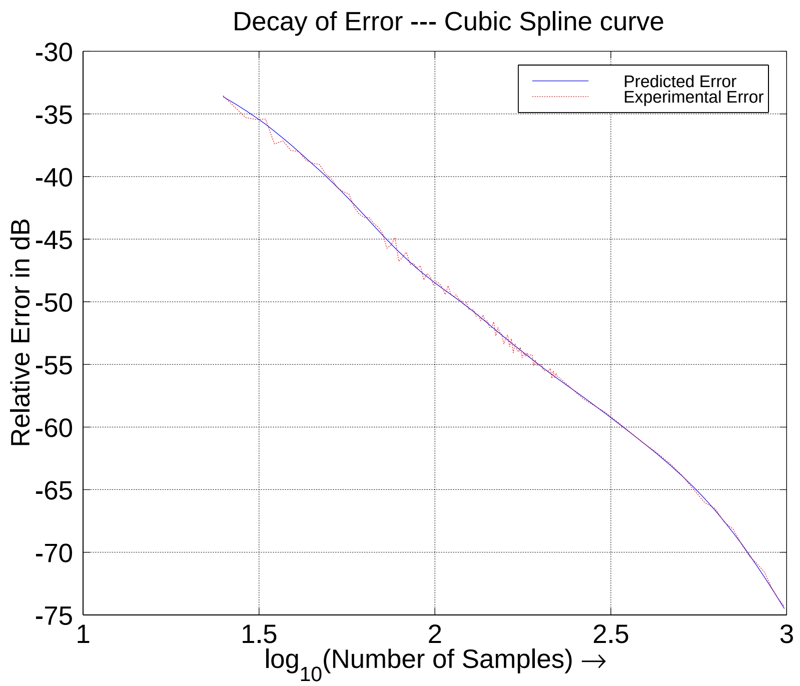
Experiment

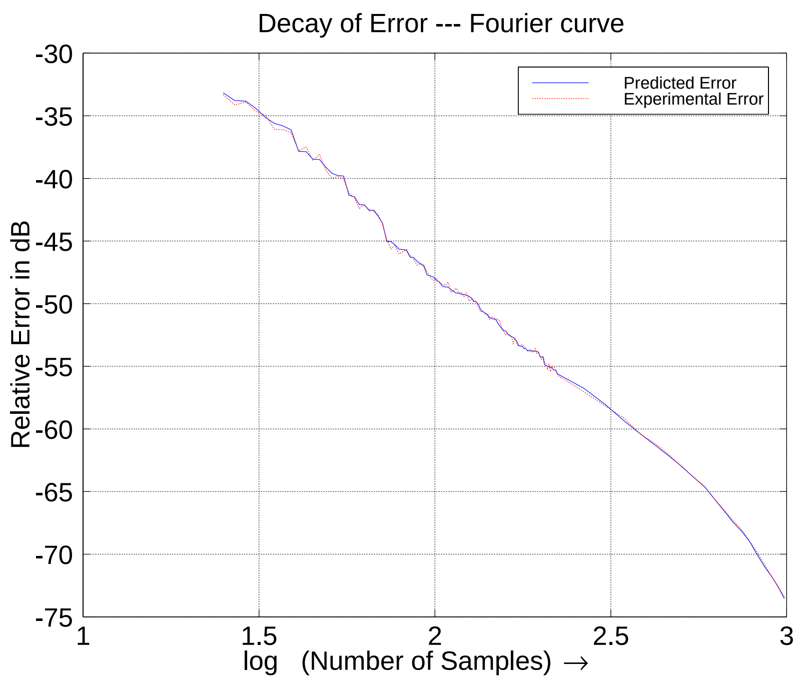
a) Initial model resampled

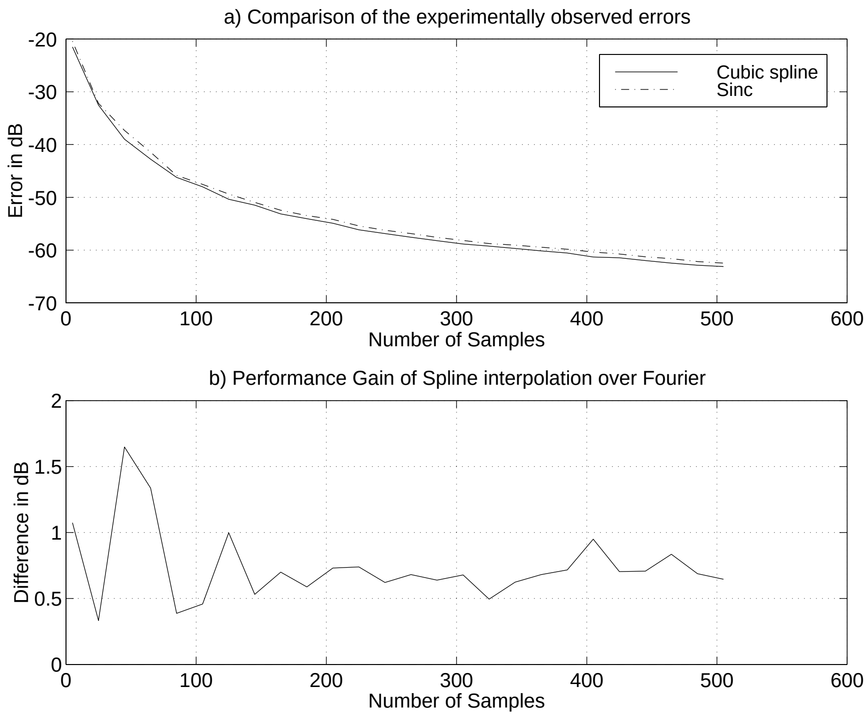
b) Interpolated with
1) Cubic spline
2) Sinc $\Leftrightarrow$ Fourier representation

c) Error compared with theoretical prediction


Actual Map
Cubic Spline
Sinc


Decay of Error --- Cubic Spline curve
Predicted Error
Experimental Error
Relative Error in dB
-30
-35
-40
-45
-50
-55
-60
-65
-70
-75
1
1.5
2
2.5
3
$\log_{10}$(Number of Samples) →


Decay of Error --- Fourier curve
Predicted Error
Experimental Error
Relative Error in dB
-30
-35
-40
-45
-50
-55
-60
-65
-70
-75
1
1.5
2
2.5
3
log (Number of Samples) →



Conclusions

- Exact expression of approximation error for periodic signals
- Allows the optimization of φ and $\tilde{\varphi}$
- Experimental verification of the error formula
- Asymptotic performance