Consistent Discretization of Linear Inverse Problems using Sparse Stochastic Processes

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***Abstract*—We introduce a novel discretization paradigm and specify MAP estimators for linear inverse problems by using the theory of continuous-domain sparse stochastic processes. We characterize the complete class of admissible priors for the discretized version of the signal and show that the said class is restricted to the family of infinitely divisible distributions. We also explain the connections between our estimators and the existing deterministic methods such as Tikhonov and ℓ_p-type (with $p \in (0,1]$) regularizations.**

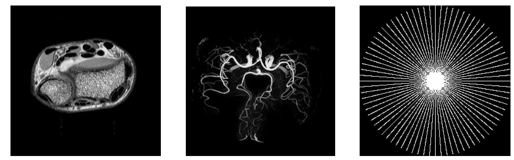
I. Introduction

Linear inverse problems (LIPs) occur in a variety of signal/image reconstruction applications and are usually solved by minimizing an energy functional. In the classical (Tikhonov-type) schemes, the energy is quadratic and one obtains a linear reconstruction. For this framework, there is a well-established connection with MAP and LMMSE estimators given that the original signal follows a multivariate Gaussian distribution (with a known covariance) and the measurement noise is additive white Gaussian [1]. As for the sparsity-promoting methods, they are often formulated as MAP estimators by considering generalized Gaussian or Laplace priors [2], [3]. These approaches, however, are solely discrete and linked to the choice of a given sparsifying transform.

II. Theoretical Contributions

We revisit the statistical formulation of the reconstruction problem by specifying a *continuous-domain signal model* that is independent from the subsequent reconstruction task. Our approach builds upon the theory of continuous-domain sparse stochastic processes [4] that defines the stochastic process s through a generalized innovation model $\mathrm{L}s = w$, where L is a differential operator and w is a continuous-domain white innovation process that is *not necessarily Gaussian*. We note that the continuous-domain formulation lends itself to an analytical treatment since it allows for the derivation of the probability density function (pdf) of the signal in any transform domain, which is much more challenging in a purely discrete framework.

In this presentation, we promote the use of continuous-domain stochastic models in the formulation of LIPs by developing a proper discretization scheme that offers the same type of error control as finite-elements. Our discretization allows to obtain a tractable representation of continuously-defined measurement problem, with minimal loss of information. This model leads to the characterization of the complete class of admissible priors (namely, infinitely divisible distributions) and to the derivation of the corresponding MAP estimators. We establish connections between our estimators and the existing deterministic methods such as Tikhonov and ℓ_p with $0 < p \leq 1$ regularizations (in particular TV-type). We also propose a general reconstruction algorithm, based on augmented Lagrangian, that handles different estimators, including the nonconvex ones.



(a) (b) (c)

Fig. 1. Data used in MR reconstruction experiment: (a) cross section of a wrist; (b) angiography image; (c) k-space sampling pattern along 40 radial lines.

	Estimation Performance (SNR in dB)		
	Gaussian	Laplace	Student's
Wrist (20 radial lines)	8.82	**11.8**	5.97
Wrist (40 radial lines)	11.30	**14.69**	13.81
Angiogram (20 radial lines)	4.30	9.01	**9.40**
Angiogram (40 radial lines)	6.31	14.48	**14.97**

III. Simulation

We consider the problem of reconstructing two MR images of size 256×256 from undersampled radial $\boldsymbol{k}$-space trajectories. The measurement functions are complex exponentials at some fixed frequencies and are defined as $\psi_m(\boldsymbol{x}) = \mathrm{e}^{2\pi \mathrm{j}\langle \boldsymbol{k}_m, \boldsymbol{x}\rangle}$ where $\boldsymbol{k}_m$ represents the sample point in $\boldsymbol{k}$-space. We use $\mathrm{sinc}(\cdot)$ for the discretization of the measurement model, which results in a system matrix $\mathbf{H}$ with the entries

$$\begin{aligned}[\mathbf{H}]_{m,\boldsymbol{k}} &= \langle \psi_m(\boldsymbol{x}), \mathrm{sinc}(\cdot - \boldsymbol{x}_{\boldsymbol{k}})\rangle \\ &= \mathrm{e}^{-\mathrm{j}2\pi\langle \boldsymbol{k}_m, \boldsymbol{x}_{\boldsymbol{k}}\rangle} \ \text{ if } \ |\boldsymbol{k}_m|_\infty \leq \tfrac{1}{2}.\end{aligned}$$

The effect of choosing a sinc function is that $\mathbf{H}$ reduces to the discrete version of complex Fourier exponentials. We choose the discrete version of L to be the gradient operator and compare different priors (with increasing sparsity) that are in the family.

We observe that the estimators based on Laplace priors yield the best reconstruction for the wrist image that has sharp edges and some amount of texture. The reconstructions using Student's priors are suboptimal because they are too sparse. On the other hand, Student's priors are suitable for reconstructing the angiogram, which is mostly composed of piecewise-smooth components.

References

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