Asymptotic properties of least squares spline filters and application to multi-scale decomposition of signals.

AKRAM ALDROUBI, MICHAEL UNSER, and MURRAY EDEN

Biomedical Engineering and Instrumentation Program
National Institutes of Health

ABSTRACT

We use B-spline functions to define a family of sequence-spaces $S^n_m$ included in the finite energy and discrete space $L_2$. We derive invariant filters that operate on finite energy signals to output their least square approximations in $S^n_m$. We obtain results on the convergence of the various filters to the ideal discrete lowpass filter providing the link with Shannon’s sampling theorem. As an application, we derive pyramidal representations of signals that can be implemented with fast algorithms and compare these representations with the Gaussian/Laplacian pyramid which is widely used in signal processing and computer vision.

INTRODUCTION

Typical application of B-spline functions in signal processing have been in magnification or minification [1], image coding and reconstruction [2]. Most of the early work dealt with exact interpolation problems where the interpolant agrees precisely with the signal samples. Less attention has been devoted to non-exact interpolation techniques which are relevant to noisy signals and the problem of under-sampling for minimum error data compression. An exception to this is the work by R. Hummel who constructs a general theory about sampling, reconstruction, and optimal filtering that is based on least square approximation [3]. More recently, Mallat [4], used polynomial spline functions to construct some examples of multi-resolution pyramids using a wavelet representation.

BEST APPROXIMATION IN THE SPACES $S^n_m$

We define discrete spaces $S^n_m$ included in the space of square summable discrete sequences $L_2$:

$$S^n_m := \left\{ v: v(k) = \sum_{i=-\infty}^{\infty} c(i)b^n_m(k-\ell i), \forall k \in \mathbb{Z}, c \in L_2 \right\},$$

where $b^n_m(k) := \beta^n(k/m)$ and where $\beta^n(x)$ is the B-spline function of order $n$ [5]. The signal $v(k)$ is a sampled polynomial spline of order $n$ with a knot spacing of $m$. Thus, $n$ represent a smoothness constraint while $m$ is a scale index representing the coarseness of the signals in $S^n_m$. Moreover, the spaces $S^n_m$ are closed sub-spaces of $L_2=S^0_0$ and have the property that for $n$ odd, if $m_1=m_2=k$ (positive integer) then $S^n_{m_1} \rightarrow S^n_{m_2}$. This property, however, does not hold for $n$ even.

The least square approximation of a signal $s(k)$ in $S^n_m$ is obtained by filtering as described in the following theorem.

Theorem 1. The least square approximation in $S^n_m$ of a signal $s(k)$ is given by:

$$y^* = h^n_m \ast [x^*_m]_{\odot m}$$

$$x^*_m = [h^n_m \ast s]_{\odot m}$$

where $h^n_m$ is a discrete spline interpolator with an expansion factor $m$ (i.e. $\gamma_m = \lceil \frac{n}{2} \rceil m$) [6] and where $h^n_m$ is the optimal pre-filter needed before the discrete cardinal-spline interpolator $h^n_m$. Their expressions are given by:

$$h^n_m := b^n_m * [(c^{-1}_m)]_{\odot m}$$

$$h^n_m := [(b^n_m * b^n_m)]_{\odot m}$$

In effect, the approximation is obtained using a pre-filtering followed by an under-sampling and then an interpolation as illustrated in Fig. 1. The proof of the theorem is given at the end of this paper.

CONVERGENCE PROPERTIES

The convergence properties of the pre-filter $h^n_m$ and the interpolator $h^n_m$ which are stated in the following theorems and corollaries provide the link between the discrete version
of the classical Whittaker-Kotel'nikov-Shannon sampling theorem [7] and the least square approximation in $S_{mn}$.

Theorem 2. The Fourier transforms of the pre-filter $\hat{h}_m^n$ converge pointwise a.e. to an ideal discrete lowpass filter as $n$ tends to infinity:

$$\lim_{n \to \infty} \hat{h}_m^n(f) = \text{Rect}_m(f) = \begin{cases} 1 & 0 < |f| < 0.5m \\ 0 & 0.5m < |f| < 1 \end{cases} \quad (6)$$

Moreover, $\hat{h}_m^n$ converge to Rect$_m(f)$ in $L_2(-0.5m, 0.5m)$ as $n$ goes to infinity.

Theorem 3. For $n$ odd, the Fourier transforms of the interpolators $H_n^m(f)$ converge pointwise a.e. to an ideal discrete lowpass filter with gain $m$ as $n$ tends to infinity:

$$\lim_{n \to \infty} H_n^m(f) = m \text{Rect}_m(f)$$

Moreover, $H_n^m$ converge to $m \text{Rect}_m(f)$ in $L_2(-0.5m, 0.5m)$ as $n$ goes to infinity.

Corollary 1. The impulse responses $\hat{h}_m^n$ converge in $L_2$ to the discrete sinc filter as $n$ tends to infinity.

Corollary 2. For $n$ odd, the interpolator $H_n^m$ converges in $L_2$ to the ideal sinc interpolator with gain $m$, as $n$ tends to infinity.

Figure 2 illustrates the convergence of the Fourier transforms of $\hat{h}_m^n$ and $H_n^m$ to the ideal discrete low-pass filters.

MULTI-SCALE REPRESENTATIONS

A multi-resolution-pyramid representation consists of several versions of the signal at different resolution levels in which the low resolution levels are described by fewer samples than the high resolution counterparts [4, 8]. They are commonly obtained by iteratively applying a filter and a down-sampler to produce the pyramid layers. As an application of our results, we derive the spline pyramid that minimizes the loss of information occurring when a discrete signal is approximated by a coarse resolution one. Using eqn. 1-5, we obtain the representations of a signal $s(k)$ with a factor of compression between two consecutive levels equal to $2 (m=2^j)$:

\begin{align*}
\hat{y}_{(j+1)} & = k_{2j}^n \hat{y}_{(j+1)} \\
y_{(j+1)} & = H_2^n \hat{y}_{(j+1)} \\
\bar{x}_{(j)} & = \sigma_{2j}^n \bar{x}_{(j)} \\

\hat{x}_0 & = \mathcal{S}
\end{align*}

where $k_{2j}^n$ and $\sigma_{2j}^n$ are convolution operators and are given by:

$$k_{2j}^n = (a_{2j+1})^{-1} * \mathcal{T}_2$$

$$\sigma_{2j}^n = (b_1)^{-1} * (b_2)^{-1} * \left[b_{2j}^n * b_{2j+1}^n\right]_{12}$$

and where $\mathcal{T}_m^n$ is given by (14). A drawback to this multi-resolution representation is that the filters $k_{2j}^n$ and $\sigma_{2j}^n$ of the first and third equation in (7) depend on the resolution level $j$. On the other hand, the second equation of (7) is independent of the resolution level and is precisely the one that defines the first pyramid level for the representation of the signal $\bar{x}_0$. This observation suggests the alternative step-wise optimal multi-resolution representation:

\begin{align*}
\hat{y}_{(j+1)} & = \hat{y}_{(j+1)} \\
y_{(j+1)} & = H_2^n \hat{y}_{(j+1)} \\
\bar{x}_{(j)} & = \mathcal{S}_j
\end{align*}

The question of how the step-wise optimal algorithm (10) compares with the optimal algorithm (7) is partially answered by the following theorem.

Theorem 4. The Fourier transforms of the filters $K_m^n$ and $O_m^n$ converge pointwise a.e. to a discrete pass-all filter as $n$ tends to infinity:

$$\lim_{n \to \infty} K_m^n(f) = 1 \quad \text{a.e. for } (-1/2, 1/2)$$

$$\lim_{n \to \infty} O_m^n(f) = 1 \quad \text{a.e. for } (-1/2, 1/2)$$

Moreover, $K_m^n$ and $O_m^n$ tend to 1 in $L_2(-1/2, +1/2)$ as $n$ goes to infinity.

AN EXPERIMENT

As an experiment, we use the "Women" image (Fig. 3) to compare the step-wise optimal spline pyramid (10) (SOSP) with the Laplacian pyramid (LP) of Burt and Adelson developed for compact image coding [8]. Fig. 3 shows the difference-image pyramid for the SOSP representations and the LP with the same intensity scaling to facilitate the
comparison. Each level in the difference-image pyramid consists of the difference between a level at a given level and its interpolated version at the next level. For this experiment we have chosen the values \( n=3 \) and \( j=1,2,3 \). Table I gives the signal to noise ratios (SNR) [9] associated with the full resolution approximation \( h_{2n}^n [s] \) together with the standard deviation or root mean square error (RMS), the entropy and the range of the difference-image. The SNRs for the representation obtained by the SOSP algorithm are better than the ones obtained by the Laplacian pyramid. As a matter of fact, the SNRs for the step-wise optimal representations at a given level (i) are comparable to the ones at level (i-1) for the LP representation. This improvement can be advantageously applied to progressive image transmission and compact image coding.

PROOF OF THEOREM 1

Since \( S_m^n \) is a closed subspace of the Hilbert space \( L_2 \), the least square approximation \( \xi \) is given by the orthogonal projection of \( s \) onto \( S_m^n \). Hence, the error \( \xi - s \) is orthogonal to \( S_m^n \). In particular, because of the definition (1), the error is orthogonal to \( b_m^n (k) \) and all of its shifted versions with shift factors that are integer multiples of \( m \):

\[
\left( (\xi - s)(k), b_m^n (k-\text{lm}) \right)_2 = 0, \quad \forall l \in \mathbb{Z}
\]

(11)

where \((.,.)_2\) denotes the usual \( L_2 \) inner product and where \( \xi(k) = \sum_{l=\infty}^{\infty} \xi(l) b_m^n (k-lm) \) is in \( S_m^n \).

Using the linearity property of the inner product and the fact that \( \xi \) is in \( S_m^n \) we rewrite (11) to get:

\[
\left( s, b_m^n (k-\text{lm}) \right)_2 =
\sum_{l=\infty}^{\infty} \xi(l) \left( b_m^n (k-\text{lm}) b_m^n (k-\text{lm}) \right)_2, \quad \forall l \in \mathbb{Z}
\]

(12)

Equation (12) can be expressed as the convolution equation:

\[
[s \ast b_m^n]_m(l) = \left( \xi * [b_m^n \ast b_m^n]_m \right)_m(l)
\]

(13)

The operator \( t_m^n(\cdot) \):

\[
t_m^n(l) := [b_m^n \ast b_m^n]_m(l)
\]

(14)

has finitely many non-zero values and defines a bounded linear operator from \( L_2 \) into itself. It can be shown that \((t_m^n)^{-1}(l)\) exists and decays exponentially fast as \( l \to \infty \) [10]. Thus, \((t_m^n)^{-1}(l)\) is absolutely summable and hence defines a bounded linear operator from \( L_2 \) into itself. Therefore, the operator \( t_m^n(\cdot) \) has the bounded inverse \((t_m^n)^{-1}\) that we use together with (13) to obtain the approximation \( \xi \):

\[
\xi = b_m^n \ast [\xi]_m
\]

\[
= b_m^n \ast \left[ (t_m^n)^{-1} \ast [s \ast b_m^n]_m \right]_m
\]

\[
= b_m^n \ast \left[ (t_m^n)^{-1} \ast [s \ast b_m^n]_m \right]_m.
\]

(15)

where in the last equality of (15), we have used the functional equality:

\[
as[b]_m \ast [s]_m = [a \ast [b]_m + [c]_m
\]

(16)

Using the functional equality (16), the fact that

\[
as[b \cdot c]_m = as[b]_m \ast [c]_m
\]

(17)

and the identity \((b_m^n, (b_m^n)^{-1})(k) = \delta(k)\) (where the existence of \((b_m^n)^{-1}\) follows from [10]), we manipulate (15) so as to exhibit a term that is a pure interpolator given by (4), we obtain:

\[
\xi = b_m^n \ast \left[ ((t_m^n)^{-1})_m \ast [s \ast b_m^n]_m \ast (b_m^n)^{-1} \right]_m
\]

\[
= b_m^n \ast \left[ ((t_m^n)^{-1})_m \ast [s \ast b_m^n]_m \right]_m.
\]

(16)

where \( \hat{h}_m^n \) is given by (5).

Similarly to Hummel [3], we can interpret \( \hat{h}_m^n \) to be the optimal pre-filter needed before the interpolator \( h_m^n \).

Fig. 1: Schematic representation of the operation for the least square approximation in \( S_m^n \).
Fig. 2: Fourier transforms of some least square spline filters. (A) Prefilters \( H_1^1(f) \) (----), \( H_2^2(f) \) (-----) and \( H_3^3(f) \) (Continuous line). (B) Interpolators \( H_2^0(f) \) (----), \( H_2^1(f) \) (-----) and \( H_2^2(f) \) (Continuous line).

Fig. 3: Error images between two consecutive levels of the SOSP pyramid and of the Laplacian pyramids for the "Women" image. (A1-A3): error/difference images of the Laplacian pyramid. (B1-B3): error/difference images of the SOSP pyramid.

<table>
<thead>
<tr>
<th>Pyramid level</th>
<th>Range</th>
<th>RMS</th>
<th>Entropy</th>
<th>SNR (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>LP-1</td>
<td>(-80, 85)</td>
<td>11.75</td>
<td>5.10</td>
<td>23.70</td>
</tr>
<tr>
<td>LP-2</td>
<td>(-69, 64)</td>
<td>12.27</td>
<td>5.39</td>
<td>19.64</td>
</tr>
<tr>
<td>LP-3</td>
<td>(-50, 52)</td>
<td>14.98</td>
<td>5.85</td>
<td>16.48</td>
</tr>
<tr>
<td>SOSP-1</td>
<td>(-57, 78)</td>
<td>6.66</td>
<td>4.35</td>
<td>28.63</td>
</tr>
<tr>
<td>SOSP-2</td>
<td>(-79, 76)</td>
<td>12.24</td>
<td>5.32</td>
<td>23.00</td>
</tr>
<tr>
<td>SOSP-3</td>
<td>(-73, 109)</td>
<td>16.21</td>
<td>5.90</td>
<td>19.50</td>
</tr>
</tbody>
</table>

Table 1: Comparison of performance measured at successive pyramid levels for the "Women" image.

REFERENCES


BIOGRAPHY

Akram Aldroubi (member of AMS '83) was born in Homs, Syria in 1958. He received the M.S. in Electrical Engineering From the Swiss Federal Institute of Technology in Lausanne, Switzerland in 1982, the M.S. and the Ph.D. in Mathematics in 1984 and 1987 respectively, from Carnegie-Mellon University. He is currently Staff Fellow at the Biomedical Engineering and Instrumentation Program, National Institutes of Health.

Michael Unser (M'89) was born in Zug, Switzerland in 1958. He received the M.S. (with honors) and the Ph.D. in Electrical Engineering in 1981 and 1984, respectively, from the Swiss Federal Institute of Technology in Lausanne, Switzerland.

He is currently a Visiting Scientist at the Biomedical Engineering and Instrumentation Program, National Institutes of Health, which he joined in 1985.

Murray Eden (M'60 - P'75) was born in Brooklyn New York in 1920. He received his B.S. degree from City College of New York in 1939 and a Ph.D. from the University of Maryland in 1951.

He is currently Chief, Biomedical Engineering and Instrumentation Branch, National Institutes of Health, and Professor of Electrical Engineering, Emeritus, Massachusetts Institute of Technology.