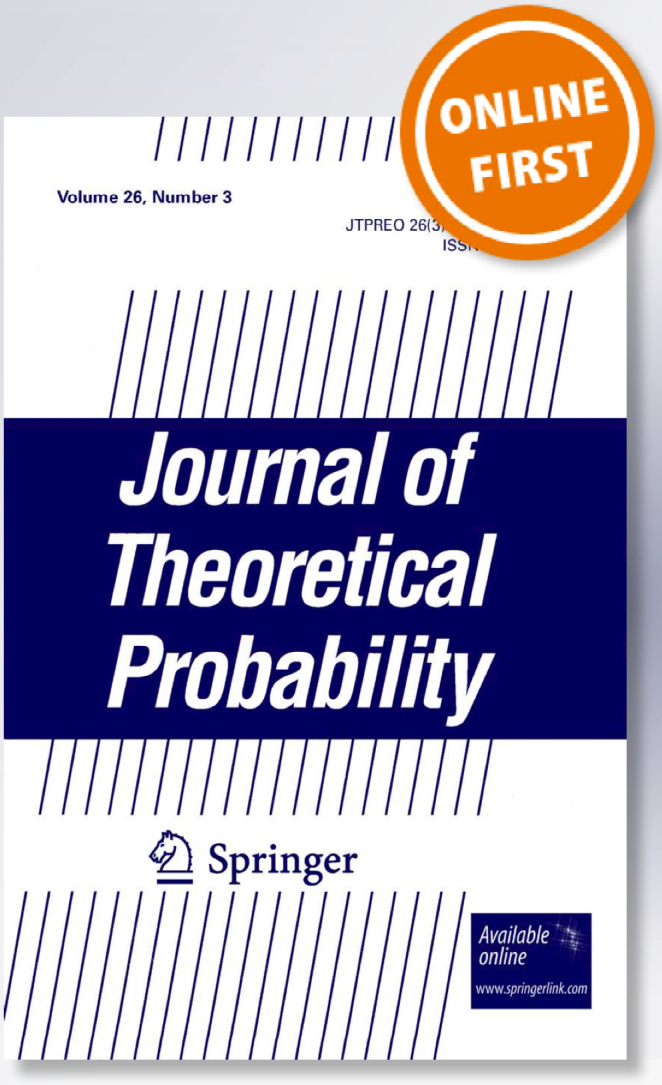
The $$n$$ n -term Approximation of Periodic Generalized Lévy Processes

Julien Fageot, Michael Unser & John Paul Ward

Journal of Theoretical Probability

ISSN 0894-9840

J Theor Probab
DOI 10.1007/s10959-018-00877-7



Springer

Your article is protected by copyright and all rights are held exclusively by Springer Science+Business Media, LLC, part of Springer Nature. This e-offprint is for personal use only and shall not be self-archived in electronic repositories. If you wish to self-archive your article, please use the accepted manuscript version for posting on your own website. You may further deposit the accepted manuscript version in any repository, provided it is only made publicly available 12 months after official publication or later and provided acknowledgement is given to the original source of publication and a link is inserted to the published article on Springer's website. The link must be accompanied by the following text: "The final publication is available at link.springer.com".

Author's personal copy

Journal of Theoretical Probability
https://doi.org/10.1007/s10959-018-00877-7

The *n*-term Approximation of Periodic Generalized Lévy Processes

Julien Fageot[1] · **Michael Unser**[1] · **John Paul Ward**[2]

Received: 15 February 2018
© Springer Science+Business Media, LLC, part of Springer Nature 2019

Abstract
In this paper, we study the compressibility of random processes and fields, called generalized Lévy processes, that are solutions of stochastic differential equations driven by d-dimensional periodic Lévy white noises. Our results are based on the estimation of the Besov regularity of Lévy white noises and generalized Lévy processes. We show in particular that non-Gaussian generalized Lévy processes are more compressible in a wavelet basis than the corresponding Gaussian processes, in the sense that their n-term approximation errors decay faster. We quantify this compressibility in terms of the Blumenthal–Getoor indices of the underlying Lévy white noise.

Keywords Generalized Lévy processes · Lévy white noises · Besov regularity · n-term approximation · Compressibility

Mathematics Subject Classification (2010) 60G20 · 41A25

1 Introduction

Stochastic models are commonly used in engineering and financial applications [12, 31,33,34]. The most widely considered stochastic models are Gaussian. However, the Gaussian assumption is too restrictive for many applications, and more general models are needed to accurately represent real data. Stochastic differential equations driven by Lévy noises are able to play this role. We call the solutions of these equations *generalized Lévy processes*. Generalized Lévy processes encompass Gaussian models

This work is an extension of the conference paper [49].

✉ John Paul Ward
jpward@ncat.edu

1 Biomedical Imaging Group, École polytechnique fédérale de Lausanne (EPFL), Station 17, 1015 Lausanne, Switzerland

2 Department of Mathematics, North Carolina A&T State University, Marteena Hall, 1601 East Market Street, Greensboro, NC 27411, USA

Published online: 24 January 2019

Author's personal copy

and the family of compound Poisson processes that are pure jump processes. They also include symmetric-alpha-stable (SαS) processes which maintain many of the desirable properties of Gaussian models; for example, they satisfy a generalized central-limit theorem [18,36].

Generalized Lévy processes with no Gaussian component are particularly relevant to applications [42]. They are called *sparse stochastic processes* [16,47] due to their ability to model sparse signals [1,7]. Each member of the family of generalized Lévy processes is associated with two parameters $0 \leq \beta' \leq \beta \leq 2$, the Blumenthal–Getoor indices. For most of the generalized Lévy processes, including Gaussian, SαS, and compound Poisson, these indices are equal; hence, we only refer to β in this introduction. As we shall see, β determines the sparsity level of the generalized Lévy process in a wavelet basis. Here, we characterize sparsity by n-term wavelet approximation rates; in other words, a process is said to be sparser than another one if it satisfies a faster rate of decay of its error of approximation.

Our interest in a sparse wavelet representation is compressibility, since it allows one to store and transmit processes more efficiently by first expanding them in a wavelet basis. We shall show that Gaussian processes (for which $\beta = \beta' = 2$) are the least compressible, while the compressibility of non-Gaussian processes increases as β decreases. Compressibility is also a useful assumption in inverse imaging problems. If an image is known to be sparse a priori, then this information can be incorporated to produce a better reconstruction algorithm [7,15,47].

Our model is the stochastic differential equation

$$\mathrm{L}s = w, \tag{1}$$

where L is a differential operator, s is a stochastic process, and w is a Lévy white noise. The two defining components are the operator and the white noise. The white noise w is a random periodic generalized function. We consider standard differential-type operators of the following type: L is assumed to be an order $\gamma > 0$ operator (typically an order γ derivative) that reduces the regularity of a function by γ.

The compressibility of a given function—and, by extension, of a stochastic process—can be quantified in terms of its Besov regularity. Besov spaces are important function spaces that can be characterized precisely by the rate of decay of their n-term approximation error [10,22]. Our strategy to access the compressibility of a stochastic process is to connect results on the Besov regularity of the underlying white noise to compressibility by (deterministic) approximation-theoretic arguments.

The question of the Besov regularity of Lévy processes—that corresponds to (1) in dimension $d = 1$ with $\mathrm{L} = \mathrm{D}$ (the derivative operator)—has been addressed in the literature for subfamilies of Lévy processes [4,9,35] as well as in the general case [26]. Several extensions by R. Schilling have been obtained for Lévy-type processes [38–40]; see also [8, Chapter 5] for a summary. In our case, we aim at considering a multidimensional setting with a general operator L. This work is therefore a continuation of our previous works on the Besov regularity of Lévy white noises [2,17,19].

The rest of the paper is organized as follows: in Sect. 2, we cover the mathematical background for the remaining sections and discuss function spaces and random processes. In Sect. 3, we define a class of admissible operators L for (1), which leads to a

Author's personal copy

definition of generalized Lévy processes on the torus. In Sect. 4, we put forth the main results of the paper: n-term approximation of periodic generalized Lévy processes. Finally, we conclude in Sect. 5 with a discussion of our results.

2 Mathematical Background

Motivated by the study of local properties of solutions of stochastic differential equations, this paper deals with periodic random processes. Therefore, we consider spaces of periodic functions in the sequel. We specify these functions on their fundamental domain, the d-dimensional torus $\mathbb{T}^d = [-1/2, 1/2)^d \subset \mathbb{R}^d$, $d \geq 1$.

2.1 Lebesgue and Sobolev Spaces

The space of continuous functions on the torus is $C(\mathbb{T}^d)$, and $C^k(\mathbb{T}^d)$ denotes the functions with $k \in \mathbb{N}$ continuous derivatives. The Lebesgue space $L_p(\mathbb{T}^d)$ is the collection of measurable functions for which

$$\|f\|_{L_p(\mathbb{T}^d)} := \left(\int_{\mathbb{T}^d} |f(t)|^p \, \mathrm{d}t \right)^{1/p} \tag{2}$$

is finite. For $p \geq 1$, (2) is a norm; for $0 < p < 1$, it is a quasi-norm. The L_2 Sobolev space of order $k \in \mathbb{Z}$ is $H_2^k(\mathbb{T}^d)$. For $k \in \mathbb{N}$, $H_2^k(\mathbb{T}^d)$ is the collection of functions in $L_2(\mathbb{T}^d)$ with k derivatives in $L_2(\mathbb{T}^d)$.

2.2 Periodic Test Functions and Generalized Functions

The standard Schwartz space of infinitely differentiable test functions is denoted as $\mathcal{S}(\mathbb{T}^d)$. The corresponding space of generalized functions is $\mathcal{S}'(\mathbb{T}^d)$. These spaces are nuclear spaces [44]. Moreover, it is known that $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$ are respectively the intersection and the union of weighted Sobolev spaces [28,43] (or more generally weighted Besov spaces, see for instance [30]). This is easily inherited for the corresponding spaces on the torus; that is,

$$\mathcal{S}(\mathbb{T}^d) = \bigcap_{k \in \mathbb{Z}} H_2^k(\mathbb{T}^d), \tag{3}$$

$$\mathcal{S}'(\mathbb{T}^d) = \bigcup_{k \in \mathbb{Z}} H_2^k(\mathbb{T}^d). \tag{4}$$

For our purpose, we consider the related spaces with mean zero. Such generalized functions are well suited to wavelet approximation since wavelets also have mean zero. In addition, this assumption simplifies the definition of stochastic processes. For example, the derivative operator becomes a bijective mapping of homogeneous Sobolev and Besov spaces. Moreover, this assumption does not impact the generality of our results: indeed, the addition of a constant term does not affect the regularity of a function.

Author's personal copy

By a 0-mean generalized function $u \in \mathcal{S}'(\mathbb{T}^d)$, we mean one for which $\langle u, 1\rangle = 0$. We use a *dot notation* to specify spaces of 0-mean (generalized) functions. For instance, the L_2 Sobolev space of 0-mean (generalized) functions of order $k \in \mathbb{Z}$ is denoted as $\dot{H}_2^k(\mathbb{T}^d)$.

Notation 1 The space of infinitely differentiable test functions with mean zero is denoted as $\dot{\mathcal{S}}(\mathbb{T}^d)$. It has the projective limit topology

$$\dot{\mathcal{S}}(\mathbb{T}^d) = \bigcap_{k\in\mathbb{Z}} \dot{H}_2^k(\mathbb{T}^d) \tag{5}$$

The corresponding space of 0-mean generalized functions (continuous linear functionals on $\dot{\mathcal{S}}(\mathbb{T}^d)$) with the weak-* topology is denoted as $\dot{\mathcal{S}}'(\mathbb{T}^d)$.

Note that $\dot{\mathcal{S}}(\mathbb{T}^d)$ has the same nuclear Fréchet space structure as $\mathcal{S}(\mathbb{T}^d)$. Also, periodic generalized functions $f \in \dot{\mathcal{S}}'(\mathbb{T}^d)$ are characterized by their Fourier coefficients $\left\langle f, \mathrm{e}^{2\pi \mathrm{i}\langle \boldsymbol{m}, \cdot\rangle}\right\rangle, \boldsymbol{m} \in \mathbb{Z}^d \backslash \{\mathbf{0}\}$. The zero term is excluded because the functions have mean 0.

2.3 Generalized Random Processes

The theory of generalized random processes was initiated independently by Gelfand and Itô [23,27] and further developed extensively in [21,24,28]. A recent introductory and concise presentation for *tempered* generalized random processes, which can easily be adapted to the case of (homogeneous) generalized functions on the torus, can be found in [5].

Definition 1 A *generalized random process* on $\dot{\mathcal{S}}'(\mathbb{T}^d)$ is a collection of real random variables $(\langle s, \varphi\rangle)_{\varphi\in\dot{\mathcal{S}}(\mathbb{T}^d)}$ that satisfy the following properties:

(i) *Linearity* $\langle s, \varphi + \lambda\psi\rangle = \langle s, \varphi\rangle + \lambda\langle s, \psi\rangle$ almost surely for every $\varphi, \psi \in \dot{\mathcal{S}}(\mathbb{T}^d)$.
(ii) *Continuity* $\langle s, \varphi_n\rangle \to \langle s, \varphi\rangle$ in probability whenever $\varphi_n \to \varphi$ in $\dot{\mathcal{S}}(\mathbb{T}^d)$.

A generalized random process is therefore a continuous and linear functional from $\dot{\mathcal{S}}(\mathbb{T}^d)$ to the space of random variables. Such an object is called a *continuous linear random functional* in [28].

Definition 2 The *characteristic functional* $\widehat{\mathscr{P}_s} : \dot{\mathcal{S}}(\mathbb{T}^d) \to \mathbb{C}$ of a generalized random process s is defined as

$$\widehat{\mathscr{P}_s}(\varphi) = \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\langle s,\varphi\rangle}\right]. \tag{6}$$

A generalized random process s specifies a characteristic functional $\widehat{\mathscr{P}_s}$ that is positive definite, continuous, and satisfies $\widehat{\mathscr{P}_s}(0) = 1$. Conversely, we can define a generalized random process by way of its characteristic functional. This is due to the structure of the nuclear Fréchet space $\dot{\mathcal{S}}'(\mathbb{T}^d)$.

- The Minlos–Bochner theorem ([24] or Theorem 2.4.1 and Theorem 2.4.3 of [28]) implies that a continuous, positive definite functional $\widehat{\mathscr{P}} : \dot{\mathcal{S}}(\mathbb{T}^d) \to \mathbb{C}$ with $\widehat{\mathscr{P}}(0) = 1$ is the Fourier transform of a probability measure μ on $\dot{\mathcal{S}}'(\mathbb{T}^d)$ endowed with its cylindrical σ-field.

Author's personal copy

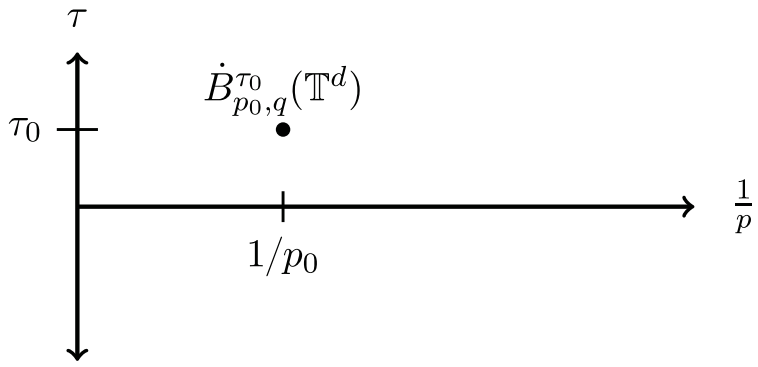


Fig. 1 Function-space diagram. A point $(1/p_0, \tau_0)$ represents all Besov spaces $\dot{B}^{\tau_0}_{p_0,q}(\mathbb{T}^d)$ for $0 < q \leq \infty$

- A probability measure on $\dot{\mathcal{S}}'(\mathbb{T}^d)$ is uniquely associated to a continuous linear random functional [28, Chapter 2].

We say that a functional $\widehat{\mathscr{P}}$ is a *characteristic functional* if it satisfies the properties of the Minlos–Bochner theorem.

Example 1 The functionals $\exp\left(-\|\varphi\|^{\alpha}_{L_\alpha(\mathbb{T}^d)}\right)$ are positive definite and continuous on $\dot{\mathcal{S}}(\mathbb{T}^d)$ for $0 < \alpha \leq 2$. Accordingly, they are characteristic functionals that specify some corresponding generalized random processes on $\dot{\mathcal{S}}'(\mathbb{T}^d)$, namely, S$\alpha$S periodic Lévy white noises.

2.4 Homogeneous Besov Spaces

The homogeneous Besov spaces $\dot{B}^{\tau}_{p,q}(\mathbb{T}^d)$ are specified by two primary parameters (p and τ) and one secondary parameter (q). The parameter $p \in (0, \infty]$ plays a role that is similar to the index defining the Lebesgue spaces $L_p(\mathbb{T}^d)$, while $\tau \in \mathbb{R}$ indicates smoothness in the sense of order of differentiability. Therefore, roughly speaking, for $\tau \in \mathbb{N}$, a function in $\dot{B}^{\tau}_{p,q}(\mathbb{T}^d)$ has τ derivatives in $L_p(\mathbb{T}^d)$. In Fig. 1, we provide a structural diagram that represents the collection of Besov spaces. Our interest in these spaces lies in the following facts:

(i) Wavelets form unconditional bases for Besov spaces;
(ii) The mapping that takes a function to its wavelet coefficients is an isomorphism between Besov function spaces and Besov sequence spaces;
(iii) The n-term approximation characterizes Besov sequence spaces. Given a sequence with a known rate of n-term approximation, we can specify which Besov sequence spaces it is in.

The classical definition of Besov spaces is taken from [45, Definition 1.27] and repeated in Definition 4. The idea is to decompose a function f by grouping dyadic frequency bands, using a partition of unity in the Fourier domain.

Definition 3 Let $\widehat{\upsilon} \in \mathcal{S}(\mathbb{R}^d)$ generate a hierarchical partition of unity outside the ball of radius $1/2$ centered at the origin. Specifically,

(i) $\widehat{\upsilon}(\boldsymbol{\omega}) = 0$ if $|\boldsymbol{\omega}| \leq 1/2$ or $|\boldsymbol{\omega}| \geq 2$

Author's personal copy

(ii) $\widehat{\upsilon}(\boldsymbol{\omega}) > 0$ if $1/2 < |\boldsymbol{\omega}| < 2$

(iii) $\sum_{j=0}^{\infty} \widehat{\upsilon}(2^{-j}\boldsymbol{\omega}) = 1$ if $1 \leq |\boldsymbol{\omega}|$.

Note that the decomposition of a function f into the components

$$\sum_{\boldsymbol{m}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} \widehat{f}(\boldsymbol{m})\widehat{\upsilon}(2^{-j}\boldsymbol{m})\mathrm{e}^{2\pi\mathrm{i}\langle\boldsymbol{m},\cdot\rangle} \tag{7}$$

is intimately related to the concept of a wavelet decomposition, with υ playing the role of a mother wavelet.

Definition 4 Suppose $0 < p, q \leq \infty$ and $\tau \in \mathbb{R}$. A generalized function $f \in \dot{\mathcal{S}}'(\mathbb{T}^d)$ with Fourier coefficients $\widehat{f}(\boldsymbol{m})$ is in $\dot{B}^{\tau}_{p,q}(\mathbb{T}^d)$ if the quantity

$$\left(\sum_{j=0}^{\infty} 2^{j\tau q} \left\| \sum_{\boldsymbol{m}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} \widehat{f}(\boldsymbol{m})\widehat{\upsilon}(2^{-j}\boldsymbol{m})\mathrm{e}^{2\pi\mathrm{i}\langle\boldsymbol{m},\cdot\rangle} \right\|^q_{L_p(\mathbb{T}^d)}\right)^{1/q} \tag{8}$$

is finite. For $q = \infty$, the norm must be suitably modified.

Besov spaces are Banach spaces for the norm (8) when p and $q \geq 1$. For p or $q < 1$, Eq. (8) is a quasi-norm and the Besov spaces are quasi-Banach spaces. The validity of the embeddings between Besov spaces is governed by Proposition 1 [45].

Proposition 1 *Let $(-\infty) < \tau_0, \tau_1 < \infty$ and $0 < p_0, p_1 < \infty$. Then, the topological embedding $\dot{B}^{\tau_1}_{p_1,p_1}(\mathbb{T}^d) \subseteq \dot{B}^{\tau_0}_{p_0,p_0}(\mathbb{T}^d)$ is valid in the following cases:*

- *if $p_0 \leq p_1$ and $\tau_0 < \tau_1$; or*
- *if $p_0 > p_1$ and $\tau_0 < \tau_1 + d\left(\frac{1}{p_0} - \frac{1}{p_1}\right)$.*

3 Generalized Lévy Processes on the Torus

The main objects of study in this paper are generalized Lévy processes s that are solutions of the stochastic differential equation $\mathrm{L}s = w$, with w a Lévy white noise. If w has no Gaussian part, then s is a sparse process. In this section, we introduce the family of Lévy white noises w and specify the class of considered operators L.

3.1 Lévy White Noises and Their Besov Regularity

Lévy white noises have been introduced as generalized random processes on the Schwartz space $\mathcal{D}'(\mathbb{R}^d)$ of generalized functions in [24]. They are commonly defined through their characteristic functional, relying on the Minlos–Bochner theorem (see Sect. 2.3).

Given a probability space Ω, a random variable $X : \Omega \to \mathbb{R}$ is infinitely divisible if for every $N \geq 1$, it is equal in law to a sum of N i.i.d. random variables [37]. Lévy white

Author's personal copy

noises are intimately connected with infinite-divisible laws, the finite-dimensional marginals of those processes being all infinitely divisible. The characteristic function of an infinitely divisible random variable X can be written as

$$\Phi_X(\xi) = \exp(\psi(\xi)) \tag{9}$$

with ψ a continuous and conditionally positive definite function with value 0 at $\xi = 0$ [37,47]. The continuous log-characteristic function of an infinitely divisible random variable is called a *Lévy exponent* or a *characteristic exponent*. Definition 5 is the adaptation of the usual definition of Lévy white noise [24] to the nuclear space $\dot{S}'(\mathbb{T}^d)$.

Definition 5 A Lévy exponent ψ specifies a generalized random process w in $\dot{S}'(\mathbb{T}^d)$ with characteristic functional

$$\widehat{\mathscr{P}}_w(\varphi) = \exp\left(\int_{\mathbb{T}^d} \psi(\varphi(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}\right), \tag{10}$$

where $\varphi \in \dot{S}(\mathbb{T}^d)$. We call such a process w a *Lévy white noise*.

Note that this defines a characteristic functional, as seen in Sect. 2.3 and [24, Chapter 3], where the arguments are easily adapted from $\mathcal{D}'(\mathbb{R}^d)$ to $\dot{S}'(\mathbb{T}^d)$.

The Lévy–Khintchine theorem [37] ensures that a Lévy exponent ψ can be decomposed as

$$\psi(\xi) = \mathrm{i}\mu\xi - \frac{\sigma^2\xi^2}{2} + \int_{\mathbb{R}} (\mathrm{e}^{\mathrm{i}\xi t} - 1 - \mathrm{i}\xi t 1_{|t|\leq 1})\nu(\mathrm{d}t) \tag{11}$$

where $\mu \in \mathbb{R}$, $\sigma^2 \geq 0$, and ν is a Lévy measure, which means that it is a measure such that $\int_{\mathbb{R}} \inf(1, t^2)\nu(\mathrm{d}t) < \infty$ and $\nu\{0\} = 0$. We say that the white noise is *Gaussian* if μ and ν are 0. In that case,

$$\widehat{\mathscr{P}}_w(\varphi) = \exp\left(-\frac{\sigma^2\|\varphi\|_2^2}{2}\right) \tag{12}$$

and we recover the usual Gaussian white noise. If $\sigma^2 = 0$ (i.e., if w has no Gaussian part), then we say that w is *sparse* [47].

We shall deduce the regularity of a solution of (1) from the regularity of the underlying white noise w. This regularity can be computed in terms of the Blumenthal–Getoor indices of this noise.

Definition 6 The *Blumenthal–Getoor indices* of a Lévy white noise with Lévy exponent f are defined as

$$\beta = \inf\left\{p \in [0,2],\ \limsup_{|\xi|\to\infty} \frac{|\psi(\xi)|}{|\xi|^p} = 0\right\}. \tag{13}$$

$$\beta' = \inf\left\{p \in [0,2],\ \liminf_{|\xi|\to\infty} \frac{|\psi(\xi)|}{|\xi|^p} = 0\right\}. \tag{14}$$

Author's personal copy

The index β was introduced by Blumenthal and Getoor in [6] to characterize the behavior of a Lévy process around the origin. Since then, it has been used to characterize many local properties of such processes, including the spectrum of singularities in multifractal analysis [14,29] and local self-similarity [18]. More importantly for our use, β allows us to characterize Lévy process [8,38] and noises [2,17,19] in terms of Besov spaces.[1]

While equivalent for all of the classical Lévy noises (including Gaussian, SαS, and compound Poisson,) the index β' differs in general from β. A counterexample where $\beta' < \beta$ can be found in [20]. Importantly, β' plays a crucial role in the Besov space characterization of Lévy noises; it is used to determine an upper bound on the Besov regularity [2].

In Theorem 1, we summarize the results obtained in our previous works [2,17,19]. The two latter references deal with processes defined over $\mathbb{R}^d$; they can easily be adapted. For a direct construction and study of Lévy white noises on the torus, see the former [19]. Here, *a.s.* means *almost surely*.

Theorem 1 *We consider a Lévy white noise w with Blumenthal–Getoor indices $0 \leq \beta' \leq \beta \leq 2$. We fix $0 < p, q \leq \infty$ and $\tau \in \mathbb{R}$.*

- *If w is Gaussian, then*

$$w \in \dot{B}^{\tau}_{p,q}(\mathbb{T}^d) \text{ a.s. if } \tau < -d/2, \text{ and} \tag{15}$$

$$w \notin \dot{B}^{\tau}_{p,q}(\mathbb{T}^d) \text{ a.s. if } \tau > -d/2. \tag{16}$$

- *If w is non-Gaussian, then*

$$w \in \dot{B}^{\tau}_{p,q}(\mathbb{T}^d) \text{ a.s. if } \tau < d\left(\frac{1}{\max(p,\beta)} - 1\right), \text{ and} \tag{17}$$

$$w \notin \dot{B}^{\tau}_{p,q}(\mathbb{T}^d) \text{ a.s. if } \tau > d\left(\frac{1}{\max(p,\beta')} - 1\right). \tag{18}$$

The Gaussian noise differs from the non-Gaussian Lévy noises in one main way: for $p \geq 2$, the critical value for the non-Gaussian case is $d(1/p - 1)$, while it is $-d/2$ for the Gaussian case. The Besov regularity of the Gaussian noise on the torus has been studied in detail in [48] using Fourier series techniques. We have re-obtained similar results in [2, Section 3] with wavelet-based methods. These two references allow us to deduce (15) and (16). The positive result (17) can be found in [19] for SαS noise and in [17] for the general case. The negative result (18) is taken from [2].

3.2 Differential Operators of Order γ

We shall consider the class of differential operators that reduce the Besov regularity of a function by some (possibly fractional) order $\gamma > 0$. Importantly, since we are interested in the regularity properties of the solutions of the differential equation

[1] The cited works deal with general Lévy-type processes that do not necessarily have stationary increments.

Author's personal copy

$\mathrm{L}s = w$, we focus on those operators that are continuous bijections from $\dot{B}^{\tau+\gamma}_{p,q}(\mathbb{T}^d)$ to $\dot{B}^{\tau}_{p,q}(\mathbb{T}^d)$. In this case, the regularity properties of the white noise allow us to deduce the regularity properties of the process s.

Definition 7 We restrict our attention to the Fourier-multiplier operators $\mathrm{L} : \dot{\mathcal{S}}'(\mathbb{T}^d) \to \dot{\mathcal{S}}'(\mathbb{T}^d)$, specified by a symbol $\widehat{L} : \mathbb{R}^d \to \mathbb{C}$, where

$$\mathrm{L}f = \mathcal{F}^{-1}\left\{\widehat{f}\,\widehat{L}\big|_{\mathbb{Z}^d}\right\}. \tag{19}$$

A bijective operator of this form is said to be *admissible*.

Remark 1 Let us point out that an admissible operator L can be initially defined as a mapping from $\dot{\mathcal{S}}(\mathbb{T}^d)$ to $\dot{\mathcal{S}}(\mathbb{T}^d)$ since there is a natural extension by duality to the space of generalized functions. Also, notice that we require the symbol $\widehat{L}$ to be defined on the continuous domain $\mathbb{R}^d$, even though the action of L is determined by the values of $\widehat{L}$ on $\mathbb{Z}^d$. Our reason will be clear in Theorem 2, where we use the noninteger values on $\mathbb{R}^d$.

Definition 8 An admissible operator L is said to be γ-*admissible* for $\gamma \in \mathbb{R}$ if $\mathrm{L} : \dot{B}^{\tau+\gamma}_{p,q}(\mathbb{T}^d) \to \dot{B}^{\tau}_{p,q}(\mathbb{T}^d)$ is a continuous bijection and L^{-1} is continuous for every $0 < p, q \leq \infty$ and $\tau \in \mathbb{R}$.

The fractional Laplacian of order $\gamma > 0$ $(-\Delta)^{\gamma/2}$ is the canonical example of a γ-admissible operator. For a generalized function

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{m}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}} \widehat{f}(\boldsymbol{m})\mathrm{e}^{2\pi\mathrm{i}\langle\boldsymbol{m},\cdot\rangle}, \tag{20}$$

the fractional Laplacian of f is

$$(-\Delta)^{\gamma/2} f(\boldsymbol{x}) = \sum_{\boldsymbol{m}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}} \widehat{f}(\boldsymbol{m})\,|\boldsymbol{m}|^{\gamma}\,\mathrm{e}^{2\pi\mathrm{i}\langle\boldsymbol{m},\cdot\rangle}. \tag{21}$$

Note that this operator maps 0-mean generalized functions to 0-mean generalized functions.

Moreover, perturbations of the fractional Laplacian are also γ-admissible. The next few results make this statement precise. The idea is the following: An operator L is γ-admissible if and only if $(-\Delta)^{\gamma/2}\mathrm{L}^{-1}$ and $(-\Delta)^{-\gamma/2}\mathrm{L}$ are automorphisms on Besov spaces.

Proposition 2 *The fractional Laplacian $(-\Delta)^{\gamma/2}$ is a γ-admissible operator.*

Proof Consider the norm of Definition 4 applied to the generalized function (21). To verify the result, we must bound the term

$$\left\| \sum_{\boldsymbol{m}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}} |\boldsymbol{m}|^{\gamma}\,\widehat{f}(\boldsymbol{m})\widehat{\upsilon}(2^{-j}\boldsymbol{m})\mathrm{e}^{2\pi\mathrm{i}\langle\boldsymbol{m},\cdot\rangle} \right\|_{L_p(\mathbb{T}^d)} \tag{22}$$

Author's personal copy

by a constant multiple of

$$2^{j\gamma}\left\|\sum_{\boldsymbol{m}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}}\widehat{f}(\boldsymbol{m})\widehat{\upsilon}(2^{-j}\boldsymbol{m})\mathrm{e}^{2\pi\mathrm{i}\langle\boldsymbol{m},\cdot\rangle}\right\|_{L_p(\mathbb{T}^d)} \tag{23}$$

for arbitrary j.

Let us define $\bar{v}=\widehat{v}(2\cdot)+\widehat{v}+\widehat{v}(2^{-1}\cdot)$, and notice that

$$|\boldsymbol{m}|^{\gamma}\,\widehat{f}(\boldsymbol{m})\widehat{\upsilon}(2^{-j}\boldsymbol{m})\mathrm{e}^{2\pi\mathrm{i}\langle\boldsymbol{m},\cdot\rangle}=|\boldsymbol{m}|^{\gamma}\,\bar{v}(2^{-j}\boldsymbol{m})\,\widehat{f}(\boldsymbol{m})\widehat{\upsilon}(2^{-j}\boldsymbol{m})\mathrm{e}^{2\pi\mathrm{i}\langle\boldsymbol{m},\cdot\rangle}. \tag{24}$$

Then we can apply Theorem 3.3.4 of [41] in the following way. The multiplier is $|\boldsymbol{m}|^{\gamma}\,\bar{v}(2^{-j}\boldsymbol{m})$ and the dilation factor is 2^{j+2}. Hence there is a constant C for which (22) is bounded by

$$C\left\|\left|2^{j+2}\cdot\right|^{\gamma}\bar{v}(2^{-j}2^{j+2}\cdot)\right\|_{H_2^k(\mathbb{R}^d)}\left\|\sum_{\boldsymbol{m}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}}\widehat{f}(\boldsymbol{m})\widehat{\upsilon}(2^{-j}\boldsymbol{m})\mathrm{e}^{2\pi\mathrm{i}\langle\boldsymbol{m},\cdot\rangle}\right\|_{L_p(\mathbb{T}^d)}. \tag{25}$$

for some sufficiently large k depending on p and d. The first term of (25) is bounded using homogeneity

$$\left\|\left|2^{j+2}\cdot\right|^{\gamma}\bar{v}(2^{-j}2^{j+2}\cdot)\right\|_{H_2^k(\mathbb{R}^d)}=2^{\gamma(j+2)}\left\||\cdot|^{\gamma}\,\bar{v}(4\cdot)\right\|_{H_2^k(\mathbb{R}^d)}. \tag{26}$$

This last expression is finite due to the smoothness of $\widehat{v}$. □

Theorem 2 *Let* L *be an admissible operator with symbol* $\widehat{L}$. *For* $\gamma>0$, *define* $m(\boldsymbol{\omega})=|\boldsymbol{\omega}|^{-\gamma}\,\widehat{L}(\boldsymbol{\omega})$. *Also, let* ζ *be any function in* $\mathcal{S}(\mathbb{R}^d)$ *satisfying*

$$0\le\zeta(\boldsymbol{x})\le 1,\quad \zeta(\boldsymbol{x})=\begin{cases}0, & |\boldsymbol{x}|\le 1/4\\ 1, & 1/2\le|\boldsymbol{x}|\le 2\\ 0, & |\boldsymbol{x}|\ge 4.\end{cases} \tag{27}$$

If

$$\sup_{j\in\mathbb{Z}_{\ge 0}}\left(\left\|\zeta m(2^j\cdot)\right\|_{H_2^{\tau}(\mathbb{R}^d)}+\left\|\zeta m(2^j\cdot)^{-1}\right\|_{H_2^{\tau}(\mathbb{R}^d)}\right)<\infty\quad \textit{for all}\quad \tau>0, \tag{28}$$

then L *is* γ*-admissible.*

Proof This follows from a sufficient condition for Fourier multipliers on Besov spaces, Theorem 3.6.3 of [41]. To summarize, if $0<p<\infty$, $0<q<\infty$, $(-\infty)<\tau<\infty$, and

$$\tau>d\left(\frac{1}{\min(1,\,p)}-\frac{1}{2}\right), \tag{29}$$

Author's personal copy

then there exists a positive constant C such that

$$\left\| \sum_{\boldsymbol{k}\in\mathbb{Z}^d} m(\boldsymbol{k})\,\widehat{f}(\boldsymbol{k})\mathrm{e}^{2\pi\mathrm{i}\langle \boldsymbol{k},\cdot\rangle} \right\|_{\dot{B}^{\tau}_{p,q}(\mathbb{T}^d)} \leq C \left(\sup_{j\in\mathbb{Z}_{\geq 0}} \left\| \zeta m(2^j\cdot) \right\|_{H^{\tau}_2(\mathbb{R}^d)} \right) \| f \|_{\dot{B}^{\tau}_{p,q}(\mathbb{T}^d)} \tag{30}$$

holds for all functions $m \in L_\infty(\mathbb{R}^d)$ and all $f \in \dot{B}^{\tau}_{p,q}(\mathbb{T}^d)$. □

Example 2 γ-Admissible Operators

(i) The derivative D is 1-admissible.
(ii) The differential operators $\mathrm{D}^\gamma + a_{\gamma-1}\mathrm{D}^{\gamma-1} + \cdots + a_0\mathrm{Id}$ with nonvanishing symbols are γ-admissible for $\gamma \in \mathbb{Z}_{\geq 0}$
(iii) The fractional derivative D^γ is γ-admissible for any $\gamma > 0$.
(iv) The fractional Laplacian $(-\Delta)^{\gamma/2}$ is γ-admissible for any $\gamma > 0$.
(v) The Matérn operator $(\mathrm{Id} - \Delta)^{\gamma/2}$ is γ-admissible for any $\gamma > 0$.

3.3 Besov Regularity of Generalized Lévy Processes

We are now in a position to deduce the Besov regularity of generalized Lévy processes. Corollary 1 directly follows from the regularity of a white noise process (Theorem 1) and from the definition of a γ-admissible operator.

Corollary 1 *We consider a stochastic process $s = \mathrm{L}^{-1}w$, where w is a Lévy white noise with Blumenthal–Getoor indices $0 \leq \beta' \leq \beta \leq 2$ and L is a γ-admissible operator for some $\gamma \geq 0$.*

- *If w is a Gaussian white noise, then*

$$s \in \dot{B}^{\tau}_{p,q}(\mathbb{T}^d) \text{ a.s. if } \tau < \gamma - d/2, \text{ and} \tag{31}$$

$$s \notin \dot{B}^{\tau}_{p,q}(\mathbb{T}^d) \text{ a.s. if } \tau > \gamma - d/2. \tag{32}$$

- *If w is non-Gaussian, then*

$$s \in \dot{B}^{\tau}_{p,q}(\mathbb{T}^d) \text{ a.s. if } \tau < \gamma + d\left(\frac{1}{\max(p,\beta)} - 1\right), \text{ and} \tag{33}$$

$$s \notin \dot{B}^{\tau}_{p,q}(\mathbb{T}^d) \text{ a.s. if } \tau > \gamma + d\left(\frac{1}{\max(p,\beta')} - 1\right). \tag{34}$$

4 The *n*-term Approximation and the Compressibility of Generalized Lévy Processes

In Sect. 3, we found an upper bound for the Besov regularity of a generalized Lévy process. We are now interested in using that result to determine its n-term approximation with respect to Daubechies wavelet bases. We begin by defining the wavelets. Then, we recall the wavelet characterization of Besov spaces.

Author's personal copy

4.1 Periodic Daubechies Wavelets

Here, mainly following [45], we introduce the family of Daubechies wavelets on the d-dimensional torus. We also give a wavelet-based characterization of homogeneous Besov spaces.

Periodizing the compactly supported Daubechies wavelets [11] results in the orthonormal basis of $L_2(\mathbb{T}^d)$ given by

$$\left\{ \Psi_{G,\boldsymbol{m}}^{j,k} \;\middle|\; j \in \mathbb{Z}_{\geq 0},\, G \in G^j,\, \boldsymbol{m} \in \mathbb{P}_j^d \right\}, \tag{35}$$

where

$$\Psi_{G,\boldsymbol{m}}^{j,k} = 2^{jd/2} \Psi_{G,0}^{0,k}(2^j \cdot - \boldsymbol{m}). \tag{36}$$

The index $j \in \mathbb{Z}_{\geq 0}$ corresponds to a scaling parameter, and G is used to denote gender. The coarsest scale is $j = 0$, which includes the scaling functions, so G^0 has 2^d elements and G^j has $(2^d - 1)$ elements for $j > 0$. The parameter k denotes the smoothness of the wavelet and determines its support. For $k > 0$, the classical Daubechies mother wavelet on the real line has support greater than one. Here, we require the support of the wavelets to be a subset of the unit cube. Consequently, the coarsest scale is scaled by 2^{ζ}, where the parameter $\zeta \in \mathbb{N}$ ensures that this condition is satisfied. For the remainder of this paper, we set ζ (as a function of k) to be the smallest integer that guarantees this condition on the support. The wavelet translates are indexed by $\boldsymbol{m}$, and the set of translations at scale j is

$$\mathbb{P}_j^d = \left\{ \boldsymbol{m} \in \mathbb{Z}^d \;\middle|\; 0 \leq m_r < 2^{j+\zeta}, r = 1, \ldots, d \right\}. \tag{37}$$

More details on the periodization of wavelet bases can be found in [45, Section 1.3]. In fact, we nearly follow the notation of that book, except that our $\Psi_{G,\boldsymbol{m}}^{j,k}$ corresponds to $\Psi_{G,\boldsymbol{m}}^{j,\mathrm{per}}$ of [45, Proposition 1.34].

Definition 9 The notation $\Psi_{G_0,\mathbf{0}}^{0,k}$ denotes a Daubechies wavelet in $C^k(\mathbb{T}^d)$. Furthermore, the Lebesgue measure of the support of $\Psi_{G_0,\mathbf{0}}^{0,k}$ is less than one.

The wavelet decomposition of $f \in L_2(\mathbb{T}^d)$ is

$$f = \sum_{j,G,\boldsymbol{m}} \lambda_{\boldsymbol{m}}^{j,G}\, 2^{-(j+\zeta)d/2} \Psi_{G,\boldsymbol{m}}^{j,k}, \tag{38}$$

where the coefficients are computed as

$$\lambda_{\boldsymbol{m}}^{j,G} = \left\langle f, 2^{(j+\zeta)d/2} \Psi_{G,\boldsymbol{m}}^{j,k} \right\rangle. \tag{39}$$

We can now state the characterization of the homogeneous Besov spaces $\dot{B}_{p,q}^{\tau}(\mathbb{T}^d)$ that will be used in the remainder of this paper. Essentially, it says that the Besov regularity of a function is characterized by a norm on its wavelet coefficients.

Author's personal copy

Definition 10 (Definition 1.3.2, [45]) Let $\tau \in \mathbb{R}$ and $0 < p, q < \infty$. The Besov sequence space $b^{\tau}_{p,q}$ is the collection of sequences

$$\lambda = \left\{ \lambda^{j,G}_{\boldsymbol{m}} \mid j \in \mathbb{Z}_{\geq 0}, G \in G^j, \boldsymbol{m} \in \mathbb{P}_j \right\} \tag{40}$$

indexed as the periodized Daubechies wavelets, with finite (quasi-)norm

$$\|\lambda\|_{b^{\tau}_{p,q}} := \left(\sum_{j=0}^{\infty} 2^{j(\tau - d/p)q} \sum_{G \in G^j} \left(\sum_{\boldsymbol{m} \in \mathbb{P}^n_j} \left| \lambda^{j,G}_{\boldsymbol{m}} \right|^p \right)^{q/p} \right)^{1/q} . \tag{41}$$

If $p = \infty$ or $q = \infty$, there is an analogous definition.

In the special case $p = q$, we have

$$\|\lambda\|_{b^{\tau}_{p,p}} := \left(\sum_{j,G,\boldsymbol{m}} \left| 2^{j(\tau - d/p)} \lambda^{j,G}_{\boldsymbol{m}} \right|^p \right)^{1/p} , \tag{42}$$

which is a weighted ℓ_p space.

Proposition 3 *Suppose $f \in \dot{B}^{-k}_{2,2}(\mathbb{T}^d)$ (which is equivalent to the Sobolev space $\dot{H}^{-k}_2(\mathbb{T}^d)$) for some $k \in \mathbb{N}$, and let $0 < p, q \leq \infty$ and $\tau \in \mathbb{R}$ such that $k > \max\left(\tau, \sigma_p - \tau\right)$, where $\sigma_p = d(1/p - 1)_+$. Then, $f \in \dot{B}^{\tau}_{p,q}(\mathbb{T}^d)$ if and only if the wavelet coefficients*

$$\left\langle f, 2^{(j+\zeta)d/2} \Psi^{j,k+1}_{G,\boldsymbol{m}} \right\rangle \tag{43}$$

are in the Besov sequence space $b^{\tau}_{p,q}$.

Proof This follows from Theorem 1.36 of [41]. □

4.2 Besov Spaces and *n*-term Approximations

Summarizing the results of Sect. 4.1, we have the following: If f is in the Besov space $B^{\tau}_{p,q}(\mathbb{T}^d)$, then we can choose k large enough so that

$$f = \sum_{j,G,\boldsymbol{m}} \lambda^{j,G}_{\boldsymbol{m}} \, 2^{-(j+\zeta)d/2} \Psi^{j,k}_{G,\boldsymbol{m}}, \tag{44}$$

where the coefficients $\lambda^{j,G}_{\boldsymbol{m}}$ are computed by (39). We are now going to determine the error in approximating f by truncating the sum (44). In order to accomplish this, we introduce the notation $\mathcal{I} = \mathbb{Z}_{\geq 0} \times G^j \times \mathbb{P}^d_j$ to represent the collection of triples $(j, G, \boldsymbol{m})$.

Author's personal copy

Definition 11 For $n \geq 1$, an n-term approximation to a function f is a finite sum of the form

$$\sum_{(j,G,\boldsymbol{m})\in\mathcal{I}'} \lambda_{\boldsymbol{m}}^{j,G}\, 2^{-(j+\zeta)d/2}\Psi_{G,\boldsymbol{m}}^{j,k}, \tag{45}$$

where $\mathcal{I}' \subset \mathcal{I}$ and $\#\mathcal{I}' = n$.

Definition 12 Let $\Sigma_{n,p,\tau}(f)$ be a best n-term approximation to f in $\dot{B}^{\tau}_{p,p}(\mathbb{T}^d)$, in other words, an n-term approximation that minimizes the approximation error in $\dot{B}^{\tau}_{p,p}(\mathbb{T}^d)$. Also, let $\sigma_{n,p,\tau}(f)$ denote the error of the approximation, with

$$\sigma_{n,p,\tau}(f) = \left\| f - \Sigma_{n,p,\tau}(f) \right\|_{\dot{B}^{\tau}_{p,p}(\mathbb{T}^d)}. \tag{46}$$

Theorem 3 *Suppose* $0 < p_0 < \infty$ *and* $\tau_0 \in \mathbb{R}$.

(i) *If* $f \in \dot{B}^{\tau_0+\Delta\tau}_{p_1,p_1}(\mathbb{T}^d)$ *for some* $\Delta\tau > 0$ *and* p_1 *satisfies* (49)*, then there is a constant* $C > 0$ *such that*

$$\sigma_{n,p_0,\tau_0}(f) \leq C n^{-\Delta\tau/d} \left\| f \right\|_{\dot{B}^{\tau_0+\Delta\tau}_{p_1,p_1}(\mathbb{T}^d)}. \tag{47}$$

(ii) *If there are constants* $C, \Delta\tau, \epsilon > 0$ *such that*

$$\sigma_{n,p_0,\tau_0}(f) \leq C n^{-\Delta\tau/d-\epsilon}, \tag{48}$$

then $f \in \dot{B}^{\tau_0+\Delta\tau}_{p_1,p_1}(\mathbb{T}^d)$*, where*

$$\frac{1}{p_1} = \frac{\Delta\tau}{d} + \frac{1}{p_0}. \tag{49}$$

Proof This proof uses Corollary 6.2 of [22], which characterizes n-term approximation spaces as Besov spaces. In particular,

$$b^{\tau_0+\Delta\tau}_{p_1,p_1} = A^{\Delta\tau/d}_{p_1}(b^{\tau_0}_{p_0,p_0}), \tag{50}$$

where $b^{\tau_0+\Delta\tau}_{p_1,p_1}$ is a Besov sequence space, and $A^{\Delta\tau/d}_{p_1}(b^{\tau_0}_{p_0,p_0})$ is an approximation space with error measured in $b^{\tau_0}_{p_0,p_0}$. Essentially, $A^{\Delta\tau/d}_{p_1}(b^{\tau_0}_{p_0,p_0})$ is the collection of sequences f for which the sequence of error terms

$$n^{\Delta\tau/d}\sigma_{n,p_0,\tau_0}(f) \tag{51}$$

is in ℓ_{p_1} with respect to a Haar-type measure on $\mathbb{N}$.

Using this characterization along with standard embedding properties of approximation spaces [13, Chapter 7], we derive our result. In particular, (50) together with the aforementioned embedding implies that

$$b^{\tau_0+\Delta\tau}_{p_1,p_1} \subset A^{\Delta\tau/d}_{\infty}(b^{\tau_0}_{p_0,p_0}). \tag{52}$$

Author's personal copy

Similarly, we have that

$$A_{\infty}^{\Delta\tau/d+\epsilon}(b_{p_0,p_0}^{\tau_0}) \subset b_{p_1,p_1}^{\tau_0+\Delta\tau}. \tag{53}$$

The fact that the continuous-domain Besov spaces are isomorphic to Besov sequence spaces (Proposition 3) completes the proof. □

4.3 The Compressibility of Generalized Lévy Processes

The compressibility of a function quantifies the speed of convergence of its n-term approximation in a wavelet basis.

Definition 13 For a generalized function $f \in \dot{B}_{p_0,p_0}^{\tau_0}(\mathbb{T}^d)$, we define its (p_0, τ_0)-*compressibility* as

$$\kappa_{p_0,\tau_0}(f) = \sup\left\{\lambda \geq 0 \,\middle|\, \sup_{n\in\mathbb{Z}_{\geq 0}} (n+1)^{\lambda} \left\| f - \Sigma_{n,p_0,\tau_0}(f)\right\|_{\dot{B}_{p_0,p_0}^{\tau_0}(\mathbb{T}^d)} < \infty\right\}. \tag{54}$$

The quantity $\kappa_{p_0,\tau_0}(f) \in [0,\infty]$ is well defined and nonnegative for $f \in \dot{B}_{p_0,p_0}^{\tau_0}(\mathbb{T}^d)$. The higher $\kappa_{p_0,\tau_0}(f)$ is, the more f is compressible. If the approximation error has a faster-than-algebraic decay, then $\kappa_{\tau_0,p_0}(f) = +\infty$.

Remark 2 One can interpret compressibility in terms of approximation spaces. The sequence $E_n(f) := \left\| f - \Sigma_{n,p_0,\tau_0}(f)\right\|_{\dot{B}_{p_0,p_0}^{\tau_0}(\mathbb{T}^d)}$ specifies how well f can be approximated by wavelets. Weighted norms of this sequence are commonly used to characterize approximation properties, and such norms specify approximation spaces A_q^{λ}. For example

$$\|f\|_{A_q^{\lambda}} := \begin{cases} \left(\sum_{n=1}^{\infty} \left(n^{\lambda} E_{n-1}(f)\right)^q \frac{1}{n}\right)^{1/q}, & 0 < q < \infty \\ \sup\left(n^{\lambda} E_{n-1}(f)\right), & q = \infty. \end{cases} \tag{55}$$

Therefore, $\kappa_{p_0,\tau_0}(f)$ is the largest λ such that $f \in A_{\infty}^{\lambda}$.

We first give a general result on the n-term approximation of a sparse process $s = \mathrm{L}^{-1}w$ in $\dot{B}_{p_0,p_0}^{\tau_0}(\mathbb{T}^d)$.

Proposition 4 *We consider a stochastic process $s = \mathrm{L}^{-1}w$ with Blumenthal–Getoor indices $0 \leq \beta' \leq \beta \leq 2$ and order $\gamma \geq 0$. If $\tau_0 \in \mathbb{R}$ and $0 < p_0 \leq \infty$ satisfy the relation*

$$\gamma > \tau_0 + d - \frac{d}{\max(p_0, \beta)}, \tag{56}$$

then we have the order relation

$$\sigma_{n,p_0,\tau_0}(s) \leq C n^{\gamma/d+1/\beta-1} \|s\|_{\dot{B}_{p_1,p_1}^{\tau_1}(\mathbb{T}^d)} \tag{57}$$

Author's personal copy

almost surely, where τ_1 *and* p_1 *are set as*

$$\tau_1 - \tau_0 = \gamma + \frac{d}{\beta} - d = \frac{d}{p_1} - \frac{d}{p_0} \tag{58}$$

and C *is a (deterministic) constant.*

Proof Due to Corollary 1, Condition (56) ensures that $s \in \dot{B}^{\tau_0}_{p_0,p_0}(\mathbb{T}^d)$. If $s \notin \dot{B}^{\tau_1}_{p_1,p_1}(\mathbb{T}^d)$, then $\|s\|_{\dot{B}^{\tau_1}_{p_1,p_1}(\mathbb{T}^d)} = \infty$ so that (57) is obvious. Otherwise, we apply Part i) of Theorem 3 with $\Delta\tau = (\tau_1 - \tau_0) = (\gamma/d + 1/\beta - 1)$ to deduce the result. □

In particular, we deduce from Proposition 4 that, under Condition (56), one has almost surely that

$$\sigma_{n,p_0,\tau_0}(s) \underset{n\to\infty}{\longrightarrow} 0. \tag{59}$$

Based on our preliminary work, we now obtain new results on the compressibility of processes s that are solutions of (1). We split the results into two cases: the Gaussian processes and the sparse processes.

Theorem 4 (Compressibility of Gaussian processes) *Let* $s = \mathrm{L}^{-1} w_{\mathrm{G}}$ *be a Gaussian process of order* γ. *For* $0 < p_0 \leq \infty$ *and if* $\tau_0 \in \mathbb{R}$ *satisfies*

$$\gamma > \tau_0 + \frac{d}{2}, \tag{60}$$

then we have, almost surely, that

$$\kappa_{p_0,\tau_0}(s) = \frac{\gamma - \tau_0}{d} - \frac{1}{2}. \tag{61}$$

Proof Fix $0 < \epsilon < (\gamma - d/2 - \tau_0)$. Then, according to Corollary 1, $s \in B^{\tau_1}_{p_1,p_1}(\mathbb{T}^d)$ with

$$\tau_1 = \gamma - \frac{d}{2} - \epsilon \text{ and } \tau_1 - \tau_0 = d\left(\frac{1}{p_1} - \frac{1}{p_0}\right). \tag{62}$$

Then, Proposition 4 implies that

$$\sigma_{n,p_0,\tau_0}(s) \leq C n^{\frac{1}{p_1} - \frac{1}{p_0}} \|s\|_{\dot{B}^{\tau_1}_{p_1,p_1}(\mathbb{T}^d)} \tag{63}$$

with $C > 0$ a constant. This shows that

$$\kappa_{p_0,\tau_0} \geq \frac{1}{p_1} - \frac{1}{p_0} = \frac{\gamma - \tau_0}{d} - \frac{1}{2} - \frac{\epsilon}{d}. \tag{64}$$

This is valid for ϵ arbitrarily small, implying that $\kappa_{p_0,\tau_0}(s) \geq ((\gamma - \tau_0)/d - 1/2)$.

We also know from (32) that s is not in $\dot{B}^{\tau_1}_{p_1,p_1}(\mathbb{T}^d)$ for $\tau_1 \geq (\gamma - d/2)$ and p_1 given by $(\tau_1 - \tau_0) = d\,(1/p_1 - 1/p_0)$. The converse of Theorem 3, Part ii), implies that there exists no constant $C > 0$ such that $\sigma_{n,p_0,\tau_0}(s) \leq C n^{-(\gamma - \tau_0 - d/2)}$. Therefore, $\kappa_{p_0,\tau_0}(s) \leq ((\gamma - \tau_0)/d - 1/2)$, which finishes the proof. □

Author's personal copy

Theorem 5 (Compressibility of sparse processes) *Let $s = \mathrm{L}^{-1} w$ be a sparse process of order γ and Blumenthal–Getoor indices $0 \leq \beta' \leq \beta \leq 2$. We assume that $\tau_0 \in \mathbb{R}$ and $0 < p_0 \leq \infty$ satisfy*

$$\gamma > \tau_0 + d - \frac{d}{p_0}. \tag{65}$$

- *If $\beta = 0$, then, almost surely,*

$$\kappa_{p_0,\tau_0}(s) = +\infty. \tag{66}$$

- *If $\beta > 0$, then, almost surely,*

$$\frac{\gamma - \tau_0}{d} + \frac{1}{\beta} - 1 \leq \kappa_{p_0,\tau_0}(s) \leq \frac{\gamma - \tau_0}{d} + \frac{1}{\beta'} - 1 \tag{67}$$

Proof We first assume that $\beta > 0$. We proceed as in the Gaussian case. Condition (65) allows us to consider $0 < \epsilon < (\gamma - \tau_0) + d(1/p_0 - 1)$ (and also implies that $s \in \dot{B}^{\tau_0}_{p_0,p_0}(\mathbb{T}^d)$, due to Corollary 1). We set τ_1, p_1 such that

$$\tau_1 = \gamma + \frac{d}{\beta} - d - \epsilon \quad \text{and} \quad \frac{d}{p_1} - \frac{d}{p_0} = \gamma + \frac{d}{\beta} - d - \tau_0 - \epsilon. \tag{68}$$

We claim that $s \in \dot{B}^{\tau_1}_{p_1,p_1}(\mathbb{T}^d)$. Based on Corollary 1, this is true if

$$\tau_1 < \gamma + \frac{d}{\max(p_1, \beta)} - d. \tag{69}$$

Then, we remark that

$$\frac{1}{p_1} = \frac{1}{\beta} + \left(\gamma - \tau_0 + \frac{d}{p_0} - d - \epsilon\right) > \frac{1}{\beta}, \tag{70}$$

so that $p_1 \leq \beta$. Moreover,

$$\tau_1 = \gamma + \frac{d}{\beta} - d - \epsilon < \gamma + \frac{d}{\beta} - d = \gamma + \frac{d}{\max\{p_1, \beta\}} - d, \tag{71}$$

as expected. Due to Theorem 3, Part (i), one deduces that

$$\kappa_{p_0,\tau_0}(s) \geq \frac{\gamma - \tau_0}{d} + \frac{1}{\beta} - 1 - \frac{\epsilon}{d} \tag{72}$$

for ϵ arbitrarily small and, therefore, the lower bound of (67) is obtained.

For the upper bound, we proceed as for the Gaussian case. For $\tau_1 > \gamma + d/\beta' - d$ and p_1 satisfying $\tau_1 - \tau_0 = d(1/p_1 - 1/p_0)$, we know from Corollary 1 that s is almost surely not in $\dot{B}^{\tau_1}_{p_1,p_1}(\mathbb{T}^d)$. Therefore, Theorem 3, Part ii) implies that there is

Author's personal copy

no constant $C > 0$ such that $\sigma_{n,p_0,\tau_0}(s) \leq Cn^{-((\gamma-\tau_0)/d+(1/\beta'-1)}$. Thus. $\kappa_{p_0,\tau}(s) \leq ((\gamma-\tau_0)/d + 1/\beta' - 1)$, concluding the proof in this case.

For $\beta = 0$, we know that $s \in \dot{B}^{\tau}_{p,p}(\mathbb{T}^d)$ when $\tau < (\gamma + d/p - d)$ (Corollary 1). For $p_1 < p_0$, we define $\tau_1 = (\tau_0 + d/p_1 - d/p_0)$. Then, using (65), we have that

$$\tau_1 = \left(\tau_0 - \frac{d}{p_0}\right) + \frac{d}{p_1} < \gamma + \frac{d}{p_1} - d \tag{73}$$

and, therefore, $s \in \dot{B}^{\tau_1}_{p_1,p_1}(\mathbb{T}^d)$. With Theorem 3, we deduce that $\kappa_{p_0,\tau_0}(s) \geq (1/p_1 - 1/p_0)$ for any p_1 arbitrarily small. Finally, this means that $\kappa_{p_0,\tau_0}(s) = +\infty$ almost surely. □

Note that Theorem 4 gives the exact value of $\kappa_{p_0,\tau_0}(s)$ in the Gaussian case, while we only provide lower and upper bounds for sparse processes in Theorem 5 in general. It is important to remark however that the compressibility is known as soon as $\beta = \beta'$, in particular when $\beta = 0$.

5 Discussion and Examples

In this section, we focus on the L_2-compressibility that is obtained for $p_0 = 2$ and $\tau_0 = 0$. This case is of special interest, in particular for signal-processing applications: the quantity $\kappa(f) = \kappa_{0,2}(f)$ measures the approximation error in the L_2-sense. We therefore reformulate Theorems 4 and 5 for this case.

Corollary 2 (L_2-compressibility of Gaussian and sparse processes) *Let $s = \mathrm{L}^{-1}w$ be a process of order $\gamma > d/2$ and of Blumenthal–Getoor indices $0 \leq \beta' \leq \beta \leq 2$.*

- *If $w = w_{\mathrm{G}}$ is a Gaussian white noise, then, almost surely,*

$$\kappa(s) = \frac{\gamma}{d} - \frac{1}{2}. \tag{74}$$

- *If w is sparse with $\beta > 0$, then, almost surely,*

$$\kappa(s) = +\infty \tag{75}$$

- *If w is sparse with $\beta > 0$, then, almost surely,*

$$\frac{\gamma}{d} + \frac{1}{\beta} - 1 \leq \kappa(s) \leq \frac{\gamma}{d} + \frac{1}{\beta'} - 1. \tag{76}$$

We introduce some classical families of Lévy white noises with their indices in Table 1. Precise definitions can be found in the proposed references. The compressibility of the associated γ-admissible processes is deduced from Corollary 2. Note that $\beta = \beta'$ in all the considered examples; hence, the compressibility is completely determined by (76).

We finish with some important remarks.

Author's personal copy

Table 1 Compressibility of Gaussian and sparse processes of order $\gamma > d/2$ based on specific Lévy white noises

White noise w	Parameter	$\psi(\xi)$	β	Compressibility
Gaussian	$\sigma^2 > 0$	$-\sigma^2\xi^2/2$	2	$\gamma - \frac{d}{2}$
Cauchy [36]	–	$-\lvert\xi\rvert$	1	γ
SαS [36]	$\alpha \in (0, 2)$	$-\lvert\xi\rvert^\alpha$	α	$\gamma + d/\alpha - d$
Compound Poisson [46]	$\lambda > 0, \mathbb{P}_{\mathrm{J}}$	$\exp(\lambda(\widehat{\mathbb{P}}_{\mathrm{J}}(\xi) - 1))$	0	∞
Laplace [32]	–	$-\log(1 + \xi^2)$	0	∞
Inverse Gaussian [3]	–	–	1/2	$\gamma + d$

- When $\beta > 0$, we have only obtained upper and lower bounds for the compressibility of a generalized Lévy process. It would be interesting to investigate the case when $\beta' < \beta$, for which the compressibility is unknown.
- In the Gaussian case, one has the exact value of $\kappa(s)$. This is due to the fact that the converse result for the Besov regularity of Gaussian white noises are known [48]. A fundamental consequence is the following: If $s = \mathrm{L}^{-1}w$ is a sparse process of order γ and $s_{\mathrm{G}} = \mathrm{L}^{-1}w_{\mathrm{G}}$ is a Gaussian process corresponding to the same operator L, then we have almost surely that

$$\kappa(s) \geq \kappa(s_{\mathrm{G}}). \tag{77}$$

  Simply stated, sparse processes are more compressible than Gaussian processes. This finally gives a functional justification for the terminology of *sparse* processes introduced in [47].
- We assume now that $\beta = \beta'$; hence, the compressibility is fully characterized. For a fixed γ, the smaller β is, the more compressible is the process. At the limit, for $\beta = 0$, the error decays faster than any polynomial. This is achieved by compound Poisson processes and also by Laplace processes, that are therefore the sparsest processes. These results are coherent with recent works of Ghourchian et al. [25] in their recent works on the entropy of sparse processes. Here, the authors have shown that, among SαS and compound Poisson white noises, the less sparse is the Gaussian white noise, and the sparsest are the compound Poisson white noises. Even though their notion of sparsity is different, we believe that their results have a strong connection with ours.
- For fixed $\beta = \beta' > 0$, the compressibility of the process increases with γ. This is a very reasonable outcome as it is well known that smoother functions are more compressible.

Acknowledgements Funding was provided by the European Research Council (Grant No. 692726 - GlobalBioIm).

References

1. Amini, A., Unser, M.: Sparsity and infinite divisibility. IEEE Trans. Inf. Theory **60**(4), 2346–2358 (2014)

Author's personal copy

2. Aziznejad, S., Fageot, J., Unser, M.: Wavelet analysis of the Besov regularity of Lévy white noises. arXiv preprint arXiv:1801.09245
3. Barndorff-Nielsen, O.E.: Processes of normal inverse Gaussian type. Finance Stoch. **2**(1), 41–68 (1997)
4. Bényi, Á., Oh, T.: Modulation spaces, Wiener amalgam spaces, and Brownian motions. Adv. Math. **228**(5), 2943–2981 (2011)
5. Biermé, H., Durieu, O., Wang, Y.: Generalized random fields and Lévy's continuity theorem on the space of tempered distributions. arXiv preprint arXiv:1706.09326 (2017)
6. Blumenthal, R.M., Getoor, R.K.: Sample functions of stochastic processes with stationary independent increments. J. Math. Mech. **10**, 493–516 (1961)
7. Bostan, E., Kamilov, U.S., Nilchian, M., Unser, M.: Sparse stochastic processes and discretization of linear inverse problems. IEEE Trans. Image Process. **22**(7), 2699–2710 (2013)
8. Böttcher, B., Schilling, R., Wang, J.: Lévy Matters III: Lévy-Type Processes: Construction, Approximation and Sample Path Properties, vol. 2099. Springer, New York (2014)
9. Ciesielski, Z., Kerkyacharian, G., Roynette, B.: Quelques espaces fonctionnels associés à des processus gaussiens. Stud. Math. **107**(2), 171–204 (1993)
10. Cohen, A., DeVore, R.A., Hochmuth, R.: Restricted nonlinear approximation. Constr. Approx. **16**(1), 85–113 (2000)
11. Daubechies, I.: Ten Lectures on Wavelets, vol. 61. SIAM, Philadelphia (1992)
12. Davis, R.A., Knight, K., Liu, J.: M-estimation for autoregressions with infinite variance. Stoch. Process. Appl. **40**(1), 145–180 (1992)
13. DeVore, R.A., Lorentz, G.G.: Constructive Approximation. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 303. Springer, Berlin (1993)
14. Durand, A., Jaffard, S.: Multifractal analysis of Lévy fields. Probab. Theory Relat. Fields **153**(1–2), 45–96 (2012)
15. Elad, M.: Sparse and Redundant Representations: From Theory to Applications in Signal and Image Processing. Springer, New York (2010)
16. Fageot, J., Amini, A., Unser, M.: On the continuity of characteristic functionals and sparse stochastic modeling. J. Fourier Anal. Appl. **20**, 1179–1211 (2014)
17. Fageot, J., Fallah, A., Unser, M.: Multidimensional Lévy white noise in weighted Besov spaces. Stoch. Process. Appl. **127**(5), 1599–1621 (2017)
18. Fageot, J., Unser, M.: Scaling limits of solutions of linear stochastic differential equations driven by Lévy white noises. J. Theor. Probab. (2018). https://doi.org/10.1007/s10959-018-0809-1
19. Fageot, J., Unser, M., Ward, J.P.: On the Besov regularity of periodic Lévy noises. Appl. Comput. Harmonic Anal. **42**(1), 21–36 (2017)
20. Farkas, W., Jacob, N., Schilling, R.L.: Function Spaces Related to Continuous Negative Definite Functions: ψ-Bessel Potential Spaces. Polska Akademia Nauk, Instytut Matematyczny, Warszawa (2001)
21. Fernique, X.: Processus linéaires, processus généralisés. Ann. l'Inst. Fourier **17**, 1–92 (1967)
22. Garrigós, G., Hernández, E.: Sharp Jackson and Bernstein inequalities for N-term approximation in sequence spaces with applications. Indiana Univ. Math. J. **53**(6), 1739–1762 (2004)
23. Gelfand, I.M.: Generalized random processes. Dokl. Akad. Nauk SSSR **100**, 853–856 (1955)
24. Gelfand, I.M., Vilenkin, N.Ya.: Generalized Functions, vol. 4. Academic Press [Harcourt Brace Jovanovich, Publishers], New York, London (1964) [1977]. Applications of Harmonic Analysis, Translated from the Russian by Amiel Feinstein (1964)
25. Ghourchian, H., Amini, A., Gohari, A.: How compressible are innovation processes? IEEE Trans. Inf. Theory **64**(7), 4843–4871 (2018). https://doi.org/10.1109/TIT.2018.2822660
26. Herren, V.: Lévy-type processes and Besov spaces. Potential Anal. **7**(3), 689–704 (1997)
27. Itô, K.: Stationary random distributions. Kyoto J. Math. **28**(3), 209–223 (1954)
28. Itô, K.: Foundations of Stochastic Differential Equations in Infinite-Dimensional Spaces. CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 47. SIAM, Philadelphia (1984)
29. Jaffard, S.: The multifractal nature of Lévy processes. Probab. Theory Relat. Fields **114**(2), 207–227 (1999)
30. Kabanava, M.: Tempered Radon measures. Rev. Mat. Complut. **21**(2), 553–564 (2008)
31. Kidmose, P.: Alpha-stable distributions in signal processing of audio signals. In: 41st Conference on Simulation and Modelling, pp. 87–94 (2000)
32. Koltz, S., Kozubowski, T.J., Podgorski, K.: The Laplace Distribution and Generalizations. Birkhauser, Boston (2001)

Author's personal copy

33. Nikias, C.L., Shao, M.: Signal Processing with Alpha-Stable Distributions and Applications. Wiley, New York (1995)
34. Rabiner, L.: A tutorial on hidden Markov models and selected applications in speech recognition. Proc. IEEE **77**(2), 257–286 (1989)
35. Roynette, B.: Mouvement brownien et espaces de Besov. Stoch. Int. J. Probab. Stoch. Process. **43**(3–4), 221–260 (1993)
36. Samoradnitsky, G., Taqqu, M .S.: Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance, vol. 1. CRC Press, Boca Raton (1994)
37. Sato, K.: Lévy Processes and Infinitely Divisible Distributions. Cambridge Studies in Advanced Mathematics, vol. 68. Cambridge University Press, Cambridge (2013)
38. Schilling, R.L.: On Feller processes with sample paths in Besov spaces. Math. Ann. **309**(4), 663–675 (1997)
39. Schilling, R.L.: Growth and Hölder conditions for the sample paths of Feller processes. Probab. Theory Relat. Fields **112**(4), 565–611 (1998)
40. Schilling, R.L.: Function spaces as path spaces of Feller processes. Math. Nachr. **217**(1), 147–174 (2000)
41. Schmeisser, H.-J., Triebel, H.: Topics in Fourier Analysis and Function Spaces. A Wiley-Interscience Publication. Wiley, Chichester (1987)
42. Shao, M., Nikias, C.L.: Signal processing with fractional lower order moments: stable processes and their applications. Proc. IEEE **81**(7), 986–1010 (1993)
43. Simon, B.: Distributions and their Hermite expansions. J. Math. Phys. **12**(1), 140–148 (2003)
44. Trèves, F.: Topological Vector Spaces, Distributions and Kernels. Academic Press, New York (1967)
45. Triebel, H.: Function Spaces and Wavelets on Domains, Volume 7 of EMS Tracts in Mathematics. European Mathematical Society (EMS), Zürich (2008)
46. Unser, M., Tafti, P.D.: Stochastic models for sparse and piecewise-smooth signals. IEEE Trans. Signal Process. **59**(3), 989–1006 (2011)
47. Unser, M., Tafti, P.D.: An Introduction to Sparse Stochastic Processes. Cambridge Univerity Press, Cambridge (2014)
48. Veraar, M.: Regularity of Gaussian white noise on the d-dimensional torus. arXiv preprint arXiv:1010.6219 (2010)
49. Ward, J.P., Fageot, J., Unser, M.: Compressibility of symmetric-α-stable processes. In: Proceedings of the Eleventh International Workshop on Sampling Theory and Applications (SampTA'15), Washington (2015)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.