Identification of Rational Transfer Functions from Sampled Data

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***Abstract*—We consider the task of estimating an operator from sampled data. The operator, which is described by a rational transfer function, is applied to continuous-time white noise and the resulting continuous-time process is sampled uniformly. The main question we are addressing is whether the stochastic properties of the time series that originates from the sample values of the process allows one to determine the operator. We focus on the autocorrelation property of the process and identify cases for which the sampling operator is injective. Our approach relies on sampling properties of almost periodic functions, which together with exponentially decaying functions, provide the building blocks of the autocorrelation measure. Our results indicate that it is possible, in principle, to estimate the parameters of the rational transfer function from sampled data, even in the presence of prominent aliasing.**

I. Introduction

Models that are based on stochastic differential equations are widely used for describing numerous physical phenomena. We consider in this work stochastic differential equations that have constant coefficients. Such equations are characterized by a rational transfer function and are equivalent to the filtering of white noise. In practice, the available data is discrete, and one is often required to estimate continuous-time parameters from sampled data. The stochastic properties of the time series that originates from such processes depend on the constant coefficients, and the question we are raising here is whether the sampling process is injective in the sense that there is a one-to-one mapping between the continuous-time and the discrete-time models.

Within the context of state-space autoregressive representation, it is known that stochastic differential equations are mapped to stochastic difference equations upon sampling. The z transform description of the difference equation is based on the exponential values of the poles (the roots of the rational transfer function); and for that reason, currently available estimation algorithms assume that there is an ambiguity in determining their imaginary part value, as the exponential function is invariant to $2\pi i$ increments in its argument. In order to overcome this ambiguity, current estimation approaches require high sampling rate values for avoiding aliasing [3]–[10]. They also restrict the imaginary part of the poles to be less than π/T where T is the sampling interval. The z transform description, however, includes non-exponential terms as well, and this fact has not been taken into account so far.

We revisit in this work the ambiguity assumption of sampled autoregressive continuous-time processes, and identify cases for which the sampling operator is injective when applied to the autocorrelation function. We will show that there is no ambiguity even in the presence of prominent aliasing. To this aim, we introduce two alternative descriptions for the poles of the model: one is used for deriving an explicit expression for the autocorrelation function, while the other is used for assigning a Lebesgue measure to subsets of poles. The building blocks of the autocorrelation function are exponentially decaying terms and almost periodic functions; and we exploit this structure for proving uniqueness of the sampled model.

II. The Problem

We consider the following stochastic process

$$x(t) = \int_0^\infty h(t-\tau;\theta)\, w(\tau)\, d\tau, \tag{1}$$

where $w(t)$ is a Gaussian or non-Gaussian white noise process. The shaping filter $h(t;\theta)$ is given in the Fourier domain by

$$H(\omega;\theta) = \frac{1}{\prod_{n=1}^{p}(i\omega - s_n)}, \tag{2}$$

where $\theta = (s_1, s_2, \dots, s_p) \in \mathbb{C}^p$ is composed of the poles of $H(\omega;\theta)$. The real part of each pole is strictly negative and complex poles appear in conjugate pairs. Assuming that $w(t)$ is white with finite variance, σ^2, and that $t \gg 0$, the autocorrelation function of $x(t)$ is given in the Fourier domain by $\Phi(\omega;\theta,\sigma^2) = \sigma^2 \left|H(\omega;\theta)\right|^2$. In this work we investigate the injective property of the sampling operator $x(t) \to \{x(n)\}$ while assuming that p is known. Specifically, we raise the following question: does the time series that originates from the sampled version of $x(t)$ allow one to recover θ?

III. Asymptotic properties of the autocorrelation function

A. Alternative representations to $H(\omega;\theta)$

We introduce two alternative parameter vectors, $\tilde{\theta}$ and $\bar{\theta}$, that will be used for deriving an explicit formula for the autocorrelation function $\varphi(t;\theta)$, and for associating subsets of θ with a measure in $\mathbb{R}^p$. Let $\theta = (s_1, \dots, s_{2m}, s_{2m+1}, \dots, s_p)$ where the first $2m$ poles are complex, and conjugate pairs appear sequentially. Additionally, for a given complex pair, we require the one with positive imaginary part to be listed first. Our first alternative representation is based on decay rates and modulation values. It extends the representation of [11] in the following manner,

$$\tilde{\theta} = (a_1, b_1, a_2, b_2, \dots, a_m, b_m, a_{m+1}, \dots, a_{p-m}), \tag{3}$$

where $a_1, \ldots a_{p-m}$ are the strictly negative real parts of the poles, and $b_1, \ldots b_m$ are the strictly positive imaginary parts. The vector $\tilde{\theta}$ is a point in $\mathbb{R}^p$ and this identification can be made unique by imposing a dictionary-type ordering:

- $0 > a_1 \geq a_2 \geq \cdots \geq a_m$;
- $0 > a_{m+1} \geq a_{m+2} \geq \cdots \geq a_{p-m}$;
- if $a_k = a_{k+1}$, then $b_{k+1} \geq b_k$.

The difference in sign between the a's and b's allows us to distinguish the two types of poles, so that there is no confusion.

The second alternative parameter vector $\bar{\theta}$ indicates multiplicities of poles and will be used for obtaining an explicit formula of autocorrelation functions

$$\bar{\theta} = (\bar{s}_1, m_1, \bar{s}_2, m_2, \ldots, \bar{s}_L, m_L). \tag{4}$$

The multiplicity of a pole $\bar{s}_l$ is represented by m_l.

Definition 1. *The collection of all parameter vectors θ is $\Omega(p)$. This is also the collection of all parameter vectors $\tilde{\theta}$ or $\bar{\theta}$.*

B. The autocorrelation function

The rational form of $H(\omega;\theta)$ is known to yield an autocorrelation function that is a sum of Hermitian symmetric exponentials, as the result of a decomposition in partial fractions [1]. The explicit formula is obtained as follows.

Proposition 1. *Let $\bar{\theta} = (\bar{s}_1, m_1 \ldots, \bar{s}_L, m_L) \in \Omega(p)$. Then,*

$$\varphi(t;\bar{\theta}) = (-1)^p \sum_{\ell=1}^{L} e^{-\lambda_l^{1/2}|t|} \sum_{n=1}^{m_l} \sum_{k=0}^{n-1} d_{l,n,k} \, |t|^{n-1-k}, \tag{5}$$

where

$$\lambda_l = \bar{s}_l^2 \tag{6}$$

$$P(\xi) = \prod_{l=1}^{L} (\xi - \lambda_l)^{m_l} \tag{7}$$

$$c_{l,n} = \lim_{\xi \to \lambda_l} \frac{1}{(m_l - n)!} \frac{\mathrm{d}^{m_l - n}}{\mathrm{d}\xi^{m_l - n}} \left(\frac{(\xi - \lambda_l)^{m_l}}{P(\xi)} \right) \tag{8}$$

$$d_{l,n,k} = \frac{(-1)^n c_{l,n} (n-1+k)!}{(n-1)! k! (n-1-k)! (2\lambda_l^{1/2})^{n+k}}, \tag{9}$$

and $\lambda_l^{1/2} \in \mathbb{C}$ denotes the principal square root of λ_l.

Definition 2. *Two parameter vectors $\theta_1, \theta_2 \in \Omega(p)$ are equivalent if there exists $\alpha \in \mathbb{R}$ such that $\varphi(n;\theta_1) + \alpha \cdot \varphi(n;\theta_2) = 0$ for all $n \in \mathbb{Z}$. If θ_1 is not equivalent to any distinct θ_2, then it is unique.*

When the uniqueness property holds, there is a one-to-one mapping between the autocorrelation function and its sampled version. The sample value of the autocorrelation function can then be estimated from the available sample values of $x(t)$. The uniqueness property is related to linear combinations of autocorrelation functions. In (5), the real parts of the parameters $\lambda_l^{1/2}$ determine exponentially decaying terms, while the imaginary parts determine periods of trigonometric polynomials. In the case of multiple poles, a polynomial term multiplies the complex exponential. Linear combinations of these functions also have the same basic structure. We generalise this structure in the following definition.

Definition 3. *We denote by X the class of functions of the form*

$$\sum_{l=1}^{L} \sum_{m=0}^{M_l} T_{l,m}(|t|) \, |t|^m \, e^{a_l |t|} \tag{10}$$

where $0 > a_1 > a_2 > \cdots > a_L$ and each $T_{l,m}$ is a trigonometric function.

We note that $T_{l,m}(t) \in \mathrm{AP}(\mathbb{R},\mathbb{R})$, which is the space of almost periodic functions. Of particular interest is the fact that uniform samples of almost periodic functions lie in the normed space of almost periodic sequences $\mathrm{AP}(\mathbb{Z},\mathbb{R})$ (cf. [2, Proposition 3.35]), and we shall exploit this fact to verify uniqueness.

Definition 4. [2, pp.94-95] *For any integer n, the mean value of $f \in \mathrm{AP}(\mathbb{Z},\mathbb{R})$ is*

$$M(f) = \lim_{k \to \infty} \frac{f(n+1) + f(n+2) + \cdots + f(n+k)}{k}. \tag{11}$$

Note that we are free to choose any integer n; however, the limit is independent of this choice. A norm for $\mathrm{AP}(\mathbb{Z},\mathbb{R})$ is given by

$$\|f\|^2_{\mathrm{AP}(\mathbb{Z},\mathbb{R})} = M\left(|f|^2\right). \tag{12}$$

Theorem 1. *If $f \in X$ and $f(n) = 0$ for all integers n, then the functions $T_{l,m}$ must also satisfy $T_{l,m}(n) = 0$.*

The value of Theorem 1 is that it essentially allows us to compare functions from X in a segmented fashion, i.e. according to decay rates. For example, suppose $\varphi(t;\theta)$ contains a term $T(|t|)\,|t|^m\, e^{a|t|}$, where $T(|n|)\,|n|^m$ is not identically 0. Then it can not be equivalent to any autocorrelation function that lacks a term with similar decay. We shall use this result to show that the uniform sampling operator is injective for large sub-collections of $\Omega(p)$.

IV. Uniqueness properties

We consider two subsets of Ω(p): $H(\omega;\theta)$ is composed of real poles only; and real and imaginary poles with minimal restrictions.

Lemma 1. *The elements of $\Omega(p)$ that are composed entirely of real poles are unique.*

Definition 5. *Let $\Omega(p)^*$ be the collection of parameter vectors $\tilde{\theta}$ satisfying:*

- *$a_{k_1} \neq a_{k_2}$ for $k_1 \neq k_2$;*
- *each b_k is an irrational multiple of π.*

Proposition 2. *As a subset of $\mathbb{R}^p$, the complement of $\Omega(p)^*$ in $\Omega(p)$ has Lebesgue measure 0.*

Proposition 3. *If an admissible vector $\tilde{\theta}_1 \in \Omega(p)^*$ is equivalent to a vector $\tilde{\theta}_2 \in \Omega(p)$, then $\tilde{\theta}_2$ must have the same number of complex pairs of poles as $\tilde{\theta}_1$. Furthermore, the complex pairs should exist at the same decay rates.*

Proposition 4. *Suppose*

$$\tilde{\theta}_1 = (a_1, b_1, \ldots, a_m, b_m, a_{m+1}, \ldots, a_{p-m}) \in \Omega(p)^* \quad (13)$$

is equivalent to

$$\tilde{\theta}_2 = (a_1, \beta_1, \ldots, a_m, \beta_m, \alpha_{m+1}, \ldots, \alpha_{p-m}) \in \Omega(p). \quad (14)$$

Then $a_l = \alpha_l$ *for all* l.

Proposition 4 implies that any vector of parameters that is equivalent to a vector of parameters in $\Omega(p)^*$ has the same real part values. The autocorrelation function in such a case is given by

$$\varphi(t;\tilde{\theta}_1) = (-1)^p \sum_{l=1}^{m} 2e^{a_l|t|}\big(\Re(\gamma_l)\cos(b_l\,|t|) - \Im(\gamma_l)\sin(b_l\,|t|)\big) + \sum_{l=m+1}^{p-m} \gamma_l e^{a_l|t|} \quad (15)$$

where for $l \leq m$

$$\gamma_l := \Bigg[-8i(a_l+ib_l)a_l b_l \prod_{l'\neq l, l'\leq m}\big((a_l+ib_l)^2-(a_{l'}+ib_{l'})^2\big) \big((a_l+ib_l)^2-(a_{l'}-ib_{l'})^2\big)\prod_{l'>m}\big((a_l+ib_l)^2-a_{l'}^2\big)\Bigg]^{-1}$$

and for $l > m$

$$\gamma_l := \Bigg[-2a_l \prod_{l'\leq m}\big(a_l^2-(a_{l'}+ib_{l'})^2\big)\big(a_l^2-(a_{l'}-ib_{l'})^2\big) \prod_{l'\neq l, l'>m}(a_l^2-a_{l'}^2)\Bigg]^{-1}.$$

Theorem 2. *Let* f_1 *and* f_2 *be functions of the form*

$$f_1(t) = \gamma_1\cos(b\,|t|) + \gamma_2\sin(b\,|t|) \quad (16)$$

$$f_2(t) = \gamma_3\cos(\beta\,|t|) + \gamma_4\sin(\beta\,|t|), \quad (17)$$

where $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ *are non-zero real numbers,* b, β *are positive real numbers, and* b *is an irrational multiple of* π. *If* $f_1(n) = f_2(n)$ *for all non-negative integers* n, *then* $\gamma_1 = \gamma_3$, $\gamma_2 = \pm\gamma_4$, *and* $b = \beta + 2\pi k$ *for some integer* k.

Lemma 2. *Let* $\tilde{\theta}_1 \in \Omega(p)^*$ *be of the form* (13) *with corresponding autocorrelation function* $\varphi(t;\tilde{\theta}_1)$ *as defined in* (15). *Let* $\tilde{\theta}_2 \in \Omega(p)$ *be of the form* (13), *with autocorrelation function* $\varphi(t;\tilde{\theta}_2)$ *as defined in* (15) *where* γ_l *is replaced by* γ_l' *and* b_l *is replaced by* β_l. *If* $\tilde{\theta}_1$ *is equivalent to* $\tilde{\theta}_2$, *then there is a positive number* σ *such that*

$$\sigma^2\gamma_l = \gamma_l' \quad \text{or} \quad \sigma^2\gamma_l^* = \gamma_l' \quad (18)$$

for every l.

Lemma 2 provides a practical criterion for determining uniqueness. According to the lemma, uniqueness translates into a set of polynomial equations that can be simplified by means of Gröbner basis algorithms. If the reduced Gröbner basis has only trivial solutions, then uniqueness is guaranteed. We utilized this property for obtaining the following results.

Theorem 3. [11] *Every element of* $\Omega(1)$ *is unique.*

Theorem 4. [11] *Every element of* $\Omega(2)^*$ *is unique.*

Theorem 5. *Every element of* $\Omega(3)^*$ *is unique.*

Finding reduced Gröbner bases for $p > 3$ is computationally demanding, and we suggest to exploit Lemma 2 for a limited number of values of k. That is, verifying uniqueness for a finite number of modulation values $b = \beta + 2\pi k$.

V. Conclusion

In this work, we investigated the injective properties of sampled continuous-time stochastic processes. We considered uniform sampling of processes with rational power spectrum and identified cases for which the sampling operator is injective when applied to the autocorrelation function. Our analysis relies on the sampling properties of almost periodic functions, which are the building blocks of the autocorrelation function of such processes. By removing a zero-measure set of vectors of parameters we derived a criterion for the uniqueness of the sampled model, and we proved the injective property of several rational operators. Our results indicate that the ambiguity assumption of sampled autoregressive models does not hold true, and that it is possible in principle to estimate the parameters of the rational operator from sampled data, even in the presence of prominent aliasing.

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