Abstract—We study 1D continuous-domain inverse problems for multicomponent signals. The prior assumption on these signals is that each component is sparse in a different dictionary specified by a regularization operators. We introduce a hybrid regularization functional matched to such signals, and prove that corresponding continuous-domain inverse problems have hybrid spline solutions, i.e., they are sums of splines matched to the regularization operators. We then propose a B-spline-based exact discretization method to solve such problems algorithmically.

The task in an inverse problem is to recover a signal \( s_0 \) based on \( M \) measurements \( y \approx \nu(s_0) \in \mathbb{R}^M \), where the measurement operator \( \nu \) models the physics of the acquisition system (forward model). This is typically achieved by minimizing the distance between the data \( y \) and the measurements \( \nu(s) \) of the reconstructed signal \( s \) (data fidelity). In order to inject some prior knowledge on the form of the signal, a regularization term is commonly added to the cost functional. In recent years, the advent of compressed sensing (CS) has led to an increasing popularity of sparsity-promoting regularization norms such as the \( \ell_1 \) norm [1], [2] for discrete signals. Then, the prior assumption is that the signal \( s_0 \) is sparse in a certain dictionary basis specified by a regularization operator. However, real-worlds signals are typically composite and are thus not sparse in a single dictionary basis. We therefore focus on multicomponent signals \( s = s_1 + \ldots + s_D \), where each component \( s_d \) is assumed to be sparse in a different dictionary basis. This framework is closely related to the data separation problem [2, Chapter 11], as well as the study of redundant dictionary bases [3]–[8]. This approach has been applied successfully in practice for imaging tasks such as morphological component analysis [9], [10] or image restoration [11].

These works dealing with multicomponent signals focus on purely discrete models. Yet many real-world signals are continuously defined, and this mismatch leads to discretization errors. This observation has lead to an abundance of research on continuous-domain problems with sparsity-promoting norms [12]–[15]. However, to the best of our knowledge, until our submitted work [16] that we present here, no such attempts have been made for multicomponent signals.

We focus on generalized total-variation regularization (gTV)

\[
\|f\|_{TV(N_0)} := \|D^{N_0} f\|_M = \sup_{\varphi \in \mathcal{S}(\mathbb{R})} \langle f, D^{N_0} \varphi \rangle,
\]

(1)

where \( D \) is the derivative operator and \( \mathcal{S}(\mathbb{R}) \) is the Schwartz space on \( \mathbb{R} \). It is known that gTV promotes sparsity in the sense that it leads to reconstructed signals that are sparse polynomial splines of order \( N_0 \) [15]. A polynomial spline of order \( N_0 \) can be expressed as

\[
s(x) = \frac{1}{(N_0-1)!} \sum_k a_k (x-x_k)^{N_0-1} + p(x) \quad \text{where} \quad a_k, x_k \in \mathbb{R} \quad \text{and} \quad p \text{ is a polynomial of degree no greater than } N_0 - 1.
\]

The sparsity of a spline refers here to the number of knots \( x_k \).

In order to deal with multicomponent signals, we introduce the hybrid regularization functional

\[
\mathcal{R}_{\text{hyb}}(f) = \min_{f_1, \ldots, f_D} \sum_{d=1}^D \alpha_d \|f_d\|_{TV(N_0,d)}
\]

(2)

where \( \alpha_d \) control the weight between each regularization term with \( \alpha_1 + \ldots + \alpha_D = 1 \). This regularization function is well suited for multicomponent signals whose components \( s_d \) are sparse in the dictionaries consisting of sparse polynomial splines of degree \( N_0,d \). We now state our main theoretical result.

**Theorem 1.** Let \( 0 < N_0,1 < \ldots < N_0,D \) and let \( \nu : f \mapsto \nu(f) \in \mathbb{R}^M \) be a weak*-continuous operator. Assume that \( \nu(p) \neq 0 \) for all polynomials of degree less than \( N_0,D \) (well-posedness assumption). Then, for any \( \lambda > 0 \), the optimization problem

\[
S = \arg \min_f \left( \|\nu(f) - y\|^2 + \lambda \mathcal{R}_{\text{hyb}}(f) \right)
\]

(3)

has a solution \( s \) of the form \( s = s_1 + \ldots + s_D + p \), where \( p \) is a polynomial of degree no greater than \( N_0,D - 1 \), and the \( s_d \) are polynomial splines of the form

\[
s_d(x) = \frac{1}{(N_0,d-1)!} \sum_{k=1}^{K_d} a_{k,d} (x-x_{k,d})^{N_0,d-1}.
\]

(4)

where \( a_{k,d}, x_{k,d} \in \mathbb{R} \). Moreover, the sparsity indices \( K_d \) verify \( K_1 + \ldots + K_D \leq M \).

Theorem 1 was proved in [16] and extends the main result of [15]. It states that Problem (3) has a sparse hybrid spline solution, i.e., a sum of different splines. A remarkable feature of Theorem 1 is that the number of components \( D \) does not affect the sparsity of the solution, which is bounded by the number of measurements \( M \).

In order to discretize Problem (3), we restrict its search space to the sum of spaces of splines with knots on a grid, i.e., \( \{ \cdot - x_n \}_{N_0,1-1}^N, \ldots, \{ \cdot - x_m \}_{N_0,D-1}^N \), where \( x_n, x_m \) lie on a uniform grid. This approach has many appealing properties: firstly, Theorem 1 guarantees that the search space is matched to the form of the solution (4). Next, critically, it leads to an exact discretization in the continuous domain: in the chosen search space, there is no discretization error. Finally, it allows for the use of B-spline as basis functions, which have compact support and thus lead to well-conditioned problems. These problems are then solved with a multiresolution algorithm introduced in [17] that uses a combination of ADMM [18] and the simplex algorithm [19].

We show some examples of our algorithm that demonstrate its power. Figure 1 is a curve fitting example, where the measurements are samples of the signal, i.e., \( \nu(f) = \{ f(x_1), \ldots, f(x_M) \} \). We notice that the reconstructed signal is very close to the ground truth signal \( s_0 \).

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Fig. 1: Curve fitting for $L_1 = D$, $L_2 = D^2$, $M = 200$, $\lambda = 1.3$, $\alpha_1 = 0.95$, $\alpha_2 = 0.05$. This figure is taken from [16].

Fig. 2: Reconstruction result with noiseless Fourier measurements for $L_1 = D$, $L_2 = D^4$, $M = 30$, $\lambda = 10^{-15}$, $\alpha_1 = 1 - 5 \times 10^{-5}$, $\alpha_2 = 5 \times 10^{-5}$. This figure is taken from [16].

REFERENCES


