CONTINUOUS-TIME AR MODEL IDENTIFICATION: DOES SAMPLING RATE REALLY MATTER?

Simona Maggio[*], *Hagai Kirshner*[**], *Michael Unser*[**]

[*] Department of Electronics, Computer Science and Systems (DEIS)
University of Bologna, Bologna, Italy
[**] Biomedical Imaging Group, École Polytechnique Fédérale de Lausanne (EPFL),
CH-1015 Lausanne, Switzerland
email: simona.maggio@unibo.it

ABSTRACT

We address the problem of identifying continuous-time auto regressive (CAR) models from sampled data. The exponential nature of CAR autocorrelation functions is taken into account by means of exponential B-splines modelling, allowing one to associate the available digital data with a CAR model. A maximum likelihood (ML) estimator is then derived for identifying the optimal parameters; it relies on an exact discretization of the sampled version of the continuous-time model. We provide both time- and frequency-domain interpretations of the proposed estimator, while introducing a weighting function that describes the CAR power spectrum by means of discrete Fourier transform values. We present experimental results demonstrating that the proposed exponential-based ML estimator outperforms currently available polynomial-based methods, while achieving Cramér-Rao lower bound values even for relatively low sampling rates.

1. INTRODUCTION

Continuous-time auto regressive (CAR) stochastic processes are widely used in control theory and in signal/image processing and analysis. Typical examples of applications are system identification and adaptive filtering [1, 2]; speech analysis and synthesis [3]; image modeling [4, 5, 6] to name a few. In practical situations, however, the available data is discrete and one is forced to estimate the underlying continuous-domain parameters from sampled values.

The problem of CAR parameter estimation from sampled data has been approached from several points of view using indirect or direct methods [7]. In indirect methods, the sampled version of the process is treated as a discrete-time auto regressive moving average (ARMA) process which can be identified by standard estimation algorithms such as least squares, maximum likelihood and maximum a posteriori [1]. CAR parameters are then recovered from the ARMA model by reverse mapping. The advantage of the indirect approach resides in the use of well-established identification methods. Its accuracy, however, is compromised since the discrete-time ARMA model does not take into account parameters dependencies. In direct methods, derivatives are replaced by weighted finite differences, providing an approximated discrete-time model. The model is then identified while keeping the original CAR parametrization [8, 9, 10]. Other direct methods are based on a frequency domain analysis [11]. Current methods are sensitive to aliasing artifacts, meaning that they require that the signals be sampled at a sufficient rate.

In this paper, we present a novel direct approach to CAR identification that is based on exponential B-spline interpolation. Our formulation can be interpreted and implemented in both time- and frequency domains, allowing one to derive a maximum likelihood estimator (MLE). The proposed framework is suitable for uniformly sampled data and it performs well over a large range of sampling rates.

The foundation of our method is an analytic form for the ARMA discrete-time model that stems from the sampled version of the CAR model. This expression is then used for developing an MLE algorithm. Specifically, we show that the autocorrelation (AC) function of a CAR model is an exponential spline and we propose here to use exponential B-splines to relate the AC sequence of the digital data with the AC function of the continuous-time process.

Our algorithm relies on an exact discretization of a CAR model; there is no prior assumptions regarding sampling rates and effective bandwidth of the process. For this reason, our method performs well even at relatively low sampling rates. We show in our experiments that the proposed algorithm outperforms polynomial B-spline models [11] while achieving the Cramér-Rao lower bound at any sampling rate. We further show that in the presence of aliasing, the proposed ML cost function has several local minima, but that it still admits a global minimum that corresponds to the MLE. A proper initialization of the algorithm is derived in this work, too.

This work was funded (in part) by the Swiss National Science Foundation under Grant 200020-121763.

2. EXPONENTIAL SPLINE MODELS FOR CAR PROCESSES

A CAR model that is driven by white Gaussian noise is specified in the time domain by the following expression:

$$P(D)y(t) = w(t), \tag{1}$$

where $y(t)$ is the CAR output signal and $w(t)$ is white Gaussian noise with zero mean and variance σ^2; $P(D) = D^N + a_1 D^{n-1} + \ldots + a_n I$ is the n-th order constant coefficient differential operator that acts on the output of the system. In order to identify this stochastic system, we parametrize its power spectrum by introducing a vector $\vec{\alpha}$ that consists of the poles of the system. Therefore, the CAR process $y(t)$ of order n is characterized by the power spectrum:

$$\Phi(j\omega; \vec{\alpha}) = \sigma^2 \left| \prod_{i=1}^{n} \frac{1}{j\omega - \alpha_i} \right|^2 \tag{2}$$

where $\vec{\alpha} = (\alpha_1, \cdots, \alpha_n)$ are the poles of the system. Stability of the system (2) is ensured if $\Re\{\alpha_i\} < 0$. The one-to-one relation between the poles, $\vec{\alpha}$, and the coefficients of the differential operator $P(D)$ allows one to express (2) as a function of real variables that can be used in the optimization process $\boldsymbol{\theta} = (a_1, \cdots, a_n)$:

$$\Phi(j\omega; \boldsymbol{\theta}) = \sigma^2 \left| \frac{1}{(j\omega)^n + a_1 (j\omega)^{n-1} + \cdots + a_n} \right|^2 \quad (3)$$

Typically, CAR identification is performed in the discrete domain where the corresponding model is an ARMA process, derived by some discretization scheme. The main contribution of this paper is the use of exponential B-splines (E-B-splines) to establish an exact link between the discrete- and the continuous- domain models. Interpolating the discrete AC sequence with E-B-splines seems to be a proper choice since the AC function of a stochastic model (CAR or CARMA) is a linear combination of exponentials, which admits an exact representation in an E-B-spline basis.

An E-B-spline [12, 13] for a given set of poles $\vec{\alpha}$ and sampling rate T is defined as follows,

$$\beta_{\vec{\alpha},T}(t) = \mathscr{F}^{-1} \left\{ \frac{1}{T^{n-1}} \prod_{i=1}^{n} \frac{1 - e^{(\alpha_i - j\omega)T}}{j\omega - \alpha_i} \right\} (t). \quad (4)$$

Given a CAR model $\Phi(j\omega; \vec{\alpha})$, one can derive the corresponding E-B-spline by localizing the Green function of the differential operator $P(D)$ [12]. These functions are dependent on the poles of the system $\vec{\alpha}$ and are compactly supported in $[0, nT)$.

The key result of this work consists in the E-B-spline based representation of the AC function of a CAR process which is expressed in the following theorem.

Theorem 1 *The autocorrelation function of a CAR process of order n, $\varphi_{\vec{\alpha}}(t) = \mathscr{F}^{-1}\{\Phi(j\omega; \vec{\alpha})\}$, is given by*

$$\varphi_{\vec{\alpha}}(t) = \sum_{k \in \mathbb{Z}} \sigma^2 p_{(-\vec{\alpha}:\vec{\alpha}),T}[k] \beta_{(-\vec{\alpha}:\vec{\alpha}),T} \left(t - (k+n)T \right) \quad (5)$$

where $\vec{\alpha}$ are the poles of the process and $\beta_{(-\vec{\alpha}:\vec{\alpha}),T}$ is the exponential B-spline with poles $(-\vec{\alpha} : \vec{\alpha})$; $p_{(-\vec{\alpha}:\vec{\alpha}),T}[k]$ is specified via its z-domain $P_{(-\vec{\alpha}:\vec{\alpha}),T}(z) = \frac{(-1)^n}{\prod_{i=1}^{n}(1-e^{\alpha_i T} z^{-1})(1-e^{-\alpha_i T} z^{-1})}$.

Theorem 1 implies that the z-transform of the AC sequence $\Phi_d(z; \vec{\alpha}, T) = \sum_{k \in \mathbb{Z}} \varphi_{\vec{\alpha}}(kT) z^{-k}$ is given by

$$\Phi_d(z; \vec{\alpha}, T) = \sigma^2 \cdot z^n \cdot B_{(-\vec{\alpha}:\vec{\alpha}),T}(z) \cdot \prod_{i=1}^{n} \frac{e^{\alpha_i T}}{(1 - e^{\alpha_i T} z)(1 - e^{\alpha_i T} z^{-1})} \quad (6)$$

where $B_{(-\vec{\alpha}:\vec{\alpha}),T}(z) = \sum_{k=0}^{2n} \beta_{(-\vec{\alpha}:\vec{\alpha}),T}(kT) z^{-k}$.

The z-transform of the AC sequence can be expressed as a function of the poles $\vec{\alpha}$ or equivalently as a function of the coefficients $\boldsymbol{\theta}$: $\Phi_d(z; \vec{\alpha}, T) \equiv \Phi_d(z; \boldsymbol{\theta}, T)$. The latter formulation introduces real variables only which lend themselves better to numerical optimization, as will be described in the following section.

Proposition 1 *The z-transform of the discrete autocorrelation $\Phi_d(z; \boldsymbol{\theta}, T)$ can be expressed as the product of a causal and an anti-causal filter:*

$$\Phi_d(z; \vec{\alpha}, T) = \lambda^2 H_d(z; \boldsymbol{\theta}, T) H_d\left(z^{-1}; \boldsymbol{\theta}, T\right), \quad (7)$$

where

$$H_d(z; \boldsymbol{\theta}, T) = \frac{\prod_{i=1}^{n-1} (1 - \zeta_i z^{-1})}{\prod_{i=1}^{n} (1 - e^{\alpha_i T} z^{-1})}. \quad (8)$$

ζ_i are the zeros of $B_{(-\vec{\alpha}:\vec{\alpha}),T}(z)$ located inside the unit circle. The constant λ^2 is the variance of the corresponding discrete innovation process (white Gaussian noise); and it is given by,

$$\lambda^2 = \sigma^2 \beta_{(-\vec{\alpha}:\vec{\alpha}),T}[1] \prod_{i=1}^{n-1} \frac{-1}{\zeta_i} \prod_{i=1}^{n} e^{\alpha_i T}. \quad (9)$$

$H_d(z; \boldsymbol{\theta}, T)$ represents the discrete-time process corresponding to the CAR system; it corresponds to a discrete ARMA$(n, n-1)$ model whose poles and zeros are interdependent (as the zeros of $B_{(-\vec{\alpha}:\vec{\alpha})T}(z)$ are also functions of $\vec{\alpha}$).

Thanks to this exact discretization, we are actually able to estimate the samples of the Fourier transform of the continuous-time signal $y(t)$ by applying an appropriate spectral weighting to the DFT of its discrete samples.

Proposition 2 *The interpolated representation of a CAR autocorrelation function is given by*

$$\varphi_{\vec{\alpha}}(t) = \sum_{k \in \mathbb{Z}} \varphi_{\vec{\alpha}}(kT) \eta_{\vec{\alpha},T}(t - kT) \quad (10)$$

where the fundamental exponential spline interpolator $\eta_{\vec{\alpha},T}(t)$ is defined in the Fourier domain as

$$\hat{\eta}_{\vec{\alpha},T}(\omega) = \frac{\hat{\beta}_{(-\vec{\alpha}:\vec{\alpha}),T}(\omega)}{B_{(-\vec{\alpha}:\vec{\alpha}),T}(e^{j\omega})} \quad (11)$$

with $\hat{\beta}_{(-\vec{\alpha}:\vec{\alpha}),T}(\omega) = \mathscr{F}\left\{\beta_{(-\vec{\alpha}:\vec{\alpha}),T}(t)\right\}(\omega)$.

Note that expression (11) introduces the correct spectral weighting for restoring the power-spectrum of the CAR model from its discrete samples: $\Phi(j\omega; \vec{\alpha}) = \hat{\eta}_{\vec{\alpha},T}(\omega) \Phi_d(e^{j\omega}; \vec{\alpha}, T)$.

Recently, Gillberg and Ljung [11] proposed a different frequency domain weighting function for CARMA systems based on polynomial splines. The spectral weighting of [11] is defined as

$$\hat{\eta}_{2n,T}(\omega) = \frac{\left| \frac{e^{j\omega T} - 1}{j\omega T} \right|^{2n}}{\left| \frac{\prod_{2n-1}(e^{j\omega T})}{(2n-1)!} \right|} \quad (12)$$

where $\prod_{2n-1}(z)$ is the Euler-Frobenius polynomial. The approach proposed in [11] was shown to perform well for relatively high sampling rates.

While (12) is derived for the limiting case $T \to 0$, the spectral weighting (11) provides an exact mapping between continuous- and discrete-time models. Further, changing the sampling interval results in scaled versions of the polynomial B-spline functions; this scaling property does not hold, however, for the exponential B-spline functions [12]. Also, the proposed exponential model is not restricted by the partition-of-unity condition, allowing for more flexibility in the estimation process.

We note that (12) depends on the number of poles n alone and it is independent of $\boldsymbol{\theta}$; furthermore it corresponds to a low-pass model. It is therefore limited to the spectral content that is captured by the DFT of the discrete-time signal. In contrast, the proposed spectral weighting (11) can assume both low-pass or band-pass configurations, allowing one to estimate CAR parameters even in cases where aliasing effects are prominent, as will be shown in Section 4. In the particular case when the continuous-time signal has base-band power spectrum that occupies frequencies that are much lower than the sampling rate, the proposed approach and the method of [11] provide similar results; this stems from the approximation properties of piece-wise polynomial models.

3. MAXIMUM LIKELIHOOD ESTIMATOR

The maximum-likelihood parameter estimation of a CAR model can be carried out by minimizing the following log-likelihood function:

$$V_N(\boldsymbol{\theta}) = \log|\Sigma| + \frac{1}{2}\mathbf{y}\cdot\Sigma^{-1}\cdot\mathbf{y}. \quad (13)$$

Here, $\mathbf{y} = (y[1], \cdots, y[N])$ is the output signal sampled data vector, $\Sigma[m,k] = \varphi_{\vec{\alpha},T}[m-k]$ is the AC matrix and N is the number of samples. When N is relatively large, calculating the determinant and inverting Σ is computational demanding. A possible way of avoiding this step is to assume $N \gg 1$. If we neglect boundary effects, $\mathbf{y}\cdot\Sigma^{-1}\cdot\mathbf{y}$ corresponds to filtering $\mathbf{y}$ by the inverse of $\lambda H_d(z;\boldsymbol{\theta},T)$ and then taking the ℓ_2 energy of the output. As for the determinant, we calculate it by applying Szegö's theorem in the following manner,

$$\lim_{N\to\infty}\log|\Sigma|^{1/N} = \frac{1}{2\pi}\int_0^{2\pi}\log\lambda^2|H_d(e^{j\omega_k};\boldsymbol{\theta},T)|^2 d\omega \quad (14)$$

Utilizing (8) and the Residue theorem, the integral on the right-hand-side of the equation reduces to $\log\lambda$; it follows that for $N \gg 1$,

$$V_N(\boldsymbol{\theta}) \cong \frac{N}{2}\log\lambda^2 + \frac{1}{2}\left\|\mathbf{y} * g\right\|^2_{\ell_2}, \quad (15)$$

where the z-transform of g is $G(z;\boldsymbol{\theta},T) = 1/H_d(z;\boldsymbol{\theta},T)$.

We can further approximate the ℓ_2 norm in (15) by a Riemann sum within the Fourier domain, which leads to a slightly different definition of the MLE cost function:

$$V_N(\boldsymbol{\theta}) \cong \sum_{k=1}^{N}\frac{|Y[k]|^2}{2\lambda^2|H_d(e^{j\omega_k};\boldsymbol{\theta},T)|^2} + \frac{1}{2}\log\lambda^2|H_d(e^{j\omega_k};\boldsymbol{\theta},T)|^2 \quad (16)$$

where $Y[k] = \frac{1}{\sqrt{N}}\sum_{l=0}^{N-1} y[k]e^{-j2\pi k\frac{l}{N}}$ is the DFT of the discretized CAR process, $\mathbf{y}$. The logarithmic term in (16) can be computed more precisely by using the integration between 0 and 2π, suggested in (14), instead of the Riemann sum. The resulting expression of the logarithmic part is $\frac{1}{2}\log(\lambda^2)$, as in (15). Equation (16) describes the joint likelihood of the DFT samples $Y[k]$, which are complex random variables. The mean of these random variables is zero and their variance can be approximated by $\lambda^2|H_d(e^{j\omega_k};\boldsymbol{\theta},T)|^2$; the approximation error decays at a rate of $\frac{1}{\sqrt{N}}$. This value of the variance becomes exact when a periodic random process is observed over an integer number of periods.

Due to the possible infinite support of g, sampling its z-transform on the unit circle may involve information loss and introduce numerical inaccuracies in the Riemann sum approximation (16). Yet, our experiments suggest that the approximation is adequate for relatively high values of N, in which one can can choose either a spatial-domain implementation of $\mathbf{y} * g$ or a DFT approximation scheme. The maximum-likelihood estimate is then given by:

$$\hat{\boldsymbol{\theta}} \cong \arg\min_{\boldsymbol{\theta}} V_N(\boldsymbol{\theta}) \quad (17)$$

The proposed MLE approach differs from currently available methods in several aspects. First, it considers exponential AC models for both the continuous and the discrete domain processes, establishing a link between the continuous and the discrete domain models. This, in turn, allows for the discrete model to stem naturally from the continuous domain formulation while no a priori assumptions are made on the digital data. Second, the log-likelihood function suggested here holds true for any value of sampling interval, rather than describing the limiting case of $T \to 0$. Finally, the log-likelihood function utilizes discrete-domain data for determining continuous-domain statistics while no approximation of continuous-domain frequency spectrum nor of an impulse response function is required.

Table 1: Local minima of the likelihood function for a CAR(2) process with poles in $-1 \pm 5i$.

Region	Log-likelihood Value	Estimated Poles	Frequency[a] [rad/time-unit]
1	-3656	$-2.4 \pm 2.1i$	0
2[b]	**-3675**	$\mathbf{-0.96 \pm 5.0}i$	**4.9**
3	-3557	$-0.46 \pm 7.8i$	7.8
4	-3067	$-0.25 \pm 11.1i$	11.1
5	-2212	$-0.16 \pm 14.0i$	14.0

[a] Frequency of maximum response.
[b] Global minimum of the log-likelihood function.

The choice of the starting point for the MLE is essential since the cost function may have several minima and an initial guess too far from the correct solution may yield a wrong convergence point. This can be avoided with an appropriate choice of the initial parameters of $\boldsymbol{\theta}$.

Since the DFT of a sampled signal comes from summing shifted versions of its continuous-domain Fourier transform, it only yields information about the shifts that fall in $[-\frac{\pi}{T}, \frac{\pi}{T}]$, while the continuous signal may also have a band-pass power spectrum, involving higher frequencies. Beside the low-pass band, possible shifts of the continuous-time response are in any band $B_k = [k\frac{\pi}{T}, (k+1)\frac{\pi}{T}]$. The cost function $V_N(\boldsymbol{\theta})$ may therefore have one local minimum in each band B_k, but the global minimum is in the band the original continuous-time signal of the model, $y(t)$, corresponds to.

An example of this kind of behaviour is reported in Table 1 for a second order CAR system, where the likelihood value of the solutions found by the MLE in five consecutive bands is shown. The global minimum among all local minima corresponds to the correct solution.

This suggests selecting K frequencies ω_k from any band until an arbitrary $K-th$ band, for example the central ones: $\omega_k = (1+2k)\frac{\pi}{2T}$. K starting points can thus be defined as parameters $\boldsymbol{\theta}_k$ for which the continuous spectrum has maximum response centered in ω_k.

The value of K may be derived either from known physical constraints of the investigated system translated in frequency domain constraints. It can also be empirically estimated for a known class of systems. When there is no knowledge about the process, this parameter can be determined iteratively as part of the identification process. One can learn K by observing the behaviour of the likelihood function in progressive bands: after the global minimum has been achieved, the value of the likelihood function at the local minimum increases in a consistent manner, allowing one to determine the number of bands to be investigated.

Once the starting points $\boldsymbol{\theta}_k$ are selected, one can perform minimization of $V_N(\boldsymbol{\theta})$ starting from every $\boldsymbol{\theta}_k$ catching all the local minima $\hat{\boldsymbol{\theta}}_k$ in the bands considered. The optimal parameter $\hat{\boldsymbol{\theta}}_{opt}$ is the one corresponding to the global minimum of the cost function. This strategy is robust since it works also for very low sampling rates, but requires several minimizations of the cost function. A flowchart describing the proposed optimization strategy is shown in Fig.1.

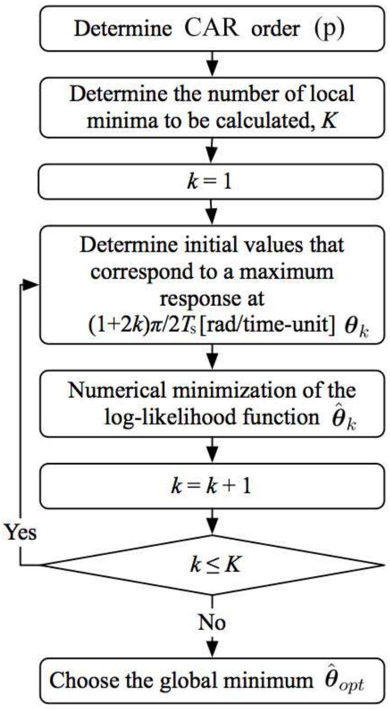


Figure 1: The proposed CAR estimation algorithm.

Table 2: MSE (dB) for band-pass and low-pass signals

		Coefficients MSE			
Model	$\frac{T}{T_{smax}}$	a_1 **Exp**	a_1 Poly[11]	a_2 **Exp**	a_2 Poly[11]
$[-1 \pm 5i]$	0.67	-24.55	-23.97	-33.00	-31.64
	0.73	-24.97	-24.34	-32.75	-32.76
	0.81	-24.91	-23.64	-33.49	-32.65
	0.90	-25.70	-23.51	-32.51	-32.28
	1.02	-25.89	-22.82	-34.42	-31.41
	1.17	-25.27	-21.71	-33.25	-28.95
	1.39	-23.98	-23.93	-27.87	-22.40
	1.69	-22.85	-19.50	-22.47	-11.62
	2.16	-26.72	-1.26	-31.75	-5.12
$[-1 \pm 10i]$	0.56	-21.08	-20.87	-35.30	-35.25
	0.62	-21.68	-21.10	-35.68	-35.84
	0.68	-22.74	-21.57	-36.44	-35.68
	0.76	-22.40	-20.83	-36.23	-34.84
	0.86	-22.56	-19.25	-36.67	-34.49
	0.99	-23.78	-16.66	-36.69	-32.72
	1.17	-24.09	-15.84	-35.23	-27.62
	1.43	-23.40	-10.70	-32.65	-10.88
	1.83	-25.15	4.40	-36.54	-3.63
$[-5, -1]$	0.50	-30.42	-28.02	-23.68	-23.39
	0.78	-31.15	-23.90	-25.71	-25.51
	1.06	-29.61	-21.31	-26.67	-27.44
	1.33	-29.24	-18.58	-25.93	-25.42
	1.61	-28.00	-16.83	-25.41	-22.62
	1.89	-25.96	-15.55	-24.39	-19.74
	2.17	-24.95	-14.49	-23.63	-17.37
	2.44	-22.05	-13.94	-20.59	-15.33
	2.72	-21.17	-13.69	-19.45	-13.95
	3.00	-19.95	-13.42	-18.55	-12.77

4. EXPERIMENTS AND RESULTS

In order to assess the performance of the proposed algorithm, we considered several CAR(2) systems, introducing both low- and band-pass power spectra. For every system 250 Monte Carlo simulations were performed. The sampled signals $y[k]$ were generated from the system $H_d(z; \boldsymbol{\theta}, T)$ driven by discrete-time white Gaussian noise of 1000 samples. The variance of the continuous-time white noise was set to unity and considered known.

For every experiment we compare the proposed MLE with the traditional ARMA$(n, n-1)$ estimator and with the polynomial spline algorithm of [11].

For every system we consider several sampling intervals. For band-pass signals, we choose sampling frequency ranging from 1.5 to 4.5 times the peak frequency, while for low-pass signals we defined a maximum frequency ω_{max} at $-10dB$ the peak frequency and considered a sampling time ranging between 0.5 and 3 times the maximum sampling time: $T_{smax} = \frac{\pi}{\omega_{max}}$. These sampling rates cover a range of both aliased and non aliased spectra.

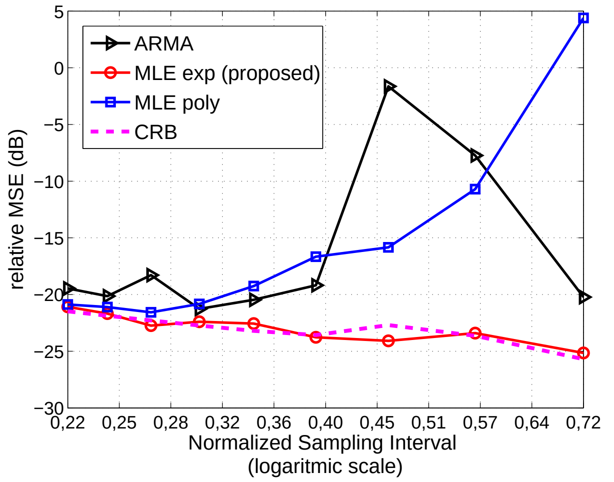
In Table 2 we compare the performance of exponential spline and polynomial spline MLE's for band-pass signals and low-pass signals. We report the relative MSE, $\mathbb{E}\left[\left(\frac{\boldsymbol{\theta}-\boldsymbol{\theta}_{true}}{\boldsymbol{\theta}_{true}}\right)\left(\frac{\boldsymbol{\theta}-\boldsymbol{\theta}_{true}}{\boldsymbol{\theta}_{true}}\right)^T\right]$ for the estimation of the first and the second coefficient of $\boldsymbol{\theta} = (a_1, a_2)^T$. The MSE is computed as mean value over 250 simulations for several sampling conditions, while the standard deviation is always below 10^{-3}. The exponential-spline-based algorithm outperforms the polynomial spline MLE in all cases; as expected, the largest gain is encountered in coarse sampling conditions.

In Fig.2 we show the results of identification of a CAR(2) system and compare the performance of ARMA, polynomial spline MLE, and the proposed MLE. The Cramér-Rao Bound (CRB) [14] is also included in the figure to show the effectiveness of the proposed algorithm. The CRB represents a lower bound for unbiased estimators; from the figure, it is evident that exponential spline based algorithm follows closely this limit.

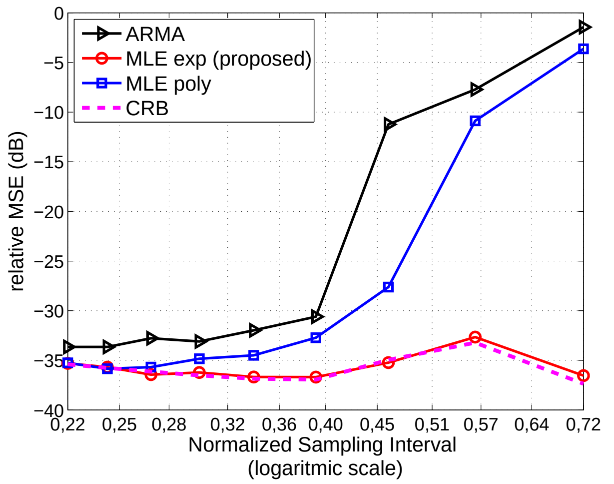
The reason for which the proposed algorithm succeeds also in the presence of aliasing is linked to its ability to locate the local minima in different frequency bands from the DFT of discrete samples. There, the method largely benefits from the use of exact discretization. Interpolating data in an exponential spline basis is the proper approach as the spectral weights in (11) are parameter's dependent and can assume band-pass configurations. The weights proposed in [11], on the other hand, are fixed and thus restricted to low-pass configurations. In Fig. 3 a CAR(2) signal output is analysed in a low sampling rate situation in the presence of strong aliasing. The correct model is located in the second band of the DFT, whereas the MLE based on polynomial splines provides a low-pass solution. By contrast, the spectral weights $\hat{\eta}_{\vec{\alpha},T}(\omega)$ for the exponential MLE adapt to the data during the search process and gathers around the ideal solution in the second band.

5. CONCLUSIONS

We presented a novel maximum likelihood algorithm for identifying continuous-time AR systems from sampled data. The proposed maximum-likelihood estimator is based on exponential B-spline interpolation of the autocorrelation sequence of the digital data. Unlike currently available identification methods, the proposed model can identify CAR parameters of both low-pass and band-pass power spectra processes, regardless of the sampling rate. The proposed formulation can yield time or frequency estimators. We presented an analysis of exponential-spline-based MLE for identification of low-pass and band-pass stochastic systems for variable sampling rates. We did also compare our algorithm with the traditional ARMA estimator and with a polynomial-based maximum-likelihood estimator, and found it consistently to perform better, especially at low sampling



(a)



(b)

Figure 2: Averaged relative MSE for ARMA, polynomial spline MLE and the proposed exponential spline MLE. The error measure refers to identification of coefficients a_1 (Fig. 2(a)) and a_2 (Fig. 2(b)) of a CAR(2) system $1/(s^2 + a_1 s + a_2)$ with poles $\vec{\alpha} = [-1 \pm 10i]$. The corresponding CRB is included in the graphs for comparison purposes.

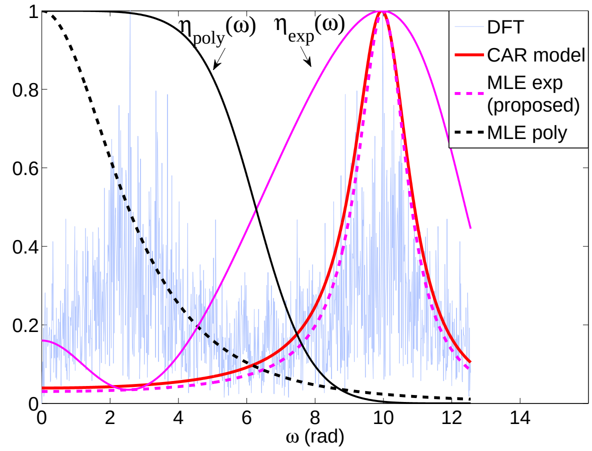


Figure 3: Analysis of CAR system $1/(s^2+a_1 s+a_2)$ with poles $\vec{\alpha} = [-1 \pm 10i]$ in prominent aliasing conditions: $T = 1$. DFT of $y[n]$ is shown in $[0, 4\pi/T]$, The left side of the figure depicts polynomial spline based MLE solution (dashed black) and its correspondent spectral weight $\eta_{poly}(\omega)$ (solid black). In the right side of the figure the exponential spline based MLE solution (dashed pink) and its correspondent spectral weight $\eta_{exp}(\omega)$ are shown (solid pink), too. The oracle CAR power spectrum is shown in a red solid line. As shown from the figure, the proposed exponential estimator performs better than the polynomial model as it is able to identify the band-pass spectrum of the original continuous-time process.

rates. When aliasing effects are prominent, the proposed MLE follows the Cramér-Rao lower bound at all rates, too. Because the cost function of the proposed MLE may have several local minima due to aliasing, we proposed a strategy to select a set of starting points in order to successfully estimate the correct band of the original CAR process. It is believed that the proposed approach provides a good alternative to the currently available methods.

REFERENCES

[1] L. Ljung, *System Identification*. Upper Saddle River, NJ: Prentice-Hall, 1999.

[2] S. Haykin, *Adaptive Filter Theory*. Prentice Hall, 4th ed., 2002.

[3] I. Lim and B. Lee, "Lossless pole-zero modeling for speech signals," *IEEE Transactions on Speech and Audio Processing*, vol. 1, no. 3, pp. 269–276, 1993.

[4] H. Kirshner and M. Porat, "On the role of exponential spline in image interpolation," *IEEE Trans. Signal Processing*, vol. 18, no. 10, pp. 2198–2208, 2009.

[5] J. Francos, A. Meiri, and B. Porat, "A unified texture model based on a 2-D Wold like decomposition," *IEEE Transactions on Signal Processing*, vol. 417, pp. 2665–2678, 1993.

[6] S. Nemirovski and M. Porat, "On texture and image interpolation using markov models," *Signal Processing: Image Communication*, vol. 24, pp. 139–157, 2009.

[7] H. Garnier and L. Wang, eds., *Identification of Continuous-time Models from Sampled Data*. Springer, 2008.

[8] H. Fan, T. Söderström, M. Mossberg, B. Carlsson, and Y. Zou, "Estimation of continuous-time AR process parameters from discrete-time data," *IEEE Transactions on Signal Processing*, vol. 47, no. 5, pp. 1232–1244, 1999.

[9] T. Söderström, "Computing stochastic continuous-time models from ARMA models," *International Journal of Control*, no. 6, pp. 1311–1326, 1991.

[10] G. C. Goodwin, J. I. Yuz, and H. Garnier, "Robustness issues in continuous-time system identification from sampled data," *Proceedings of the 16th IFAC World Congress*, 2005.

[11] J. Gillberg and L. Ljung, "Frequency-domain identification of continuous-time ARMA models from sampled data," *Automatica*, vol. 45, pp. 1371–1378, 2009.

[12] M. Unser and T. Blu, "Cardinal exponential splines: Part I - theory and filtering algorithms," *IEEE Transactions on Signal Processing*, vol. 5373, no. 4, 2005.

[13] M. Unser and T. Blu, "Cardinal exponential splines: Part II - think analog, act digital," *IEEE Transactions on Signal Processing*, vol. 53, no. 4, 2005.

[14] P. Stoica and R. Moses, *Introduction to Spectral Analysis*. Prentice Hall, 1997.