Supplementary Material for "Improved Variational Denoising of Flow Fields with Application to Phase-Contrast MRI Data"

Proof of Proposition 1. Let us first remark that in Fourier domain

$$\mathcal{F}\{\mathbf{J}\mathbf{f}\}(\boldsymbol{\omega}) = \hat{\mathbf{f}}\mathrm{j}\boldsymbol{\omega}^{\top},$$

where $\hat{\mathbf{f}}$ is the Fourier transform of $\mathbf{f}$. For the scaling invariance, we note that the scaling operator $\mathrm{S}_a : \mathbf{f} \mapsto \mathbf{f}(\cdot/a)$ commutes with the Jacobian (up to a multiplicative constant) as it holds that

$$\begin{aligned}\mathcal{F}\{\mathbf{J}\{\mathrm{S}_a\mathbf{f}\}\}(\boldsymbol{\omega}) &= |a|^d\hat{\mathbf{f}}(a\boldsymbol{\omega})\mathrm{j}\boldsymbol{\omega}^{\top}\\ &= (1/a)\left(|a|^d\hat{\mathbf{f}}(a\boldsymbol{\omega})\mathrm{j}(a\boldsymbol{\omega})^{\top}\right)\\ &= (1/a)\,\mathcal{F}\{\mathrm{S}_a\{\mathbf{J}\mathbf{f}\}\}(\boldsymbol{\omega}).\end{aligned}$$

Since the Schatten p-norms are 1-homogeneous functions, we obtain that

$$\begin{aligned}\mathrm{TV}_p(\mathrm{S}_a\mathbf{f}) &= \frac{1}{|a|}\int_{\mathbb{R}^d}\|\{\mathbf{J}\mathbf{f}\}(\mathbf{x}/a)\|_{\mathcal{S}_p}\,\mathrm{d}\mathbf{x}\\ &= |a|^{d-1}\int_{\mathbb{R}^d}\|\mathbf{J}\mathbf{f}(\mathbf{u})\|_{\mathcal{S}_p}\,\mathrm{d}\mathbf{u}\\ &= |a|^{d-1}\,\mathrm{TV}_p(\mathbf{f}),\end{aligned}$$

where the second equality follows from a simple change of variables.

As for rotation by a matrix ξ, we have (R_ξ is the rotation operator)

$$\begin{aligned}\mathcal{F}\{\mathbf{J}\{\mathrm{R}_\xi\mathbf{f}\}\}(\boldsymbol{\omega}) &= \xi\hat{\mathbf{f}}(\xi\boldsymbol{\omega})\mathrm{j}\boldsymbol{\omega}^{\top}\\ &= \xi\hat{\mathbf{f}}(\xi\boldsymbol{\omega})\mathrm{j}\boldsymbol{\omega}^{\top}\xi^{\top}\xi\\ &= \xi\hat{\mathbf{f}}(\xi\boldsymbol{\omega})\mathrm{j}(\xi\boldsymbol{\omega})^{\top}\xi\\ &= \mathcal{F}\{\mathrm{R}_\xi\{\mathbf{J}\mathbf{f}\}\xi\}(\boldsymbol{\omega}).\end{aligned}$$

Then, we write that

$$\begin{aligned}
\mathrm{TV}_p(\mathrm{R}_\xi \mathbf{f}) &= \int_{\mathbb{R}^d} \|\mathbf{J}\{\mathrm{R}_\xi \mathbf{f}\}(\mathbf{x})\|_{\mathcal{S}_p} \,\mathrm{d}\mathbf{x} \\
&= \int_{\mathbb{R}^d} \left\|\xi^\top \{\mathbf{J}\mathbf{f}\}(\xi\mathbf{x})\xi\right\|_{\mathcal{S}_p} \mathrm{d}\mathbf{x} \\
&= \int_{\mathbb{R}^d} \|\mathbf{J}\mathbf{f}(\xi\mathbf{x})\|_{\mathcal{S}_p} \,\mathrm{d}\mathbf{x},
\end{aligned}$$

since the Schatten norms are unitarily invariant[1]. Now, applying a change of variable $\mathbf{u} = \xi\mathbf{x}$, with $\mathrm{d}\mathbf{u} = |\det \xi|\mathrm{d}\mathbf{x} = \mathrm{d}\mathbf{x}$, we arrive at the desired result:

$$\mathrm{TV}_p(\mathrm{R}_\xi \mathbf{f}) = \mathrm{TV}_p(\mathbf{f})$$

Translation invariance is straight forward to show by using a change of variable. □

[1] R. Bhatia, Matrix Analysis. Springer, 1997.