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Scaling Limits of Solutions of Linear Stochastic Differential Equations Driven by Lévy White Noises

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Abstract Consider a random process s that is a solution of the stochastic differential equation $\mathrm{L}s = w$ with L a homogeneous operator and w a multidimensional Lévy white noise. In this paper, we study the asymptotic effect of zooming in or zooming out of the process s. More precisely, we give sufficient conditions on L and w such that $a^H s(\cdot/a)$ converges in law to a non-trivial self-similar process for some H, when $a \to 0$ (coarse-scale behavior) or $a \to \infty$ (fine-scale behavior). The parameter H depends on the homogeneity order of the operator L and the Blumenthal–Getoor and Pruitt indices associated with the Lévy white noise w. Finally, we apply our general results to several famous classes of random processes and random fields and illustrate our results on simulations of Lévy processes.

Keywords Lévy white noises · Linear SDE · Scaling limit · Self-similar processes

Mathematics Subject Classification (2010) 60G18 · 60G20 · 60G51

1 Introduction

A random process s is self-similar if there exists H, called the self-similarity index of s, such that the rescaled process $a^H s(\cdot/a)$ has the same probability law than s for every $a > 0$. A Lévy process is a stochastically continuous random process $X = (X(t))_{t\in\mathbb{R}}$ that vanishes at 0 and with stationary and independent increments. When the marginals of $X(t)$ are symmetric-α-stable (SαS), the process X is self-similar [46]. More precisely, for the SαS process X_α with $0 < \alpha \leq 2$, we have that

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$$a^{1/\alpha} X_\alpha(t/a) \overset{(d)}{=} X_\alpha(t), \tag{1}$$

for every $t \in \mathbb{R}$ and $a > 0$. The self-similarity index of X_α is therefore $H = 1/\alpha$. The case $\alpha = 2$ corresponds to the Brownian motion. However, the Lévy process X is no longer self-similar when the noise is not stable.

The study of self-similar processes (indexed by $t \in \mathbb{R}$) and fields (indexed by $\boldsymbol{x} \in \mathbb{R}^d$ with $d \geq 2$) is a branch of probability theory [17]. They have been applied in areas such as signal and image processing [7,19,43] or traffic networks [34,40], among others [36,37]. Many prominent random processes are self-similar, including fractional Brownian motion [38], its higher-order extensions [42], infinite-variance stable processes [46], and their fractional versions [29]. The case of random fields has also been investigated in both the Gaussian [2,4,15,35,52] and the α-stable case [1,2,6].

As already seen below, the self-similarity is intimately linked with stable laws [46], since they are the only possible probabilistic limits of the renormalized sum of independent and identically distributed random variables. This is the well-known (generalized) central limit theorem [24, Sect. XVII-5], with the consequence that self-similar processes are often scaling limits of many discretization schemes and stochastic models [5,11,16,32,50].

The self-similarity imposes a strong constraint on the law of the random process. In particular, it intimately links the behaviors at coarse and fine scales. We have mentioned how the self-similar models have been successfully used, but it can also appear to be too restrictive. One advantage of the family of Lévy processes and their generalizations is to overcome this restriction.

In this paper, we focus on the impact of rescaling operations for a broad class of random processes that are asymptotically or locally self-similar. These processes are specified as the solutions of a stochastic differential equation (SDE) of the form

$$\mathrm{L}s = w, \tag{2}$$

where w a multidimensional Lévy white noise and L is a differential operator on the functions from $\mathbb{R}^d$ to $\mathbb{R}$. We assume moreover that the operator L is homogeneous of some order $\gamma \geq 0$, in the sense that $\mathrm{L}\{\varphi(\cdot/a)\} = a^{-\gamma}(\mathrm{L}\varphi)(\cdot/a)$ for any function φ and $a > 0$. Typically, in dimension $d = 1$, the derivative is homogeneous of order 1. Our aim is to study the statistical behavior of the rescaling $\boldsymbol{x} \mapsto s(\boldsymbol{x}/a)$ of a solution of (2) when $a > 0$ is varying. Our two main questions are:

- What is the asymptotic behavior of $s(\cdot/a)$ when we zoom out the process (*i.e.*, when $a \to 0$)?
- What is the local behavior when we zoom in (*i.e.*, when $a \to \infty$)?

Our main contribution is to identify sufficient conditions such that the rescaling $a^H s(\cdot/a)$ of a solution of (2) has a non-trivial self-similar asymptotic limit as a goes to 0 or ∞. When this limit exists, the parameter H is unique and depends essentially on the degree of homogeneity γ of L and on the Blumenthal–Getoor and Pruitt indices β_∞ and β_0 of w [8,44]. The indices β_0 and β_∞ are used in the literature to characterize the asymptotic and local behaviors of Lévy processes and more generally Lévy-type processes [9]. We summarize the main results of our paper in Theorem 1.1. Precise definitions and rigorous statements are given later.

Theorem 1.1 *Let* L *be a* γ*-homogeneous operator and* w *a Lévy process with indices* β_∞ *and* β_0*. Under some technical conditions, the solution* s *to the equation* $\mathrm{L}s = w$ *has the following properties.*

- *The process* s *is asymptotically self-similar of order* $H_\infty = \gamma + d(1/\beta_0 - 1)$*, in the sense that* $a^{H_\infty} s(\cdot/a)$ *converges to a self-similar process of order* H_∞ *when* $a \to 0$.
- *The process* s *is locally self-similar of order* $H_{\text{loc}} = \gamma + d(1/\beta_\infty - 1)$*, in the sense that* $a^{H_{\text{loc}}} s(\cdot/a)$ *converges to a self-similar process of order* H_{loc} *when* $a \to \infty$*.*

□

The paper is organized as follows: In Sect. 2, we introduce the framework of generalized random processes, while the general class of random processes of interest is addressed in Sect. 3. We precise and demonstrate Theorem 1.1 in Sect. 4, where we identify sufficient conditions under which the asymptotic behavior of $a^H s(\cdot/a)$ is known at coarse and fine scales. We also investigate the necessity of these conditions. Finally, we apply our results in Sect. 5 to specific classes of random processes, for different types of white noises and operators.

2 Generalized Random Processes

The theory of generalized random processes was introduced independently in the 1950s by Itô [30] and Gelfand [26]. Among the benefits of this theory to the construction and study of random processes, we mention:

- *Its generality* It allows one to define the broadest class of linear processes, including processes with no pointwise interpretation such as Lévy white noises.
- *The availability of an infinite-dimensional Bochner theorem* The characteristic functional (see Definition 2.2) characterizes the law of a generalized random process in the same way the characteristic function does for random variables. This allows for the construction of generalized random processes via their characteristic functional (see Theorem 2.3 below).
- *The availability of an infinite-dimensional Lévy continuity theorem* The convergence in law of a sequence of random vectors is equivalent to the pointwise convergence of the corresponding characteristic functions. This result is also true for the generalized random processes defined over the extended space of tempered distribution $\mathcal{S}'(\mathbb{R}^d)$ (see Theorem 2.5). This provides a powerful tool to show the convergence in law of random processes. We shall exploit this tool extensively in this paper.

2.1 Generalized Random Processes and Their Characteristic Functional

The space of rapidly decaying and infinitely differentiable functions is denoted by $\mathcal{S}(\mathbb{R}^d)$ and endowed with its usual Fréchet topology. Its continuous dual, the space of tempered distribution, is $\mathcal{S}'(\mathbb{R}^d)$. We fix a probability space $(\Omega, \mathcal{F}, \mathscr{P})$. The space of real-valued random variables $L^0(\Omega)$ is endowed with the Fréchet topology associated with the convergence in probability.

Definition 2.1 A *generalized random process* s on $\mathcal{S}'(\mathbb{R}^d)$ is mapping $\varphi \mapsto \langle s, \varphi \rangle$ from $\mathcal{S}(\mathbb{R}^d)$ to $L^0(\Omega)$ that is linear and sequentially continuous in probability. □

The generalized random process s in Definition 2.1 is specified as a random linear functional over the space $\mathcal{S}(\mathbb{R}^d)$. Alternatively, one can see s as a random variable with values in $\mathcal{S}'(\mathbb{R}^d)$; that is, a measurable map from Ω to $\mathcal{S}'(\mathbb{R}^d)$, where $\mathcal{S}'(\mathbb{R}^d)$ is endowed with the Borelian σ-field associated with the strong topology. This equivalence is a consequence of the structure of the nuclear Fréchet space of $\mathcal{S}(\mathbb{R}^d)$ [25,31]. It means in particular that one should look at s as a random tempered generalized function.

Definition 2.2 The *characteristic functional* of a generalized random process s is defined as $\widehat{\mathscr{P}}_s(\varphi) = \mathbb{E}[\mathrm{e}^{\mathrm{i}\langle s, \varphi \rangle}]$ for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$. □

As announced in Introduction of Sect. 2, we present the generalizations of the Bochner and Lévy continuity theorems for generalized random processes.

Theorem 2.3 *A functional* $\widehat{\mathscr{P}} : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$ *is the characteristic functional of a generalized random process if and only if it is continuous and positive definite over* $\mathcal{S}(\mathbb{R}^d)$ *and normalized as* $\widehat{\mathscr{P}}(0) = 1$. □

This result is known as the Minlos–Bochner theorem. It is valid for any functional defined over a nuclear Fréchet space such as $\mathcal{S}(\mathbb{R}^d)$ [41] and more generally over an inductive limit of nuclear Fréchet spaces such as $\mathcal{D}(\mathbb{R}^d)$, the space of compactly supported smooth functions [25, Theorem II.3.3]. The nuclear structure of $\mathcal{S}(\mathbb{R}^d)$ is at the heart of the theory of generalized random processes.

Definition 2.4 A sequence of generalized random processes (s_n) converges in law to the generalized process s if, for any $\varphi_1, \dots, \varphi_N \in \mathcal{S}(\mathbb{R}^d)$, the sequence of random vectors $(\langle s_n, \varphi_1 \rangle, \dots, \langle s_n, \varphi_N \rangle)$ converges in law to the random vector $(\langle s, \varphi_1 \rangle, \dots, \langle s, \varphi_N \rangle)$. □

Theorem 2.5 *A sequence of generalized random processes* (s_n) *converges in law to the generalized random process s if and only if* $\widehat{\mathscr{P}}_{s_n}(\varphi) \underset{n\to\infty}{\longrightarrow} \widehat{\mathscr{P}}_s(\varphi)$ *for any* $\varphi \in \mathcal{S}(\mathbb{R}^d)$. □

This result was proved by X. Fernique for the space of compactly supported smooth functions $\mathcal{D}(\mathbb{R}^d)$ [25, Theorem III.6.5]. The case of a nuclear Fréchet space (including $\mathcal{S}(\mathbb{R}^d)$) is simpler and can be deduced from the results of Fernique. It is also proved by P. Boulicaut in [10] together with a converse to this result: The weak convergence of probability measures on the dual of a Fréchet space $\mathcal{F}$ is equivalent with the pointwise convergence of the characteristic functionals on $\mathcal{F}$ if and only if $\mathcal{F}$ is nuclear [10, Theorem 5.3]. A comprehensive and self-contained exposition of the theory of generalized random processes, including proofs of Theorems 2.3 and 2.5, can be found in [3].

2.2 Lévy White Noises and Infinite Divisibility

The construction of continuous-domain Lévy white noises and related processes is intimately linked with the infinite divisibility of the finite-dimensional marginals of those processes [27,47]. A random variable (and more generally a random vector) is *infinitely divisible* if it can be decomposed (in law) as the sum of N i.i.d. random variables for every N. The logarithm of the characteristic function $\widehat{\mathscr{P}}_X$ of an infinitely divisible random variable X is called its *Lévy exponent*, denoted by Ψ; *i.e.*, $\widehat{\mathscr{P}}_X(\xi) = \exp(\Psi(\xi))$.

Initially, the family of Lévy white noises was introduced over the space $\mathcal{D}'(\mathbb{R}^d)$ of generalized functions [27, Chapter III]. There is a one-to-one correspondence between infinitely divisible random variables and Lévy white noises in $\mathcal{D}'(\mathbb{R}^d)$. Indeed, the characteristic functional of a Lévy white noise is of the form $\widehat{\mathscr{P}}_w(\varphi) = \exp(\int_{\mathbb{R}^d} \Psi(\varphi(\boldsymbol{x}))\mathrm{d}\boldsymbol{x})$ where Ψ is a Lévy exponent. However, a Lévy white noise on $\mathcal{D}'(\mathbb{R}^d)$ is not necessarily tempered (a counterexample is given in [18, Sect. 3.1]). The characterization of tempered white noises has been obtained recently and is based on Theorem 2.6.

Theorem 2.6 *Let Ψ be a continuous function from $\mathbb{R}$ to $\mathbb{C}$. The functional*

$$\widehat{\mathscr{P}}(\varphi) = \exp\left(\int_{\mathbb{R}^d} \Psi(\varphi(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}\right) \tag{3}$$

is the characteristic functional of a generalized random process in $\mathcal{S}'(\mathbb{R}^d)$ if and only if Ψ is a Lévy exponent and the infinitely divisible random variable X with Lévy exponent Ψ has a finite ϵ-moment for some $\epsilon > 0$, such that $\mathbb{E}[|X|^\epsilon] < \infty$. □

We have shown that the existence of a finite absolute moment is sufficient for w being tempered in [18, Theorem 3]. More recently, Dalang and Humeau [12, Theorem 3.13] have proved that this condition is also necessary. This provides a one-to-one correspondence between tempered Lévy noise and infinitely divisible random variable having a finite absolute moment, which justifies the following definition.

Definition 2.7 A generalized random process w whose characteristic functional has form (3) with Ψ satisfying the two conditions of Theorem 2.6 is called a *Lévy white noise* in $\mathcal{S}'(\mathbb{R}^d)$. □

The finiteness of absolute moments is strongly related to the behavior of the Lévy exponent at the origin or, equivalently, the asymptotic behavior of the Lévy measure μ associated with Ψ (see Sect. 5.1 for a short reminder on (symmetric) Lévy measures). Especially, the condition $\mathbb{E}[|X|^p] < \infty$ is equivalent to $\int_{|t|\geq 1} |t|^p \mu(\mathrm{d}t) < \infty$ for every $p > 0$ [47, Sect. 5.25]. The point in Theorem 2.6 is that ϵ can be arbitrarily small; hence, this requirement for being tempered is extremely mild and satisfied by any Lévy noise encountered in practice.

A Lévy white noise is stationary in the sense that $w(\cdot - \boldsymbol{x}_0)$ and w have the same law for every $\boldsymbol{x}_0 \in \mathbb{R}^d$. It is moreover independent at every point, meaning that $\langle w, \varphi_1\rangle$ and $\langle w, \varphi_2\rangle$ are independent whenever φ_1 and φ_2 have disjoint supports.

As we shall see, one particular subclass of Lévy white noise plays a crucial role as potential scaling limits of general Lévy white noises: the SαS (symmetric-α-stable) white noises.

Definition 2.8 Let $0 < \alpha \leq 2$. A Lévy white noise w_α is a *SαS white noise* if its characteristic functional has the form

$$\widehat{\mathscr{P}}_{w_\alpha}(\varphi) = \exp(-C\|\varphi\|_\alpha^\alpha) \tag{4}$$

for some $C > 0$ and every $\varphi \in \mathcal{S}(\mathbb{R}^d)$, where $\|\varphi\|_\alpha = (\int_{\mathbb{R}^d} |\varphi(\boldsymbol{x})|^\alpha \mathrm{d}\boldsymbol{x})^{1/\alpha}$. □

Functional (4) is a characteristic functional and corresponds to (3) with Lévy exponent $\Psi(\xi) = -C|\xi|^\alpha$. For every $\varphi \in \mathcal{S}(\mathbb{R}^d)$, the random variable $X = \langle w_\alpha, \varphi \rangle$ is SαS with characteristic function $\widehat{\mathscr{P}}_X(\xi) = \exp(-C\|\varphi\|_\alpha^\alpha |\xi|^\alpha)$. For $\alpha = 2$, one recognizes the Gaussian law. When $\alpha < 2$, by contrast, the considered random variables have infinite variances. More information on non-Gaussian SαS random variables and processes can be found on [46].

2.3 Indices of Lévy White Noises

Definition 2.9 We consider the following quantities associated with a Lévy exponent Ψ:

$$\beta_0 = \sup\left\{p \in [0,2], \quad \limsup_{|\xi|\to 0} \frac{|\Psi(\xi)|}{|\xi|^p} < \infty\right\}, \tag{5}$$

$$\beta_\infty = \inf\left\{p \in [0,2], \quad \limsup_{|\xi|\to \infty} \frac{|\Psi(\xi)|}{|\xi|^p} < \infty\right\}. \tag{6}$$

We call β_0 the *Pruitt index* and β_∞ the *Blumenthal–Getoor index* of Ψ. □

The Blumenthal–Getoor index β_∞ was initially introduced in [8] to study the behavior of Lévy processes at the origin. It is connected to the local regularity of Lévy processes [22], and more generally of Feller processes [48,49]. The index β_0 is the asymptotic counterpart of β_∞ in the sense that it relies to the behavior of Lévy processes at infinity. It was considered by Pruitt [44] and is highly connected to the existence of moments of the infinitely divisible random variable with Lévy exponent Ψ. Many global regularity properties of Lévy [20] and Feller processes [49] are captured by the knowledge of β_0 and β_∞. Moreover, one can often characterized β_0 with the Lévy measure associated with Ψ (see [14, Sect. 3]). The fact that a Lévy white noise on $\mathcal{S}'(\mathbb{R}^d)$ always has a finite moment $\epsilon > 0$ finite (see Theorem 2.6) imposes that $\beta_0 > 0$. Consequently, we should always consider indices such that $0 < \beta_0 \leq 2$.

Consider a Lévy exponent Ψ with indices $0 < \beta_0, \beta_\infty \leq 2$ and fix $0 < p \leq 2$. The notation $f(\xi) \underset{0/\infty}{\sim} g(\xi)$ means that $g(\xi) \neq 0$ for $\xi \neq 0$ and that $f(\xi)/g(\xi)$ converges to 1 when ξ goes to $0/\infty$. From the definition, we easily see if $\Psi(\xi) \underset{\infty}{\sim} -C|\xi|^p$, then

$\beta_\infty = p$. Similarly, if $\Psi(\xi) \underset{0}{\sim} -C\,|\xi|^p$, then $\beta_0 = p$. Therefore, the indices β_0 and β_∞ are, respectively, the only possible power law behavior of the Lévy exponent in the origin or asymptotically, respectively. Finally, if the Lévy exponent is bounded as $|\Psi(\xi)| \leq C\,|\xi|^p$, then $\beta_\infty \leq p \leq \beta_0$.

3 Linear SDE Driven by Lévy White Noises

The main goal of this section is to introduce the class of random processes of interest for the study of the local and asymptotic self-similarity. A linear differential operator L and a white noise in $\mathcal{S}'(\mathbb{R}^d)$ being given, we consider the linear SDE

$$\mathrm{L}s = w. \tag{7}$$

We say that a solution exists if there is a generalized random process s in $\mathcal{S}'(\mathbb{R}^d)$ such that the processes Ls and w are equal in law (or equivalently, the same characteristic functional). The general framework to solve (7) is based on the existence of inverse operators with adequate properties [54, Chapter 4]. In this section, we first construct generalized random processes that are solutions of (7) (Sect. 3.1). Then, we introduce the homogeneous operators and the class of studied processes, called γ-order linear processes in Sect. 3.2.

3.1 Construction of Linear Processes

Let L be a continuous and linear operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$. Then, its adjoint is the operator L^* from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ defined as $\langle \mathrm{L}^*\varphi_1, \varphi_2\rangle = \langle \varphi_1, \mathrm{L}\varphi_2\rangle$ for every $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$.

Proposition 3.1 (Specification of a linear process). *Consider a linear and continuous operator* L *from* $\mathcal{S}(\mathbb{R}^d)$ *to* $\mathcal{S}'(\mathbb{R}^d)$ *and a Lévy white noise* w *on* $\mathcal{S}'(\mathbb{R}^d)$*. Assume the existence of a topological vector space* $\mathcal{X}$ *such that*

- *The adjoint* L^* *of* L *admits a left inverse operator* T *that is linear and continuous from* $\mathcal{S}(\mathbb{R}^d)$ *to* $\mathcal{X}$*;*
- *The characteristic functional* $\widehat{\mathscr{P}}_w$ *of* w *can be extended as a continuous and positive-definite functional on* $\mathcal{X}$*.*

Then, there exists a generalized random process s *whose characteristic functional is, for every* $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\widehat{\mathscr{P}}_s(\varphi) = \widehat{\mathscr{P}}_w(\mathrm{T}\varphi)$*. Moreover, we have that* $\mathrm{L}s \overset{(d)}{=} w$*.* □

By considering a more general $\mathcal{X}$, this result refines the original theorem that was first presented in [54, Sect. 3.5.4] albeit with some unnecessary restrictions on $\mathcal{X}$. The principle is simply to check the conditions of the Minlos–Bochner theorem. We give the proof for the sake of completeness.

Proof Set $\widehat{\mathscr{P}}(\varphi) = \widehat{\mathscr{P}}_w(\mathrm{T}\varphi)$. From the assumption on $\widehat{\mathscr{P}}_w$ and T, we easily deduce that $\widehat{\mathscr{P}}$ is well defined and continuous over $\mathcal{S}(\mathbb{R}^d)$ by composition. It is positive definite by composition of linear and a positive-definite functions. Finally, $\mathrm{T}\{0\} = 0$, hence $\widehat{\mathscr{P}}(0) = \widehat{\mathscr{P}}_w(0) = 1$. Therefore, $\widehat{\mathscr{P}}$ is the characteristic functional of a generalized random process s according to Theorem 2.3. For the last point, we simply remark that, T being a left inverse of L^*, $\widehat{\mathscr{P}}_{\mathrm{L}s}(\varphi) = \mathbb{E}[\mathrm{e}^{\mathrm{i}\langle \mathrm{L}s,\varphi\rangle}] = \mathbb{E}[\mathrm{e}^{\mathrm{i}\langle s,\mathrm{L}^*\varphi\rangle}] = \widehat{\mathscr{P}}_w(\mathrm{TL}^*\varphi) = \widehat{\mathscr{P}}_w(\varphi)$, which is equivalent to $\mathrm{L}s \overset{(d)}{=} w$. □

Definition 3.2 A generalized random process constructed via Proposition 3.1 is called a *linear process*. □

In practice, for given L and w, one has to determine an adequate space $\mathcal{X}$ in order to correctly define the process s. The choice of $\mathcal{X}$ is generally driven by the white noise. For instance, we will consider the case $\mathcal{X} = L^\alpha(\mathbb{R}^d)$ when dealing with SαS white noises. The optimal choice of $\mathcal{X}$ for a given w is investigated in [21] based on the results of [45]. Then, the main issue becomes the existence of a left inverse with the adequate stability, mapping $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{X}$.

3.2 Homogeneous Operators and γ-Order Linear Processes

This section is dedicated to the specification of random processes concerned by Theorem 1.1. We start with some definitions. For $u \in \mathcal{S}'(\mathbb{R}^d)$, $\boldsymbol{x}_0 \in \mathbb{R}^d$, and $a > 0$, we define $u(\cdot - \boldsymbol{x}_0)$ as the tempered distribution such that $\langle u(\cdot - \boldsymbol{x}_0), \varphi\rangle = \langle u, \varphi(\cdot + \boldsymbol{x}_0)\rangle$ for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Similarly, $u(\cdot/a)$ is the tempered distribution such that $\langle u(\cdot/a), \varphi\rangle = \langle u, a^d \varphi(a\cdot)\rangle$.

Definition 3.3 Consider a linear and continuous operator L from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$. We say that L is γ*-homogeneous* for some $\gamma \in \mathbb{R}$ if $\mathrm{L}\{\varphi(\cdot/a)\} = a^{-\gamma}(\mathrm{L}\varphi)(\cdot/a)$ for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $a > 0$. □

For instance, the derivative is a 1-homogeneous operator. We shall now focus on operators L that are: (1) linear shift invariant, (2) continuous from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$, and (3) γ-homogeneous for some $\gamma \geq 0$. Moreover, inspired by Proposition 3.1, the adjoint operator L^* should have a left inverse with some stability property. We shall essentially consider two cases, assuming the existence of a left inverse T as in Proposition 3.1 for $\mathcal{X} = \mathcal{R}(\mathbb{R}^d)$, the space of rapidly decaying measurable functions (see below), or $\mathcal{X} = L^p(\mathbb{R}^d)$ for some p such that $0 < p \leq 2$. These spaces naturally arise as domains of continuity of the characteristic functional of Lévy white noises.

The space $\mathcal{R}(\mathbb{R}^d)$ is defined as $\mathcal{R}(\mathbb{R}^d) = \{f \text{ measurable}, \ (1 + |\cdot|)^N f \in L^2(\mathbb{R}^d) \text{ for all } N \in \mathbb{N}\}$. It is endowed with a natural Fréchet topology, as a projective limit of the Hilbert spaces $L^2_N(\mathbb{R}^d) = \{f, \ (1 + |\cdot|)^N f \in L^2(\mathbb{R}^d)\}$, $N \in \mathbb{N}$ (see [39, Chapter IV] for more details on Fréchet spaces).

Fix a linear, shift-invariant, continuous, and γ-homogeneous operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$, with $\gamma \geq 0$. We consider two cases.

- *Condition (C1)* The adjoint L^* admits a $(-\gamma)$-homogeneous left inverse that continuously maps $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{R}(\mathbb{R}^d)$.
- *Condition (C2)* The adjoint L^* admits a $(-\gamma)$-homogeneous left inverse that continuously maps $\mathcal{S}(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ for some $0 < p \leq 2$.

Note that (C1) is more restrictive than (C2) since $\mathcal{R}(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$ for any $0 < p \leq 2$.

One shall construct the random processes of interests thanks to Proposition 3.1. We start by giving some new results on the continuity of the characteristic functionals of Lévy white noises that allow for new compatibility conditions between an operator L and a Lévy white noise w in the general case and for p-admissible white noise (see Definition 3.4 below).

Definition 3.4 We say that a Lévy exponent Ψ is *p-admissible* for $0 < p \leq 2$ if $|\Psi(\xi)| \leq C\,|\xi|^p$ for some $C > 0$ and every $\xi \in \mathbb{R}$. By extension, a Lévy white noise with a p-admissible Lévy exponent is said to be *p-admissible* itself. □

Proposition 3.5 *Let w be a Lévy white noise over $\mathcal{S}'(\mathbb{R}^d)$. Then, the characteristic functional $\widehat{\mathscr{P}}_w$ of w can be extended as a continuous and positive-definite functional over $\mathcal{R}(\mathbb{R}^d)$. If moreover w is p-admissible for some $0 < p \leq 2$, then $\widehat{\mathscr{P}}_w$ can be extended as a continuous and positive-definite functional over $L^p(\mathbb{R}^d)$.* □

Proof The proof for $L_p(\mathbb{R}^d)$ is similar to the one for $\mathcal{R}(\mathbb{R}^d)$; hence, we omit it. We only need to prove the continuity, because the positive-definiteness follows simply by density of $\mathcal{S}(\mathbb{R}^d)$ in $\mathcal{R}(\mathbb{R}^d)$. The positive-definiteness of $\widehat{\mathscr{P}}_w$ in $\mathcal{S}(\mathbb{R}^d)$ implies that

$$\left|\widehat{\mathscr{P}}(\varphi_2) - \widehat{\mathscr{P}}(\varphi_1)\right| \leq 2\left|1 - \widehat{\mathscr{P}}(\varphi_2 - \varphi_1)\right| \tag{8}$$

for any $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ (see, for instance, [25, Sect. II.5.1] or [55, Sect. IV.1.2, Proposition 1.1]). For every $z \in \mathbb{C}$ with $\Re\{z\} \leq 0$, one has $|\mathrm{e}^z - 1| \leq |z|$. Since $\Re\{\Psi\} \leq 0$, this implies that

$$\left|1 - \widehat{\mathscr{P}}_w(\varphi)\right| \leq \left|1 - \mathrm{e}^{\int_{\mathbb{R}^d} \Psi(\varphi(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}}\right| \leq \left|\int_{\mathbb{R}^d} \Psi(\varphi(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}\right| \leq \int_{\mathbb{R}^d} |\Psi(\varphi(\boldsymbol{x}))|\mathrm{d}\boldsymbol{x}. \tag{9}$$

Moreover, according to [18, Corollary 1], w being tempered, there exists $C > 0$ and $\epsilon > 0$ such that $|\Psi(\xi)| \leq C(|\xi|^\epsilon + |\xi|^2)$. Putting the ingredients together, we then easily show that

$$\left|\widehat{\mathscr{P}}_w(\varphi_2) - \widehat{\mathscr{P}}_w(\varphi_1)\right| \leq C\left(\|\varphi_2 - \varphi_1\|_\epsilon^\epsilon + \|\varphi_2 - \varphi_1\|_2^2\right). \tag{10}$$

Hence, $\widehat{\mathscr{P}}_w$ can be extended continuously to functions $\varphi \in \mathcal{R}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \cap L^\epsilon(\mathbb{R}^d)$ as expected. □

Remark In [54, Definition 4.4], an alternative definition of the p-admissibility was introduced. Since this definition requires an additional bound on the derivative of the Lévy exponent and is used for the construction of random processes only for $1 \leq p \leq 2$, the second part of Proposition 3.5 is a generalization of [54, Theorem 8.2]. Proposition 3.5 allows for new criteria to solve SDEs driven by Lévy white noises.

Theorem 3.6 *Let w be a Lévy white noise on $\mathcal{S}'(\mathbb{R}^d)$ with Lévy exponent Ψ and* L *be a linear, γ-homogeneous, and continuous operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ for $\gamma \geq 0$. We consider two cases.*

- *Condition (C1) The adjoint* L^* *admits a* $(-\gamma)$*-homogeneous left inverse* T *that maps* $\mathcal{S}(\mathbb{R}^d)$ *to* $\mathcal{R}(\mathbb{R}^d)$.
- *Condition (C2) There exists* $0 < p \leq 2$ *such that (i) the adjoint* L^* *admits a* $(-\gamma)$*-homogeneous left inverse* T *that maps* $\mathcal{S}(\mathbb{R}^d)$ *to* $L^p(\mathbb{R}^d)$ *and (ii)* Ψ *is* p*-admissible.*

when (C1) or (C2) are satisfied, there exists a generalized random process s *whose characteristic functional is* $\widehat{\mathscr{P}}(\varphi) = \widehat{\mathscr{P}}_w(\mathrm{T}\varphi)$. *Moreover, s is a solution of (7).* □

Proof The result follows from the application of Proposition 3.1 with $\mathcal{X} = \mathcal{R}(\mathbb{R}^d)$ and $\mathcal{X} = L^p(\mathbb{R}^d)$, respectively. The assumptions on $\widehat{\mathscr{P}}_w$ are satisfied due to Proposition 3.5. □

Definition 3.7 A generalized random process s constructed with the method of Theorem 3.6 is called a γ*-order linear process*. We summarize the situation described in Theorem 3.6 with the (slightly abusive) notation $s = \mathrm{L}^{-1}w$. □

Remark The Lévy exponent of a SαS white noise is $\Psi(\xi) = -C\,|\xi|^\alpha$ for some constant $C > 0$ and thus is α-admissible. The construction of a process s such that $\mathrm{L}s = w_\alpha$ therefore relies on the existence of a left inverse T of L^* that maps continuously $\mathcal{S}(\mathbb{R}^d)$ into $L^\alpha(\mathbb{R}^d)$.

4 Scaling Limits of γ-Order Linear Processes

In this section, we study the statistical behavior of γ-order linear processes at coarse and fine scales. We recall that for a generalized random process s and a nonnegative real number a, the process $s(\cdot/a)$ is defined by $\langle s(\cdot/a), \varphi\rangle = a^d\langle s, \varphi(a\cdot)\rangle$.

- We zoom out the process when $a < 1$. In particular, we consider the limit case $a \to 0$ and call it the *coarse-scale behavior* of s.
- We zoom in the process when $a > 1$. Again, we pay attention to the limit case $a \to \infty$, which one call the *fine-scale behavior* of s.

In general, we shall see that $s(\cdot/a)$ has no non-trivial limits itself when $a \to 0/\infty$. However, we will encounter situations where $a^H s(\cdot/a)$ has a stochastic limit for some $H \in \mathbb{R}$. When it exists, the coefficient H is unique and determines the renormalization procedure required to observe the convergence phenomenon.

In what follows, we first treat the case of SDEs driven by SαS white noises as a preparatory example. Their solutions are actually self-similar and have therefore straightforward scaling limit behaviors (Sect. 4.1). We then give sufficient conditions on the Lévy exponent to determine the coarse- and fine-scale behaviors of γ-order linear processes. These results are presented in Sect. 4.2 and are the main contributions of this paper. Finally, we question the necessity of our conditions such that the scaling limit exists in Sect. 4.3.

4.1 Linear Processes Driven by SαS White Noises

When the white noise is stable, the change of scale has by definition no effect on the noise up to renormalization. Under reasonable assumptions on the operator L, we extend this fact to solutions of SDEs driven by SαS white noises. This property is referred to as self-similarity.

Definition 4.1 A generalized random process s is said to be *self-similar of order* H if $a^H s(\cdot/a) \overset{(d)}{=} s$ for all $a > 0$. The parameter H is called the *Hurst exponent* of s. □

The coarse-scale and fine-scale behaviors of a self-similar process are obvious, since the law of the process is not changed by scaling, up to renormalization. The self-similarity property is directly inferred from the characteristic functional of the process. Indeed, since $\widehat{\mathscr{P}}_{a^H s(\cdot/a)}(\varphi) = \mathbb{E}[\mathrm{e}^{\mathrm{i}\langle a^H s(\cdot/a), \varphi\rangle}] = \mathbb{E}[\mathrm{e}^{\mathrm{i}\langle s, a^{d+H}\varphi(a\cdot)\rangle}] = \widehat{\mathscr{P}}_s(a^{d+H}\varphi(a\cdot))$, we deduce that s is self-similar of order H if and only if $\widehat{\mathscr{P}}_s(\varphi) = \widehat{\mathscr{P}}_s(a^{d+H}\varphi(a\cdot))$ for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $a > 0$. This equivalence and some other considerations on self-similar processes can be found in [54, Sect. 7.2].

Proposition 4.2 *Let* $\gamma \geq 0$ *and* $0 < \alpha \leq 2$. *We assume that* $s = \mathrm{L}^{-1} w_\alpha$ *is a* γ*-order linear process driven by a SαS white noise. Then,* s *is self-similar with Hurst exponent* $H = \gamma + d(\frac{1}{\alpha} - 1)$. □

Proof By definition of a γ-order linear process, there exists a linear operator T, left inverse of L^*, that is $(-\gamma)$-homogeneous, continuous from $\mathcal{S}(\mathbb{R}^d)$ to $L^\alpha(\mathbb{R}^d)$, and such that $\widehat{\mathscr{P}}_s(\varphi) = \exp(-C\|\mathrm{T}\varphi\|_\alpha^\alpha)$. Then, we have

$$\widehat{\mathscr{P}}_s(a^{d+H}\varphi(a\cdot)) = \exp\left(-C\|a^{\gamma+d/\alpha}\mathrm{T}\{\varphi(a\cdot)\}\|_\alpha^\alpha\right) \overset{(i)}{=} \exp\left(-C\|a^{d/\alpha}\{\mathrm{T}\varphi\}(a\cdot)\|_\alpha^\alpha\right)$$
$$\overset{(ii)}{=} \exp\left(-C\|\mathrm{T}\varphi\|_\alpha^\alpha\right) = \widehat{\mathscr{P}}_s(\varphi),$$

where we used, respectively, the $(-\gamma)$-homogeneity of T and the change of variable $\boldsymbol{y} = a\boldsymbol{x}$ in (i) and (ii). This implies that s is self-similar with the Hurst exponent $H = \gamma + d(1/\alpha - 1)$. □

4.2 Linear Processes at Coarse and Fine Scales

Here we consider the general problem of characterizing the coarse and large-scale behaviors of γ-order linear processes. We analyze the coarse- and fine-scale behavior separately even if the methods of proof are similar, in order to emphasize the different assumptions: The relevant parameter of the underlying white noise is the index β_0 at coarse scales and β_∞ at fine scales.

We have seen in Sect. 4.1 that two ingredients are sufficient to make a linear process self-similar: the self-similarity of the Lévy noise and the homogeneity of the left inverse operator T of L^*. Moreover, the self-similarity of a Lévy noise is equivalent to the stability of the underlying infinitely divisible random variable [46]. Even if γ-order linear processes are not self-similar in general, one can often recover the self-similarity by asymptotically zooming the process in or out.

Definition 4.3 We say that the generalized random process s is *asymptotically self-similar of order* $H_\infty \in \mathbb{R}$ (*locally self-similar of order* $H_{\mathrm{loc}} \in \mathbb{R}$, respectively) if the rescaled processes $a^{H_\infty}s(\cdot/a)$ ($a^{H_{\text{loc}}}s(\cdot/a)$, respectively) converge in law to a non-trivial generalized random process as $a \to 0$ ($a \to \infty$, respectively). □

The terminology of Definition 4.3 is justified by Proposition 4.3.

Proposition 4.4 *If the generalized random process s is asymptotically self-similar of order H_∞ (locally self-similar of order H_{loc}, respectively), then the limit is self-similar of order H_∞ (H_{loc}, respectively).* □

Proof Assume that s is asymptotically self-similar of order H_∞. Then, there exists a process s_∞ such that $a^{H_\infty}s(\cdot/a)$ converges in law to s_∞. Let $b > 0$ and set $s_b = b^{H_\infty}s(\cdot/b)$. Clearly, s_b is also asymptotically self-similar with limit $b^{H_\infty}s_\infty(\cdot/b)$. Moreover, $a^{H_\infty}s_b(\cdot/a) = (ab)^{H_\infty}s(\cdot/(ab))$. This latter quantity has the same limit in law than $a^{H_\infty}s(\cdot/a)$ as $a \to 0$, which is s_∞. As a consequence, we have shown that $b^{H_\infty}s_\infty(\cdot/b)$ and s_∞ have the same law for any $b > 0$ and the limit is H_∞-self-similar. The proof is identical in the local case. □

One now considers the following question: When is a generalized Lévy process asymptotically self-similar, when is it locally self-similar, and, if so, what are the asymptotic and local Hurst exponents H_∞ and H_{loc}? Theorems 4.5 and 4.6 answer these two questions.

Theorem 4.5 (Coarse-scale behavior of γ-order linear processes). *We consider a Lévy white noise w with Lévy exponent Ψ and index $\beta_0 \in (0, 2]$ and a γ-homogeneous operator* L *with $\gamma \geq 0$. We assume that there exists an operator* T *such that* (T, Ψ) *satisfies one of the following set of conditions.*

- *Condition (C1)* T *is a $(-\gamma)$-homogeneous left inverse of* L^**, continuous from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{R}(\mathbb{R}^d)$; or*
- *Condition (C2)* T *is a $(-\gamma)$-homogeneous left inverse of* L^**, continuous from $\mathcal{S}(\mathbb{R}^d)$ to $L^{\beta_0}(\mathbb{R}^d)$ and Ψ is β_0-admissible.*

Let $s = \mathrm{L}^{-1}w$ be the γ-order linear process with characteristic functional $\widehat{\mathscr{P}}_s(\varphi) = \widehat{\mathscr{P}}_w(\mathrm{T}\varphi)$. If $\Psi(\xi) \underset{0}{\sim} -C\,|\xi|^{\beta_0}$ for some constant $C > 0$, we have the convergence in law

$$a^{\gamma+d\left(\frac{1}{\beta_0}-1\right)}s(\cdot/a) \xrightarrow[a\to 0]{(d)} s_{\mathrm{L},\beta_0}, \tag{11}$$

where $\mathrm{L}s_{\mathrm{L},\beta_0} \overset{(d)}{=} w_{\beta_0}$ is a $S\alpha S$ white noise with $\alpha = \beta_0$. In particular, s is asymptotically self-similar of order $H_\infty = \gamma + d(\frac{1}{\beta_0} - 1)$. □

Proof First, assuming that (C1) or (C2) holds, Theorem 3.6 implies that both $\widehat{\mathscr{P}}_w(\mathrm{T}\varphi)$ and $\widehat{\mathscr{P}}_{w_{\beta_0}}(\mathrm{T}\varphi) = \exp(-\,C\|\mathrm{T}\varphi\|_{\beta_0}^{\beta_0})$ are characteristic functionals; hence, the processes s and s_{L,β_0} are well defined.

By Theorem 2.5, we know moreover that the convergence in law (11) is equivalent to the pointwise convergence of the characteristic functionals. Hence, we have to prove that, for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$\log \widehat{\mathscr{P}}_{a^{\gamma+d(1/\beta_0-1)}s(\cdot/a)}(\varphi) \underset{a\to 0}{\longrightarrow} -C\|\mathrm{T}\varphi\|_{\beta_0}^{\beta_0}. \tag{12}$$

We fix $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Then, we have

$$\langle a^{\gamma+d(1/\beta_0-1)} s(\cdot/a), \varphi\rangle = \langle w, a^{\gamma+d/\beta_0}\varphi(a\cdot)\rangle \overset{(i)}{=} \langle w, \mathrm{T}\{a^{\gamma+d/\beta_0}\varphi(a\cdot)\}\rangle \overset{(ii)}{=} \langle w, a^{d/\beta_0}\{\mathrm{T}\varphi\}(a\cdot)\rangle, \tag{13}$$

where we have used that $\langle s, \varphi\rangle = \langle w, \mathrm{T}\varphi\rangle$ and the $(-\gamma)$-homogeneity of T in (i) and (ii), respectively. Therefore, we have

$$\log \widehat{\mathscr{P}}_{a^{\gamma+d(1/\beta_0-1)}s(\cdot/a)}(\varphi) = \log \widehat{\mathscr{P}}_w(a^{d/\beta_0}\{\mathrm{T}\varphi\}(a\cdot)) = \int_{\mathbb{R}^d} \Psi(a^{d/\beta_0}\{\mathrm{T}\varphi\}(a\boldsymbol{x}))\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^d} \left(a^{-d}\Psi(a^{d/\beta_0}\mathrm{T}\varphi(\boldsymbol{y})\right)\mathrm{d}\boldsymbol{y}. \tag{14}$$

By assumption on Ψ, we have moreover that, for every $\boldsymbol{y} \in \mathbb{R}^d$,

$$a^{-d}\Psi\left(a^{d/\beta_0}\mathrm{T}\varphi(\boldsymbol{y})\right) \underset{a\to 0}{\longrightarrow} -C\,|\mathrm{T}\varphi(\boldsymbol{y})|^{\beta_0}. \tag{15}$$

We split here the proof in two parts, depending on whether T and Ψ follow (C1) or (C2).

- We start with (C2). The β_0-admissibility of Ψ implies that

$$\left|a^{-d}\Psi\left(a^{d/\beta_0}\mathrm{T}\varphi(\boldsymbol{y})\right)\right| \leq C'\,|\mathrm{T}\varphi(\boldsymbol{y})|^{\beta_0} \tag{16}$$

  for some $C' > 0$ and every $\boldsymbol{y} \in \mathbb{R}^d$. The right term of (16) is integrable by assumption on T. Therefore, the Lebesgue dominated convergence theorem applies and (12) is showed.
- We assume now (C1). In that case, we do not have a full bound on Ψ. We know, however, that $\mathrm{T}\varphi$ is bounded; hence, $\|\mathrm{T}\varphi\|_\infty < \infty$. Since Ψ is continuous and behaves like $(-C\,|\cdot|^{\beta_0})$ at the origin, there exists $C' > 0$ such that $|\Psi(\xi)| \leq C'\,|\xi|^{\beta_0}$ for every $|\xi| \leq \|\mathrm{T}\varphi\|_\infty$. Hence, for all $a \leq 1$, we have $\left|a^{d/\beta_0}\mathrm{T}\varphi(\boldsymbol{y})\right| \leq 1$, and (16) is still valid. Again, we deduce (12) from the Lebesgue dominated convergence theorem.

Finally, the limit process s_{L,β_0} is self-similar with order $H_\infty = \gamma + d(\frac{1}{\beta_0} - 1)$ according to Proposition 4.2. □

Theorem 4.6 (Fine-scale behavior of γ-order linear processes). *Under the same assumptions as in Theorem 4.5 but replacing β_0 by $\beta_\infty \in (0, 2]$, we consider $s = \mathrm{L}^{-1}w$ a γ-order linear process. If the Lévy exponent Ψ of w satisfies $\Psi(\xi) \underset{\infty}{\sim} -C\,|\xi|^{\beta_\infty}$ for some constant $C > 0$, then we have the convergence in law*

$$a^{\gamma+d\left(\frac{1}{\beta_\infty}-1\right)} s(\cdot/a) \underset{a\to\infty}{\overset{(d)}{\longrightarrow}} s_{\mathrm{L},\beta_\infty}, \tag{17}$$

where $\mathrm{L}s_{\mathrm{L},\beta_\infty} \overset{(d)}{=} w_{\beta_\infty}$ *is a SαS white noise with* $\alpha = \beta_\infty$. *In particular, s is locally self-similar of order* $H_{\mathrm{loc}} = \gamma + d(\frac{1}{\beta_\infty} - 1)$. □

Proof The proof is similar to the one of Theorem 4.5, and we only develop the parts that differ. If T and Ψ satisfy (C2), the proof follows exactly the line of Theorem 4.5. We should therefore assume that T maps continuously $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{R}(\mathbb{R}^d)$. Restarting from (14) with β_∞ instead of β_0, we split the integral into two parts and get

$$\begin{aligned}\log \widehat{\mathscr{P}}_{a^{\gamma+d(1/\beta_\infty-1)}s(\cdot/a)}(\varphi) &= \int_{\mathbb{R}^d} \mathbb{1}_{|\mathrm{T}\varphi(\boldsymbol{y})|a^{d/\beta_\infty}\geq 1} a^{-d}\Psi(a^{d/\beta_\infty}\mathrm{T}\varphi(\boldsymbol{y}))\mathrm{d}\boldsymbol{y}\\ &\quad+\int_{\mathbb{R}^d} \mathbb{1}_{|\mathrm{T}\varphi(\boldsymbol{y})|a^{d/\beta_\infty}< 1} a^{-d}\Psi(a^{d/\beta_\infty}\mathrm{T}\varphi(\boldsymbol{y}))\mathrm{d}\boldsymbol{y}\\ &:= I(a)+J(a).\end{aligned} \tag{18}$$

Control of $I(a)$ We have, by assumption on Ψ, that $\mathbb{1}_{|\mathrm{T}\varphi(\boldsymbol{y})|a^{d/\beta_\infty}\geq 1} a^{-d}\Psi(a^{d/\beta_\infty}\mathrm{T}\varphi(\boldsymbol{y})) \underset{a\to\infty}{\longrightarrow} -C\,|\mathrm{T}\varphi(\boldsymbol{y})|^{\beta_\infty}$. Moreover, since the continuous function Ψ behaves like $(-C\,|\cdot|^{\beta_\infty})$ at infinity, there exists a constant C' such that $|\Psi(\xi)| \leq C'\,|\xi|^{\beta_\infty}$ for every $|\xi| \geq 1$. Moreover, the function $\mathrm{T}\varphi$, which is in $\mathcal{R}(\mathbb{R}^d)$, is bounded. Hence, we have, when $a \geq \|\mathrm{T}\varphi\|_\infty^{-1}$, that $\left|\mathbb{1}_{|\mathrm{T}\varphi(\boldsymbol{y})|a^{d/\beta_\infty}\geq 1} a^{-d}\Psi(a^{d/\beta_\infty}\mathrm{T}\varphi(\boldsymbol{y}))\right| \leq C'\,|\mathrm{T}\varphi(\boldsymbol{y})|^{\beta_\infty}$ for all $\boldsymbol{y}\in\mathbb{R}^d$. The function on the right is integrable; therefore, the Lebesgue dominated convergence applies and we obtain that $I(a) \underset{a\to\infty}{\longrightarrow} -C\|\mathrm{T}\varphi\|_{\beta_\infty}^{\beta_\infty}$.

Control of $J(a)$ As seen in (10), there exists $C' > 0$ and $\epsilon > 0$ such that $|\Psi(\xi)| \leq C'(|\xi|^\epsilon + |\xi|^2)$. Without loss of generality, one can choose $\epsilon < \beta_\infty$. Then, for $|\xi| \leq 1$, we have $|\Psi(\xi)| \leq 2C'\,|\xi|^\epsilon$ and, therefore,

$$\left|\int_{\mathbb{R}^d} \mathbb{1}_{|\mathrm{T}\varphi(\boldsymbol{y})|a^{d/\beta_\infty}< 1} a^{-d}\Psi(a^{d/\beta_\infty}\mathrm{T}\varphi(\boldsymbol{y}))\mathrm{d}\boldsymbol{y}\right| \leq 2C' a^{d(\epsilon/\beta_\infty-1)}\|\mathrm{T}\varphi\|_\epsilon^\epsilon. \tag{19}$$

Since $\mathcal{R}(\mathbb{R}^d) \subset L^\epsilon(\mathbb{R}^d)$ and $\epsilon < \beta_\infty$, we have $\|\mathrm{T}\varphi\|_\epsilon^\epsilon < \infty$ and $a^{d(\epsilon/\beta_\infty-1)} \underset{a\to\infty}{\longrightarrow} 0$, which implies that $J(a) \underset{a\to\infty}{\longrightarrow} 0$. We have shown that $\log \widehat{\mathscr{P}}_{a^{\gamma+d(1/\beta_\infty-1)}s(\cdot/a)}(\varphi) = I(a) + J(a) \underset{a\to\infty}{\longrightarrow} -C\|\mathrm{T}\varphi\|_{\beta_\infty}^{\beta_\infty}$, as expected. Finally, the limit process $s_{\mathrm{L},\beta_\infty}$ is self-similar with order $H_{\mathrm{loc}} = \gamma + d(\frac{1}{\beta_\infty} - 1)$ according to Proposition 4.2. □

4.3 Discussion and Converse Results

In this section, we investigate the generality of our results by questioning the hypotheses in Theorems 4.5 and 4.6. One should only consider the case of γ-homogeneous L operators whose adjoint has a $(-\gamma)$-homogeneous stable inverse T. We start with preliminary results.

- The renormalization procedures in Theorems 4.5 and 4.6 have to be compared with the index $H = \gamma + d(1/\alpha - 1)$ of a γ-order self-similar process (see Proposition 4.2). In particular, the γ-order linear processes studied in this section are asymptotically self-similar with index $\gamma + d(1/\beta_{0/\infty} - 1)$, where $\beta_{0/\infty} = \beta_0$ or β_∞. One can say that the lack of self-similarity of s vanishes asymptotically or locally.

- (C1) has to be understood as the sufficient assumption on the operator T such that the process s with characteristic functional $\widehat{\mathscr{P}}_w(\mathrm{T}\varphi)$ is well defined without any additional assumption on the Lévy white noise w. Therefore, (C1) is restrictive for the operator but applies to any Lévy noise.
- This is in contrast to (C2). Here, the restriction on T is minimal since the process $s_{\mathrm{L},\beta_{0/\infty}}$ should be well defined, and, therefore, T should at least map $\mathcal{S}(\mathbb{R}^d)$ into $L^{\beta_{0/\infty}}(\mathbb{R}^d)$. It means that (C2) gives sufficient assumptions on the Lévy white noise such that the minimal assumption on T is also sufficient.
- When the variance of the noise is finite, we have in particular that $\beta_0 = 2$. Under the assumptions of Theorem 4.5, the process $a^{\gamma - d/2} s(\cdot/a)$ converges to a Gaussian self-similar process. This can be seen as a central limit theorem for γ-order finite-variance linear processes. This finite-variance result was already established in our previous work [19, Theorem 4.2]. Theorem 4.5 is a generalization for the infinite-variance case.
- For important classes of Lévy white noises, the parameter β_∞ is 0 and Theorem 4.6 does not apply. This includes (generalized) Laplace white noises and compound Poisson white noises (see Sect. 5.1). In that case, one does not expect the underlying process to be locally self-similar. This is made more precise and proved when L satisfies (C1) in Proposition 4.7.

Proposition 4.7 *Let w be a white noise with Blumenthal–Getoor index $\beta_\infty = 0$. Assume that* L *is a γ-homogeneous operator and that there exists a $(-\gamma)$-homogeneous left inverse* T *of* L^* *that is continuous from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{R}(\mathbb{R}^d)$. Let $s = \mathrm{L}^{-1} w$ be the γ-order linear process with characteristic functional $\widehat{\mathscr{P}}_s(\varphi) = \widehat{\mathscr{P}}_w(\mathrm{T}\varphi)$. Then, for every $H \in \mathbb{R}$, $a^H s(\cdot/a) \xrightarrow[a\to\infty]{(d)} 0$.* □

Proof Due to Theorem 2.5, we have to show that, for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\log \widehat{\mathscr{P}}_{a^H s(\cdot/a)}(\varphi) \underset{a\to\infty}{\longrightarrow} 0$. Proceeding as in Theorem 4.5, we easily show that

$$\log \widehat{\mathscr{P}}_{a^H s(\cdot/a)}(\varphi) = \int_{\mathbb{R}^d} a^{-d} \Psi(a^{d+H} \mathrm{T}\varphi(\boldsymbol{y})) \mathrm{d}\boldsymbol{y}. \tag{20}$$

According to (10), there exists $\epsilon, C' > 0$ such that $|\Psi(\xi)| \leq C' |\xi|^\epsilon$ for $|\xi| \leq 1$. Without loss of generality, one can assume that $\epsilon < \frac{d}{d+|H|}$. This implies in particular that $\epsilon(d+H) - d < 0$. The knowledge that $\beta_\infty = 0$ is enough to deduce that $\Psi(\xi)$ is also dominated by $|\xi|^\epsilon$ for $|\xi| \geq 1$. Thus, there exists $C > 0$ such that $|\Psi(\xi)| \leq C |\xi|^\epsilon$ for every $\xi \in \mathbb{R}$. Restarting from (20), we obtain that

$$\left|\log \widehat{\mathscr{P}}_{a^H s(\cdot/a)}(\varphi)\right| \leq C \int_{\mathbb{R}^d} a^{\epsilon(d+H)-d} \, |\mathrm{T}\varphi(\boldsymbol{y})|^\epsilon \, \mathrm{d}\boldsymbol{y} = C \|\mathrm{T}\varphi\|_\epsilon^\epsilon a^{\epsilon(d+H)-d}, \tag{21}$$

which vanishes when $a \to \infty$ due to our choice for ϵ. This concludes the proof. □

In Theorems 4.5 and 4.6, we assume some asymptotic behaviors of the Lévy exponent at 0 or at ∞. We see here that under reasonable conditions, this assumption is necessary for (11) or (17) to occur.

Proposition 4.8 *Let $s = \mathrm{L}^{-1}w$ be a γ-order linear process with $\gamma \geq 0$. We also assume that s behaves at coarse scale as*

$$a^{H_\infty} s(\cdot/a) \underset{a\to 0}{\longrightarrow} s_{\mathrm{L},\alpha} \tag{22}$$

where $H_\infty \in \mathbb{R}$ and $\mathrm{L}s_{\mathrm{L},\alpha} = w_\alpha$ is a SαS white noise with $0 < \alpha \leq 2$. Then, $H_\infty = \gamma + d(1/\alpha - 1)$. If moreover Ψ is bounded by $|\cdot|^\alpha$ at the origin, then $\Psi(\xi) \underset{0}{\sim} -C\,|\xi|^\alpha$ for some $C > 0$.
Assume now that s behaves at fine scale as

$$a^{H_{\mathrm{loc}}} s(\cdot/a) \underset{a\to \infty}{\longrightarrow} s_{\mathrm{L},\alpha} \tag{23}$$

where $H_{\mathrm{loc}} \in \mathbb{R}$ and $\mathrm{L}s_{\mathrm{L},\alpha} = w_\alpha$ is a SαS white noise with $0 < \alpha \leq 2$. Then, $H_{\mathrm{loc}} = \gamma + d(1/\alpha - 1)$. If moreover Ψ is bounded by $|\cdot|^\alpha$ at ∞, then $\Psi(\xi) \underset{\infty}{\sim} -C\,|\xi|^\alpha$ for some $C > 0$. □

Proof Due to (22), s is asymptotically self-similar; hence, its limit is self-similar of order H_∞ (Proposition 4.4). We also now that $s_{\mathrm{L},\alpha}$ is self-similar of order $\gamma + d(1/\alpha - 1)$ with Proposition 4.2. Thus, $H_\infty = \gamma + d(1/\alpha - 1)$.

By γ-homogeneity, we have $\mathrm{L}\{a^{\gamma + d(1/\alpha - 1)} s(\cdot/a)\} = a^{d(1/\alpha - 1)} w(\cdot/a)$. Hence, applying the linear operator L each side, (22) implies that $a^{d(1/\alpha-1)} w(\cdot/a)$ converges in law to w_α. In particular, $\widehat{\mathscr{P}}_w(a^{d/\alpha}\varphi(a\cdot))$ converges to $\widehat{\mathscr{P}}_{w_\alpha}(\varphi) = \exp(-C\|\varphi\|_\alpha^\alpha)$ for $\varphi \in \mathcal{S}(\mathbb{R}^d)$. We show now that this convergence can be extended to functions $f \in \mathcal{R}(\mathbb{R}^d)$. Indeed, for $a > 0$, $f \in \mathcal{R}(\mathbb{R}^d)$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\begin{aligned}\left|\widehat{\mathscr{P}}_w(a^{d/\alpha} f(a\cdot)) - \widehat{\mathscr{P}}_{w_\alpha}(f)\right| &\leq \left|\widehat{\mathscr{P}}_w(a^{d/\alpha} f(a\cdot)) - \widehat{\mathscr{P}}_w(a^{d/\alpha}\varphi(a\cdot))\right| + \left|\widehat{\mathscr{P}}_w(a^{d/\alpha}\varphi(a\cdot)) - \widehat{\mathscr{P}}_{w_\alpha}(\varphi)\right| \\ &\quad + \left|\widehat{\mathscr{P}}_{w_\alpha}(\varphi) - \widehat{\mathscr{P}}_{w_\alpha}(f)\right| \\ &= (i) + (ii) + (iii)\end{aligned} \tag{24}$$

Using the arguments of the proof of Proposition 3.5 and Theorem 4.5, we see that (iii) $\leq 2\left|1 - \widehat{\mathscr{P}}_{w_\alpha}(f - \varphi)\right| \leq 2C\|f - \varphi\|_\alpha^\alpha$. With the same ideas, we have

$$\begin{aligned}(i) &\leq 2\left|1 - \widehat{\mathscr{P}}_w(a^{d/\alpha}(f(a\cdot) - \varphi(a\cdot)))\right| \leq 2\int_{\mathbb{R}^d} a^{-d}\left|\Psi(a^{d/\alpha}(f(\boldsymbol{y}) - \varphi(\boldsymbol{y})))\right| \mathrm{d}\boldsymbol{y} \\ &\leq C'\|f - \varphi\|_\alpha^\alpha,\end{aligned} \tag{25}$$

where the last inequality is obtained by exploiting that $|\Psi|$ is bounded by $|\cdot|^\alpha$ at the origin. The second term (ii) vanishes when $a \to 0$ for $\varphi \in \mathcal{S}(\mathbb{R}^d)$ fixed. It means it suffices to select $\varphi \in \mathcal{S}(\mathbb{R}^d)$ such that $\|f - \varphi\|_\alpha^\alpha$ is small and then $a > 0$ such that (ii) is small (this is possible because $\varphi \in \mathcal{S}(\mathbb{R}^d)$; hence, (ii) vanishes when $a \to 0$) to make $\left|\widehat{\mathscr{P}}_w(a^{d/\alpha} f(a\cdot)) - \widehat{\mathscr{P}}_{w_\alpha}(f)\right|$ arbitrarily small as expected.

Let us now consider $f = \mathbb{1}_{[0,1]^d}$. Then, we have $\log \widehat{\mathscr{P}}_w(a^{d/\alpha} f(a\cdot) = a^{-d}\Psi(a^{d/\alpha}) \underset{a\to 0}{\longrightarrow} \log \widehat{\mathscr{P}}_{w_\alpha}(f) = -C$. With $f = -\mathbb{1}_{[0,1]^d}$, we have similarly that $a^{-d}\Psi(-\,a^{d/\alpha})$ converges to $-\,C$. Finally, setting $\xi = \pm a^{d/\alpha}$, we obtain that $\Psi(\xi) \underset{|\xi|\to 0}{\sim} -C\,|\xi|^\alpha$.

The proof for the local case is very similar. The only difference is for the control of $K(a) := \int_{\mathbb{R}^d} a^{-d} \left|\Psi(a^{d/\alpha}(f(\boldsymbol{y}) - \varphi(\boldsymbol{y})))\right| \mathrm{d}\boldsymbol{y}$ in (25) when $a \to \infty$. Then, the result follows from the same decomposition and arguments used in (18). Indeed, we have

$$\begin{aligned} K(a) &= \int_{\mathbb{R}^d} \mathbb{1}_{a^{d/\alpha}|f(\boldsymbol{y})-\varphi(\boldsymbol{y})|\geq 1} a^{-d} \left|\Psi(a^{d/\alpha}(f(\boldsymbol{y}) - \varphi(\boldsymbol{y})))\right| \mathrm{d}\boldsymbol{y} \\ &\quad + \int_{\mathbb{R}^d} \mathbb{1}_{a^{d/\alpha}|f(\boldsymbol{y})-\varphi(\boldsymbol{y})|< 1} a^{-d} \left|\Psi(a^{d/\alpha}(f(\boldsymbol{y}) - \varphi(\boldsymbol{y})))\right| \mathrm{d}\boldsymbol{y} \\ &:= I(a) + J(a). \end{aligned} \tag{26}$$

We bound $I(a) \leq C\|f-\varphi\|_\alpha^\alpha$ because Ψ is bounded by $|\cdot|^\alpha$ at infinity. We also have that $J(a)$ vanishes when $a \to \infty$, as we see by bounding $|\Psi|$ by $|\cdot|^\epsilon$ with $\epsilon < \alpha$ [ϵ exists because w is tempered, see (10)]. □

As a final remark, we point out that there exist Lévy exponents Ψ that are oscillating between two different power laws at infinity. Some examples are constructed in [23, Examples 1.1.15 and 1.1.16]. These examples coupled with Proposition 4.8 imply that one cannot hope to have local self-similarity for any γ-order linear process.

5 Application to Specific Classes of SDEs and Simulations

5.1 Examples of Lévy White Noises

We introduce classical families of Lévy white noises that allow us to illustrate our results.

From infinitely divisible random variables to Lévy white noises Consider a Lévy white noise w on $\mathcal{S}'(\mathbb{R}^d)$ and a family of functions $\varphi_n \in \mathcal{S}(\mathbb{R}^d)$ that converges to $\mathbb{1}_{[0,1]^d}$ for the topology of $\mathcal{R}(\mathbb{R}^d)$ (see Sect. 3.2). Since the characteristic functional of w is continuous over $\mathcal{R}(\mathbb{R}^d)$ (Proposition 3.5), one can show that the sequence $(\langle w, \varphi_n\rangle)$ is a Cauchy sequence in $L^0(\Omega)$. It therefore converges to some random variable denoted by $X = \langle w, \mathbb{1}_{[0,1]^d}\rangle$. This random variable is infinitely divisible, with characteristic function

$$\widehat{\mathscr{P}}_X(\xi) = \exp\left(\int_{\mathbb{R}^d} \Psi(\xi \mathbb{1}_{[0,1]^d}(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}\right) = \exp\left(\Psi(\xi)\right). \tag{27}$$

The latter equality in (27) comes from the fact that $\Psi(0) = 0$. The law of w is fully characterized by the law of $\langle w, \mathbb{1}_{[0,1]^d}\rangle$. This principle is made rigorous and extended to many more test functions in [21].

By convention, the terminology for the random variable $\langle w, \mathbb{1}_{[0,1]^d}\rangle$ is inherited by the underlying white noise w. We have already exploited this principle for the definition of SαS white noises, with the particular case of the Gaussian white noise. Another example is the *Cauchy white noise* that corresponds to the case $\alpha = 1$.

Compound Poisson White Noises A *compound Poisson white noise* is such that $\langle w, \mathbb{1}_{[0,1]^d}\rangle$ is a compound Poisson random variable with characteristic function of the form [54, Sect. 4.4.2] $\widehat{\mathscr{P}}_{\langle w,\mathbb{1}_{[0,1]^d}\rangle}(\xi) = \exp(\lambda(\widehat{\mathscr{P}}_{\mathrm{Jump}}(\xi) - 1))$, where $\lambda > 0$ and $\widehat{\mathscr{P}}_{\mathrm{Jump}}$ is the characteristic function of a probably law $\mathscr{P}_{\mathrm{Jump}}$ such that $\mathscr{P}_{\mathrm{Jump}}\{0\} = 0$ (no singularity at the origin). The notation $\widehat{\mathscr{P}}_{\mathrm{Jump}}$ is motivated by the fact that the underlying probability law is the common law of the jumps of the compound Poisson white noise [53]. The Lévy exponent of a compound Poisson white noise is bounded; hence, its index is $\beta_\infty = 0$. The Pruitt index β_0 can take any value in $(0, 2]$ and is equal to 2 if $\langle w, \mathbb{1}_{[0,1]^d}\rangle$ is symmetric with a finite variance. When the law of the jump is Gaussian (Cauchy, respectively), we call w a *Poisson–Gaussian* white noise (a *Poisson–Cauchy* white noise, respectively).

Generalized Laplace White Noises Another interesting infinitely divisible family is given by the generalized Laplace laws. We follow here the notations of [33]. A *generalized Laplace white noise* is such that $\langle w, \mathbb{1}_{[0,1]^d}\rangle$ is a generalized Laplace variable whose characteristic function is given by $\widehat{\mathscr{P}}_{\langle w,\mathbb{1}_{[0,1]^d}\rangle}(\xi) = \frac{1}{(1+\xi^2)^c} = \exp\left(- c\log(1+\xi^2)\right)$ with $c > 0$. When $c = 1$, we recognize the Laplace law. The Blumenthal–Getoor and Pruitt indices of generalized Laplace white noises are $\beta_\infty = 0$ (since Ψ grows asymptotically slower than any polynomial) and $\beta_0 = 2$ (symmetric finite-variance white noise), respectively.

Layered Stable White Noises Finally, we consider the family of white noises introduced by Houdré and Kawai in [28] to illustrate the richness of the Lévy family. We first need some notation. A *Lévy measure* is a measure ν on $\mathbb{R}$ such that $\nu\{0\} = 0$ and $\int_{\mathbb{R}} \inf(1, t^2)\nu(\mathrm{d}t) < \infty$. Then, for ν a symmetric Lévy measure, the function $\Psi(\xi) = -\int_{\mathbb{R}}(1 - \cos(\xi t))\nu(\mathrm{d}t)$ is a Lévy exponent. This is a particular case of the Lévy–Khintchine decomposition of a Lévy exponent [54, Theorem 4.2]. Then, for $\alpha, \beta \in (0, 2)$, we consider the measure

$$\nu_{\alpha,\beta}(\mathrm{d}t) = \mathbb{1}_{|t|\leq 1}\frac{\mathrm{d}t}{|t|^{\alpha+1}} + \mathbb{1}_{|t|>1}\frac{\mathrm{d}t}{|t|^{\beta+1}}. \tag{28}$$

We easily check that $\nu_{\alpha,\beta}$ is a symmetric Lévy measure and define therefore the Lévy exponent $\Psi_{\alpha,\beta}(\xi) = -\int_{\mathbb{R}}(1 - \cos(\xi t))\nu_{\alpha,\beta}(\mathrm{d}t)$. When $\alpha = \beta$, we recover a SαS white noise with Lévy measure $\nu_\alpha(\mathrm{d}t) = \mathrm{d}t/|t|^{\alpha+1}$. The Lévy white noise with exponent $\Psi_{\alpha,\beta}$ is called an *layered stable white noise*. Its interest for our purpose is that it displays all the possible joint behaviors of the Lévy exponent at the origin and at infinity, as shown in Proposition 5.1. Many additional properties of layered stable laws and processes have been studied in [28].

Proposition 5.1 *For $0 < \alpha, \beta < 2$, the Lévy exponent $\Psi_{\alpha,\beta}$ satisfies $\Psi_{\alpha,\beta}(\xi) \underset{\infty}{\sim} -C_\infty |\xi|^\alpha$ and $\Psi_{\alpha,\beta}(\xi) \underset{0}{\sim} -C_0 |\xi|^\beta$ with $C_0, C_\infty > 0$ some constants.* □

Table 1 Some Lévy white noises with their Blumenthal–Getoor indices

White noise	Parameter	$\Psi(\xi)$	β_0	β_∞
Gaussian	$\sigma^2 > 0$	$-\sigma^2\xi^2/2$	2	2
Non-Gaussian SαS	$\alpha \in (0,2)$	$-\lvert\xi\rvert^\alpha$	α	α
Generalized Laplace	$c > 0$	$-c\log(1+\xi^2)$	2	0
Symmetric finite-variance compound Poisson	$\lambda > 0$, $\mathscr{P}_{\text{Jump}}$	$\lambda(\widehat{\mathscr{P}}_{\text{Jump}}(\xi) - 1)$	2	0
Compound Poisson with SαS jumps	$\lambda > 0, \alpha \in (0,2)$	$\lambda(\mathrm{e}^{-\lvert\xi\rvert^\alpha} - 1)$	α	0
Layered stable	$\alpha, \beta \in (0,2)$	$\Psi_{\alpha,\beta}(\xi)$	α	β

Proof We have

$$\begin{aligned}\Psi_{\alpha,\beta}(\xi) &= -\int_{|t|\leq 1}(1-\cos(\xi t))\frac{\mathrm{d}t}{|t|^{\alpha+1}} - \int_{|t|>1}(1-\cos(\xi t))\frac{\mathrm{d}t}{|t|^{\beta+1}} \\ &:= \Psi_1(\xi) + \Psi_2(\xi).\end{aligned} \tag{29}$$

Then, by the change of variable $x = \xi t$, we have that

$$\Psi_1(\xi) = -\left(\int_{|x|\leq|\xi|}(1-\cos x)\frac{\mathrm{d}x}{|x|^{\alpha+1}}\right)|\xi|^\alpha \underset{\infty}{\sim} -\left(\int_{\mathbb{R}}(1-\cos x)\frac{\mathrm{d}x}{|x|^{\alpha+1}}\right)|\xi|^\alpha \tag{30}$$

while $|\Psi_2(\xi)| \leq \int_{|t|>1} 2\frac{\mathrm{d}t}{|t|^{\beta+1}} = o(|\xi|^\alpha)$, implying the expected asymptotic behavior with $C_\infty = \int_{\mathbb{R}}(1-\cos x)\frac{\mathrm{d}x}{|x|^{\alpha+1}}$. Similarly, we have that $\Psi_2(\xi) \underset{0}{\sim} -(\int_{\mathbb{R}}(1-\cos x)\frac{\mathrm{d}x}{|x|^{\beta+1}})\,|\xi|^\beta$ while $|\Psi_1(\xi)| \leq \frac{1}{2}(\int_{|t|\leq 1}\frac{\mathrm{d}t}{|t|^{\alpha-1}})\,|\xi|^2 = o(|\xi|^\beta)$, where we have used that $|1-\cos(\xi t)| \leq \frac{\xi^2 t^2}{2}$. This implies the behavior of Ψ at the origin with $C_0 = \int_{\mathbb{R}}(1-\cos x)\frac{\mathrm{d}x}{|x|^{\beta+1}}$. □

Proposition 5.1 implies that $(\beta_\infty, \beta_0) = (\alpha, \beta)$. Therefore, γ-order linear processes based on a layered stable white noise share the interesting following property: While failing to be self-similar, they offer a transition from a local self-similarity of order $H_{\text{loc}} = \gamma + d(1/\alpha - 1)$ to an asymptotic self-similarity of order $H_\infty = \gamma + d(1/\beta - 1)$. This can be of interest for modeling purposes.

Summary By studying the behavior of the Lévy exponent around the origin and at ∞ (as we did for $\Psi_{\alpha,\beta}$), one easily obtains the indices of the Lévy white noises of Table 1.

5.2 Lévy Processes and Sheets

The canonical basis of $\mathbb{R}^d$ is $(\boldsymbol{e}_k)_{k=1\dots d}$. We denote by D_k the partial derivative along the direction $\boldsymbol{e}_k$. Then, a *Lévy sheet* in dimension d is a solution of

$$\mathrm{L}s = \mathrm{D}_1 \cdots \mathrm{D}_d s = w \tag{31}$$

with w a d-dimensional Lévy white noise [13]. When $d = 1$, one recognizes the family of Lévy processes that corresponds to the differential equation $\mathrm{D}s = w$ in dimension $d = 1$.

The linear operator $\mathrm{L} = \mathrm{D}_1 \cdots \mathrm{D}_d$ is continuous from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$ and d-homogeneous. Its adjoint $\mathrm{L}^* = (-1)^d \mathrm{D}_1 \cdots \mathrm{D}_d$ admits the natural $(-\,d)$-homogeneous (left and right) inverse defined by $(\mathrm{L}^*)^{-1}\varphi(\boldsymbol{x}) = \int_{(-\infty,x_1)\times\cdots\times(-\infty,x_d)} \varphi(\boldsymbol{t})\mathrm{d}\boldsymbol{t}$ for $\boldsymbol{x} = (x_1, \dots, x_d)$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Unfortunately, $(\mathrm{L}^*)^{-1}$ is unstable in the sense that it does not map $\mathcal{S}(\mathbb{R}^d)$ in any $L^p(\mathbb{R}^d)$ space, $0 < p \leq 2$ (and, *a fortiori*, not in $\mathcal{R}(\mathbb{R}^d)$). We can however correct $(\mathrm{L}^*)^{-1}$ to transform it into a stable *left* inverse. For this, we define T as the adjoint of the operator $\mathrm{T}^*\varphi(\boldsymbol{x}) = \int_{(0,x_1)\times\cdots\times(0,x_d)} \varphi(\boldsymbol{t})\mathrm{d}\boldsymbol{t}$. The operator T is $(-\,d)$-homogeneous and continuous from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{R}(\mathbb{R}^d)$ [18, Sect. 4.2]. We satisfy therefore the Condition (C1) of Theorem 3.6 and define $s = (\mathrm{D}_1 \cdots \mathrm{D}_d)^{-1} w$ with characteristic functional $\widehat{\mathscr{P}}_s(\varphi) = \widehat{\mathscr{P}}_w(\mathrm{T}\varphi)$ for any white noise w.

This way of defining s can be interpreted in terms of boundary conditions—it imposes that $s(\boldsymbol{x}) = 0$ almost surely for every $\boldsymbol{x} = (x_1, \dots, x_d)$ such that one of the x_k is 0. In particular, in dimension $d = 1$, it imposes that $s(0) = 0$ almost surely. Here, the random variable $s(\boldsymbol{x})$, not well defined from the specification of s as a generalized random process, is understood as the limit in probability of random variables $\langle s, \varphi\rangle$ where φ approximates the shifted Dirac impulse $\delta(\cdot - \boldsymbol{x})$ in an adequate sense. This extension is possible for the same reasons one can consider the random variable $\langle w, \mathbb{1}_{[0,1]^d}\rangle$ for any Lévy noise, which has been already discussed below. Applying the results of Sect. 4.2, we directly deduce Proposition 5.2.

Proposition 5.2 *Consider w a Lévy white noise with indices $0 < \beta_0, \beta_\infty \leq 2$, and $s = (\mathrm{D}_1 \cdots \mathrm{D}_d)^{-1} w$ as above. Then,*

- *If $\Psi(\xi) \underset{0}{\sim} -C\,|\xi|^{\beta_0}$ for some $C > 0$, then $a^{d/\beta_0} s(\cdot/a) \underset{a\to 0}{\longrightarrow} s_{\mathrm{D}_1\cdots\mathrm{D}_d,\beta_0}$;*
- *If $\Psi(\xi) \underset{\infty}{\sim} -C\,|\xi|^{\beta_\infty}$ for some $C > 0$, then $a^{d/\beta_\infty} s(\cdot/a) \underset{a\to\infty}{\longrightarrow} s_{\mathrm{D}_1\cdots\mathrm{D}_d,\beta_\infty}$.*

Here, $s_{\mathrm{D}_1,\dots,\mathrm{D}_d,\alpha} = (\mathrm{D}_1 \cdots \mathrm{D}_d)^{-1} w_\alpha$, where w_α is a SαS white noise. □

We illustrate our results on dimension 1 with some simulations of Lévy processes. First, we consider three Lévy processes driven, respectively, by the Laplace white noise, the Poisson–Gaussian white noise, and the Poisson–Cauchy white noise. We look at the processes at three different scales by representing them on [0, 1], [0, 10], and [0, 1000]. We only generate one process of each type and represent it on the different intervals: This corresponds to zooming out it. The theoretical prediction at large scale is as follows: The Laplace and Poisson–Gaussian processes should be statistically indistinguishable from the Brownian motion, while the Poisson–Cauchy

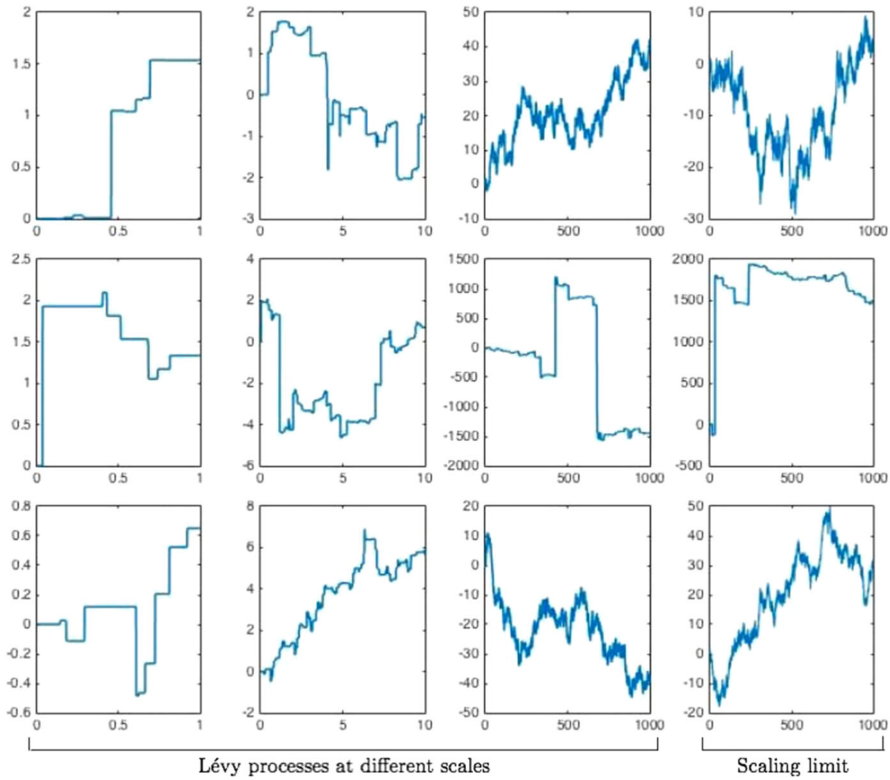

Fig. 1 Lévy processes at three different scales and comparison with the corresponding self-similar process at large scale according to Theorem 4.5

process should be statistically indistinguishable from the Cauchy process (also called Lévy flight). We see in Fig. 1 that this is observed on simulations. For comparison purposes, we also represent one realization of the expected limit process.

We now illustrate the difference between fine-scale and coarse-scale behaviors. To do so, we consider a Lévy white noise w, sum of a Gaussian and a Cauchy white noise that are independent. Then, we have $\beta_0 = 1$ and $\beta_\infty = 2$. The prediction is that the Lévy process driven by w converges to the Brownian motion at fine scales and to the Lévy flight at coarse scales. Again, the theoretical prediction is observed on simulations in Fig. 2, where one realization of the process is represented on $[0, 1/10]$ (fine scale), $[0, 10]$ (intermediate scale), and $[0, 1000]$ (coarse scale).

5.3 Fractional Lévy Processes and Fields

In dimension d, we consider the SDE

$$\mathrm{L}s = (-\Delta)^{\gamma/2} s = w, \tag{32}$$

where $(-\Delta)^{\gamma/2}$ is the fractional Laplacian whose Fourier multiplier is $\|\boldsymbol{\omega}\|^\gamma$ with $\gamma \geq 0$ and $\gamma/2 \notin \mathbb{N}$. The fractional Laplacian is self-adjoint and γ-homogeneous. For (p, γ) satisfying

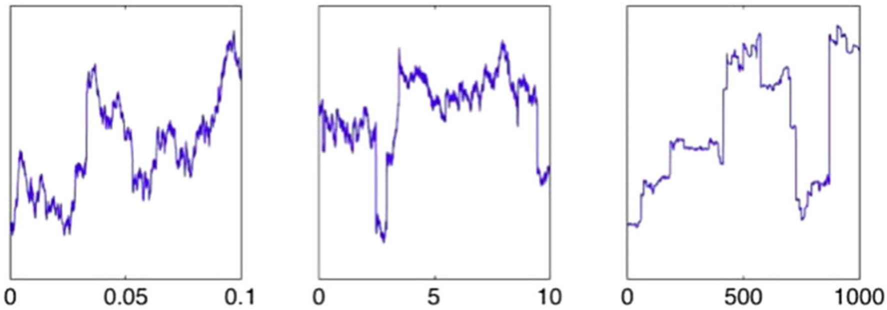

Fig. 2 Sum of a Lévy flight and a Brownian motion at thee different scales

$$p \geq 1 \text{ and } (\gamma + d/p - 1) \notin \mathbb{N}, \tag{33}$$

$(-\Delta)^{\gamma/2}$ admits a (unique) $(-\gamma)$-homogeneous left inverse $\mathrm{T}_{\gamma,p}$ that continuously map $\mathcal{S}(\mathbb{R}^d)$ into $L^p(\mathbb{R}^d)$ [51, Theorem 3.7]. For such p, if the Lévy white noise is p-admissible, we satisfy Condition (C2) of Theorem 3.6 and define $s = ((-\Delta)^{\gamma/2})^{-1}w$ with characteristic functional $\widehat{\mathscr{P}_s}(\varphi) = \widehat{\mathscr{P}_w}(\mathrm{T}_{\gamma,p}\varphi)$. The process s is called a fractional Lévy process (a fractional Lévy field when $d \geq 2$). Again, the direct application of the results of Sect. 4.2 yields Proposition 5.3.

Proposition 5.3 *For (p, γ) satisfying (33), consider a p-admissible Lévy white noise w with indices $0 < \beta_0, \beta_\infty \leq 2$, and $s = ((-\Delta)^{\gamma/2})^{-1}w$ as above. Then,*

- *If $\Psi(\xi) \underset{0}{\sim} -C\,|\xi|^{\beta_0}$ for some $C > 0$, then $a^{\gamma+d(1/\beta_0-1)}s(\cdot/a) \underset{a\to 0}{\longrightarrow} s_{(-\Delta)^{\gamma/2},\beta_0}$;*
- *If $\Psi(\xi) \underset{\infty}{\sim} -C\,|\xi|^{\beta_\infty}$ for some $C > 0$, then $a^{\gamma+d(1/\beta_\infty-1)}s(\cdot/a) \underset{a\to \infty}{\longrightarrow} s_{(-\Delta)^{\gamma/2},\beta_\infty}$.*

Here, $s_{(-\Delta)^{\gamma/2},\alpha} = (-\Delta)^{-\gamma/2}w_\alpha$, where w_α is a $S\alpha S$ white noise. □

In dimension $d = 1$, identical results can be derived for the fractional derivative $\mathrm{L} = \mathrm{D}^\gamma$ in a very similar fashion. This includes in particular the fractional Brownian motions [38] and its Lévy-driven generalizations. The construction of stable inverses of the adjoint of D^γ is the subject of [54, Sect. 5.5.1].

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