Relationship Between High-Resolution Methods and Discrete Fourier Transform

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ABSTRACT: In this paper, we establish a link between the Discrete Fourier Transform (DFT) and 2 high-resolution methods, MUSIC and the Tufts and Kumaresan's method (TK). Existence and location of the extraneous peaks of MUSIC, of the "noise" zeros of TK, are related to the minima of the DFT of the rectangular window filtering the data. Other properties of the "noise" zeros are given, in relation to polynomial theory.

I- INTRODUCTION

Time series harmonic analysis is a crucial problem in a number of practical situations. However, very often, data lengths are short, so that the separation between the frequencies to be retrieved may be shorter than the Fourier resolution limit. In this case, the so-called "high-resolution" methods are needed. Various such methods have been devised. In this paper, we will consider the MUSIC method [1], and Tufts and Kumaresan's method (TK) [2] [3]. In both methods, a "model order" is chosen. It corresponds to a rectangular windowing of the data. The sources are obtained as peaks in a spectrum (MUSIC) or zeros of a polynomial (TK). The number of such peaks (resp. zeros) is equal to the model order. The order is generally chosen in order to overestimate the number of frequencies, so that a number of extraneous peaks (resp. zeros) are introduced. These are generally attributed to the noise. We show that they originate in fact in the shape of the spectrum of the above-mentioned rectangular window.

We first show that MUSIC can be viewed as a special case of the periodogram, this latter being understood as the average spectrum of the data, taken over overlapping rectangular windows of width equal to the model order. The secondary peaks of the MUSIC spectrum are thus related to the minima of the Fourier transform of this rectangular window.

The TK method relies on the properties of the minimum-norm vector with first coefficient equal to 1 belonging to the so-called "noise subspace". It is well-known that the zeros of the associated polynomial can be divided into two subsets, namely the "source" zeros and the "noise" zeros. The "noise" zeros have been observed to be approximately uniformly distributed on a circle centered in 0, and of radius <1.

This fact has never been explained, though [4] performs an explicit computation of the location of the roots in the similar case of the maximum-entropy method, but only in the single frequency case.

On another hand, [5] considers this problem from the point of view of measure theory. A probability measure is introduced, which assigns the same weight to each "noise" root. For the 2 frequencies case, [5] proves that for almost every value of the difference between these two frequencies, this measure converges weakly to the uniform measure on the unit circle when the model order tends to infinity.

In this paper, we present results on the structure of the "noise" polynomial, which leads to explain properties of their zeros. We then can relate the location of the zeros to the shape of the Fourier transform of the above-mentioned rectangular window.

We first show that the "noise" zeros are in fact roots of a lacunary polynomial. The properties of lacunary polynomials give an explanation to the distribution of the zeros.

We can then prove that the "noise" roots converge uniformly to the unit circle, except possibly for m-1 of them, m being the number of frequencies. The proof will not be given here, due to lack of space.

We study in detail the 1 frequency case, giving a direct proof of the convergence and the angular "equi" distribution of the roots.

For any number of frequencies, we give a first-order approximation of the lacunary polynomial, leading to prove the convergence of the roots to the unit circle. An explicit expression of this polynomial in the 2 frequencies case is given.

Finally, we give an heuristic explanation to the angular distribution of the zeros: since the "noise" zeros lie near the unit circle, the spectrum of the coefficients of the lacunary polynomial can be expected to have minima close to the location of these zeros. Using the above-mentioned first-order approximation, it is possible to prove that these minima are approximately angularly equispaced. The zeros are then likely to be also so. On another hand we have proven that the coefficients of the noise polynomial behaved more or less like the sum of the signal itself plus its derivative. The angular equispacing of the minima of its spectrum then corresponds to the minima of the spectrum of the rectangular window of width equal to the model order.

II- Problem Position

Throughout this paper, only noiseless data will be considered.

Let \( x \) be a signal composed of a sum of \( N \) complex sine waves, observed on \( N \) samples.

\[
x_k = \sum d_i \exp(j \omega_k) \quad \text{for } k = 0, \ldots, N-1
\]

where \( \omega_k, d_i \) are resp. the pulsations and complex amplitudes of the \( i \)-th complex exponential.

III- Periodogram

In this paragraph, we recall the definition and properties of the periodogram.

The periodogram is commonly defined as the squared modulus of the discrete Fourier transform of the data. However, very often, overlapping windowing of the data is introduced and an averaging over the periodograms of the windowed data is used. This can be written in matrix form as follows. Letting
\[
X = \begin{bmatrix}
X_0 & X_1 & \cdots & X_{N-1}
\end{bmatrix}^T
\]
and \( P(\omega) \) denotes the averaged periodogram, we have \( P(\omega) = \frac{1}{(N-L+1)} \left[ 1, e^{j\omega_0}, \ldots, e^{j(N-L-1)\omega_0} \right] \left[ X^*X \right] \left[ 1, e^{j\omega_0}, \ldots, e^{j(N-L-1)\omega_0} \right]^* \)

where the superscript * denotes conjugate transpose. Let us investigate the shape of \( P(\omega) \). Using (1) we obtain:

\[
A = S_L A
\]

where \( A \) is the matrix containing the amplitudes of the windowed signals. Thus \( XX^* = S_L A A^* S_L^* \).

Note that \( A \) can be expressed as follows:

\[
A = \begin{bmatrix}
d_1 & e^{j\omega_0} & e^{j(N-L-1)\omega_0} \\
& \ddots & \ddots \\
& & d_m & e^{j\omega_0} & e^{j(N-L-1)\omega_0}
\end{bmatrix}
\]

and thus \( \frac{1}{(N-L+1)} XX^* \) tends towards \( S_L \left[ d_1, 1, e^{j\omega_0}, \ldots, e^{j(N-L-1)\omega_0} \right] \) when \( N \) tends to infinity, where \( \left[ d_1, 1, e^{j\omega_0}, \ldots, e^{j(N-L-1)\omega_0} \right] \) is \( \frac{1}{\sqrt{N}} \left[ 1, 1, e^{j\omega_0}, \ldots, e^{j(N-L-1)\omega_0} \right] \).

Recall that the periodogram was given by \( P(\omega) \), with \( P(\omega) \) proportional to \( \left[ 1, \ldots, e^{j(N-L)\omega_0} \right] \left[ S_L A A^* S_L^* \right] \left[ 1, \ldots, e^{j(N-L-1)\omega_0} \right]^* \).

We can thus consider \( f(\omega) \) as the "periodogram" of a special signal whose amplitudes would be characterized by \( P \) instead of \( A \). This explains why the MUSIC spectrum exhibits regularly spaced secondary peaks: they originate in the implicit windowing of the data, as it is the case for the periodogram.

Note that for well-separated frequencies \( AA^* \cong \mathbb{I} \) and \( PP^* \cong \mathbb{I} \). Then \( f(\omega) \) tends to be proportional to \( P(\omega) \).

\section{Tufts and Kumaresan's method}

In this paragraph, we first recall the principle of the TK method, and then relate the "noise" roots to the periodogram of the data.

\subsection{Principle of the method}

We defined in the above section the "signal" subspace as the one spanned by \( S_L \). The subspace which is orthogonal to the "signal" subspace will be termed "noise" subspace. Any vector of the "noise" subspace is thus orthogonal to \( S_L^* \). Consider a polynomial having as coefficients the elements of such a vector. The special structure of the columns of \( S_L \) implies that the \( \{ e^{j\omega_0} \} \) belongs to the roots of this polynomial. There is an infinity of such polynomials. TK method singles out the minimum-norm vector belonging to the noise subspace, with first coefficient equal to 1. The associated polynomial has thus \( L-1 \) roots besides the \( \{ e^{j\omega_0} \} \). These are termed the "noise" roots.

\subsection{Problem position}

It has been observed that the \( L-1 \) "noise" roots of the associated polynomial behave "regularly" in some sense: they seem to be approximately equidistributed on a circle concentric to the unit circle, and of radius \(<1\). This property facilitates the separation between these zeros and the \( \{ e^{j\omega_0} \} \).

However, while it has long been proven [3] that the noise roots are located inside the unit circle, their other properties have never been given an explanation, except by [4] for \( m=1 \) in the context of the maximum entropy method which leads to polynomials having similar properties.

In another respect, we have proven that the noise roots converge uniformly to the unit circle, except for at most \( m \) of them (we in fact give an upper bound of the distance of those roots to the unit circle). The proof will not be given here, due to lack of space.

We are going to study the properties of the "noise" polynomial coefficients (polynomial whose roots are the "noise" roots). They originate in the following fact [3]. The coefficient vector of the noise polynomial is the AR prediction-error filter for a data sequence formed by the coefficients of the
"signal" polynomial surrounded by zeros. We establish that the "noise" roots are also roots of a laguerre polynomial. This latter property gives the key to the understanding of the location of these zeros.

We then put forth a complete study of the case m = 1.

For any number of sources, we give a first-order approximation of the laguerre polynomial.

For m = 2, we give an explicit expression for the laguerre polynomial.

Finally, we give an heuristic explanation of the almost equiangular distribution of the zeros, noting that since the zeros lie mostly near the unit circle, the spectrum of the noise polynomial coefficients can be expected to exhibit minima close to the location of these zeros. We have shown that the coefficients of this polynomial were the sum of two terms, one of them a linear sum of the same sine waves as the signal itself, the other one a linear sum of their derivatives. This hints that the spectrum of these coefficients has minima distributed as those of the spectrum of the rectangular window of width equal to the model order. This latter result can be proven using the first-order approximation of the laguerre polynomial.

V.3 The "noise" polynomial coefficients viewed as the AR solution to a special problem

Seeking the minimum-norm vector with first coefficient equal to 1 belonging to the "noise" subspace is equivalent to seeking a (L-m-1)-th degree monic polynomial with m given zeros and minimum-norm. The product of two polynomials can be written in matrix form as follows:

\[
\begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & a_1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_m \\
0 & 0 & \cdots & a_m \\
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
\vdots \\
1 \\
1 \\
\end{bmatrix}
= 
\begin{bmatrix}
a_{m+1} \\
a_{m+2} \\
\vdots \\
a_{2m} \\
a_{2m+1} \\
\end{bmatrix}

\]

where \(a_1, \ldots, a_m\) (resp. \(b_0, \ldots, b_{2m}\)) are the coefficients of the "signal" (resp. "noise") polynomial \(A(z)\) (resp. \(B_{1m}(z)\)). Assuming \(a_1 = 0\) involves no loss in generality. The minimum-norm condition gives: \(b^*A^*A\) minimum under the constraint \(b_0 = 1\). The solution to this problem is given by:

\[
A^*A b_{1m} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ a_1 & a_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_m & a_m & \cdots & a_m \\ b_0 & b_0 & \cdots & b_0 \\ \end{bmatrix} 
\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ \end{bmatrix} = 
\begin{bmatrix} 1 = b_{L-1} \\ b_{L-2} \\ \vdots \\ b_{2m} \\ b_0 \\ \end{bmatrix} 
\]

where \(a_1, \ldots, a_m\) (resp. \(b_0, \ldots, b_{2m}\)) are the coefficients of the "signal" (resp. "noise") polynomial \(A(z)\) (resp. \(B_{1m}(z)\)). Assuming \(a_1 = 0\) involves no loss in generality. The minimum-norm condition gives: \(b^*A^*A\) minimum under the constraint \(b_0 = 1\). The solution to this problem is given by:

\[
A^*A = \begin{bmatrix} r_m & r_m & \cdots & r_m \\ r_m & r_m & \cdots & r_m \\ \vdots & \vdots & \ddots & \vdots \\ r_m & r_m & \cdots & r_m \\ \end{bmatrix} 
\]

Therefore, since \(A^*A\) is proportional to \(\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & a_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_m \\ 0 & 0 & \cdots & a_m \\ \end{bmatrix}\), the data sequence \(\ldots, 0, 0, 0, b_0 b_1, \ldots, b_{L-1}, 0, \ldots, 0, \ldots\) follows a linear recursion relationship with coefficients \(\{1\}\). It is clearly seen that since the zeros of the "signal" polynomial have modulus equal to 1, the \(\{1\}\) are the coefficients of \((A(z))^2\).

Therefore, we can write

\[
b_{kL+1} = \sum_{k=0}^{\infty} \left( \lambda_k + n \mu_k \right) e^{j n \phi_k} 
\]

The 2m values \(\lambda_k\) and \(\mu_k\) are determined by writing \(b_{1L} \ldots, b_{L-1L}, b_{L-m}, \ldots, b_{L-1L}\) are zero and \(b_{L-m+1} = 1\).

V.5 A laguerre polynomial

\[
A^*A b_{1m} = \begin{bmatrix} \end{bmatrix} 
\]

\[
\begin{bmatrix} 1 \end{bmatrix} = b_{L-1} \]

Therefore, we can write

\[
b_{L+1} = \sum_{k=0}^{\infty} (\lambda_k + n \mu_k) e^{j n \phi_k} 
\]

The 2m values \(\lambda_k\) and \(\mu_k\) are determined by writing \(b_{1L} \ldots, b_{L-1L}, b_{L-mL} \ldots, b_{L-mL}\) are zero and \(b_{L-m+1} = 1\).

V.6 Case m = 1

We give here an exhaustive study of the location of the roots. Taking \(a_1 = 0\) involves no loss in generality. Using (3), we obtain \(b_{1L} = (\lambda + n \mu)\). We have \(b_{1L} = 0\), thus \(\lambda = \mu\). We also have \(b_{L-m+1} = 1\). Thus \(\lambda = \mu = 1/2\) (L-m).

Consequently

\[
B_{1m}(z) = \frac{1}{(L-m)} \left[ \frac{1 + \mu}{1 + \mu} \right]^{2} 
\]

\[
= \frac{1}{(L-m)} \left[ \frac{(1+1/2)^2}{1+1/2} \right] 
\]

\[
= \frac{1}{(L-m)} \left[ \frac{(3/2)^2}{3/2} \right] 
\]

\[
= \frac{1}{(L-m)} \left[ \frac{9/4}{3/2} \right] 
\]

\[
= \frac{1}{(L-m)} \left[ \frac{3}{2} \right] 
\]

Following [4], [6], we can assert that the "noise" roots \(\xi_k\) are such that

\[
(2(L-m))^{1/(L-m)} \leq |\xi_k| \leq (L-m)^{-1/(L-m)} 
\]

(See appendix for a proof).

Moreover, (17) (p.165) implies that each root lies within the sector \((2k+1)(\pi)/(L-m))\) for \(k = 0, \ldots, (L-m-1)/2\).

Therefore, we have proven that for \(m = 1\) the noise zeros converge to the unit circle when \(L\) tends to infinity, while being approximately angularly distributed.

V.7 General case

Let us write

\[
A(z)^2 B_{1m}(z) = 1/L Q_L(z) + z^{L-1}/L R_L(z) + A(z) 
\]

There, letting \(\xi_n = e^{i\theta_n}\), we write, using Lagrange's interpolation polynomials:

\[
Q_L(z) = \sum_{m=1}^{m} A'_{(\xi_n)} (z) \xi_n - \xi_n 
\]

\[
R_L(z) = \sum_{m=1}^{m} A'_{(\xi_n)} (z) \xi_n - \xi_n 
\]
and thus, since $\zeta_n$ are 2-fold zeros of the lacunary polynomial:

\[ Q_L(\zeta_n) + \zeta_n^{-L} R_L(\zeta_n) = 0 \]

\[ \sum_{k \neq n} R_L(\zeta_k) A(\zeta_k) \zeta_k^{-L} \left( \zeta_k - \zeta_n \right)^{-1} \]

\[ + (L-1) \zeta_n^{-L} R_L(\zeta_n) + L \zeta_n^{-L} A(\zeta_n) = 0 \]

for $n=1 \ldots m$. The second equation implies that $R_L(\zeta_n)$ is bounded over $L$. Therefore, $R_L(z) = Q_L(z)$ are two polynomials whose coefficients are bounded over $L$, wherefrom the Kumaresan polynomial $V_L(z)$ follows

\[ V_L(z) = A(z) B_L(z) \]

\[ = z L - 1 + \frac{1}{L} Q_L(z) + \frac{L^{-1}}{L} R_L(z) \]

This is a useful property which shows in the limit $L \to \infty$ that, when its modulus is evaluated over the unit circle, the Kumaresan polynomial behaves as a constant except near the roots of $A$. More precisely,

\[ \left| V_L(e^{i\theta}) \right|^2 \ll 1 + \frac{2}{L} \int_{-\pi}^{\pi} \frac{R_L(e^{i\theta})}{A(e^{i\theta})} + \frac{2}{L} \int_{-\pi}^{\pi} e^{-i(L-1)\theta} Q_L(e^{i\theta}) \]

Thus, away from the $\{e^{i\theta_j}\}$, the first (constant) term dominates, while the second provides a slowly varying $1/L$ correction and the third a fast varying one, behaving like the Fourier transform of the rectangular window of width $L-1$.

V. 8 Case $m = 2$

Note first that choosing $\omega_1 = -\omega_2 = \theta$ involves no loss of generality. We now solve the system of equations leading to the 4 coefficients of $R_L(z)$ and $Q_L(z)$ and obtain

\[ R_L(z) = \frac{1 - S(\theta) \cos(L \theta) + z S(\theta) \cos((L-1) \theta) \cos \theta \cos \theta}{D(\theta)} = \frac{S(\theta) \cos(L \theta) - \cos((L-1) \theta) \cos \theta \cos \theta}{D(\theta)} \]

\[ Q_L(z) = \frac{S(\theta) \cos(L \theta) - \cos((L-1) \theta) \cos \theta \cos \theta}{D(\theta)} \]

where $S(\theta)$ and $D(\theta)$ are defined as follows

\[ S(\theta) = \sin(L \theta) \]

\[ D(\theta) = \frac{L - 1}{2} (1 - S(\theta))^2 \]

VI. Conclusion

We have established a link between the shape of the periodogram and, on the one hand, the location of the peaks in the MUSIC spectrum, on the other hand the location of the "noise" zeros in the TK method. In the latter case, the zeros are proven to root a lacunary polynomial. Further work will use this property to obtain a better understanding of the behavior of the roots. A future paper will present the proof of the convergence of the roots to the unit circle (except for a finite number of them) when the model order tends to infinity.

REFERENCES