Supplementary Material for
“Improved Variational Denoising of Flow Fields
with Application to Phase-Contrast MRI Data”

Proof of Proposition 1. Let us first remark that in Fourier domain

$$\mathcal{F}\{\mathbf{J} f\}(\omega) = \hat{f}_j \omega^\top,$$

where $\hat{f}$ is the Fourier transform of $f$. For the scaling invariance, we note
that the scaling operator $S_a : f \mapsto f(\cdot/a)$ commutes with the Jacobian (up
to a multiplicative constant) as it holds that

$$\mathcal{F}\{\mathbf{J}\{S_a f\}\}(\omega) = |a|^d \hat{f}(a\omega) j \omega^\top
= (1/a) \left( |a|^d \hat{f}(a\omega) j (a\omega)^\top \right)
= (1/a) \mathcal{F}\{S_a \{\mathbf{J} f\}\}(\omega).$$

Since the Schatten $p$-norms are 1-homogeneous functions, we obtain that

$$\text{TV}_p(S_a f) = \frac{1}{|a|} \int_{\mathbb{R}^d} \|\{\mathbf{J} f\}(\chi/a)\|_{\mathcal{S}_p} d\chi
= |a|^{d-1} \int_{\mathbb{R}^d} \|\mathbf{J} f(\chi)\|_{\mathcal{S}_p} d\chi
= |a|^{d-1} \text{TV}_p(f),$$

where the second equality follows from a simple change of variables.

As for rotation by a matrix $\xi$, we have ($R_\xi$ is the rotation operator)

$$\mathcal{F}\{\mathbf{J}\{R_\xi f\}\}(\omega) = \xi \hat{f}(\xi \omega) j \omega^\top
= \xi \hat{f}(\xi \omega) j \omega^\top \xi^\top \xi
= \xi \hat{f}(\xi \omega)(\xi \omega)^\top \xi
= \mathcal{F}\{R_\xi \{\mathbf{J} f\}\}(\omega).$$
Then, we write that

\[
TV_p(R\xi f) = \int_{\mathbb{R}^d} \| J\{R\xi f\}(x) \|_{S_p} \, dx
\]

\[
= \int_{\mathbb{R}^d} \left\| \xi^\top \{Jf\}(\xi x) \xi \right\|_{S_p} \, dx
\]

\[
= \int_{\mathbb{R}^d} \| Jf(\xi x) \|_{S_p} \, dx,
\]

since the Schatten norms are unitarily invariant\(^1\). Now, applying a change of variable \( u = \xi x \), with \( du = |\det \xi| dx = dx \), we arrive at the desired result:

\[
TV_p(R\xi f) = TV_p(f)
\]

Translation invariance is straightforward to show by using a change of variable.

\[\square\]