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A sampling theory for non-decaying signals ☆

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ABSTRACT

The classical assumption in sampling and spline theories is that the input signal is square-integrable, which prevents us from applying such techniques to signals that do not decay or even grow at infinity. In this paper, we develop a sampling theory for multidimensional non-decaying signals living in weighted L_p spaces. The sampling and reconstruction of an analog signal can be done by a projection onto a shift-invariant subspace generated by an interpolating kernel. We show that, if this kernel and its biorthogonal counterpart are elements of appropriate hybrid-norm spaces, then both the sampling and the reconstruction are stable. This is an extension of earlier results by Aldroubi and Gröchenig. The extension is required because it allows us to develop the theory for the ideal sampling of non-decaying signals in weighted Sobolev spaces. When the d-dimensional signal and its $d/p + \varepsilon$ derivatives, for arbitrarily small $\varepsilon > 0$, grow no faster than a polynomial in the L_p sense, the sampling operator is shown to be bounded even without a sampling kernel. As a consequence, the signal can also be interpolated from its samples with a nicely behaved interpolating kernel.

1. Introduction

The sampling theory has a rich history [1,2] starting with the classical Whittaker–Shannon–Kotel'nikov sampling theorem [3]. The crucial fact behind the sampling theorem is that bandlimited signals live in the *shift-invariant (spline-like) space* generated by the *sinc* function. Sampling theory has been extended for the general shift-invariant spaces generated by splines or wavelets, which exhibit better localization properties than the ideal sinc kernel [4–6]. Although a large body of work has been dedicated to sampling in shift-invariant spaces, [7–20] just to name a few, a general theory for sampling *non-decaying* signals seems to be still missing. There were some attempts to generalize the sampling theorem for bandlimited signals of polynomial growth [21–24], but the bandlimitedness requirement is too restrictive. Aldroubi and

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Gröchenig developed in [10] a theory for sampling signals in weighted L_p spaces with *moderate* weights. Although the authors did not emphasize this property, their theoretical framework can also handle the sampling of growing signals with a well-behaved sampling kernel as moderate weights might be decaying. However, in the absence of a prefilter, the *ideal* sampling of weighted L_p signals cannot be done stably. To the best of our knowledge, none of the existing theories apply to the ideal sampling of general functions that are non-decreasing, such as realizations of Brownian and Lévy processes, which happen to be intimately linked to splines [25–27]. Intuitively, however, there seems to be no fundamental reason to prevent one from sampling a continuous signal even if it is not decaying.

In the first half of this paper, we develop a theory for the *regular* sampling of *multidimensional* non-decaying signals that are modeled as elements of weighted L_p spaces in which the growth of the signals is controlled by a decaying weighting function. These signals can be sampled and then reconstructed by a projection onto the shift-invariant subspace generated by an interpolating kernel φ where the non-decaying coefficients are allowed to be in the corresponding weighted ℓ_p space. We shall show that both the sampling and the reconstruction are stable through Riesz-type bounds, provided that the kernel φ is an element of an appropriate weighted *hybrid-norm space* [28,29]. These results are extensions of what has been presented in [10] where the kernel φ is required to be in a weighted Wiener *amalgam space* [30–38], which is not sensitive to the power p of the function spaces. By relaxing amalgam spaces to hybrid-norm spaces, we can control p which, together with the weighting function, dictates the order of growth of the signals. These extensions are nontrivial and, more importantly, essential for us to develop the theory of ideal sampling.

The second half of the paper deals with the ideal sampling of non-decaying signals when a sampling kernel is not available. For this, we switch from the weighted L_p space to a weighted Sobolev space of signals whose *weak* derivatives up to some (fractional) order are in the weighted L_p space. We shall prove that the sampling of such a signal is bounded if the weighting function decays polynomially and the order of the weighted Sobolev space is above d/p, where d is the dimension. The beauty of this result is that, when p tends to infinity, a signal can be sampled if its ε derivatives are bounded by a polynomial for an arbitrarily small $\varepsilon > 0$. This condition is just slightly stronger than requiring the signal to be continuous. Interestingly, the result on the boundedness of the ideal sampling makes an intermediate step towards the proof of the Poisson summation formula for a general class of growing functions [39]. Once the signal can be stably sampled, we shall show that it can also be stably interpolated with an interpolating kernel lying in some appropriate hybrid-norm space.

The outline of this paper is as follows: In Section 2, we introduce notations and definitions of the relevant function spaces. In Section 3, we provide several Riesz-type bounds that establish the sampling theory for non-decaying signals in weighted L_p spaces. In Section 4, the boundedness of the ideal sampling operator is proved for non-decaying signals in weighted Sobolev spaces and the spline interpolation of such a signal is discussed. Proofs of most results in Section 3 are given in Section 5.

2. Notations and definitions

Throughout the paper, we deal with *complex-valued* multidimensional functions with bold letters denoting variables in $\mathbb{R}^d$ or $\mathbb{Z}^d$, where $d \geq 1$ is a fixed dimension. The Euclidean norm of a vector $\boldsymbol{x} \in \mathbb{R}^d$ is denoted by $\|\boldsymbol{x}\|$. The complex conjugate of a number $z \in \mathbb{C}$ is denoted by $\bar{z}$. The constants throughout the paper are denoted by C with subscripts that indicate the dependence of the constants on some parameters. For example, $C_{x,y,z}$ is a constant that depends only on x, y, and z.

Following the convention in signal processing, we use parentheses for functions and square brackets for sequences to specify their values at a particular point. Especially, for a function f, $f[\cdot]$ denotes the sequence $\{f(\boldsymbol{k})\}_{\boldsymbol{k}\in\mathbb{Z}^d}$. Also, $f^\vee$ denotes the reflection $f(-\cdot)$. For $1 \leq p \leq \infty$, we denote by p' the Hölder conjugate that satisfies $\frac{1}{p} + \frac{1}{p'} = 1$. As usual, the p-norms of a function f and a sequence c are respectively defined as

$$\|f\|_{L_p(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |f(\boldsymbol{x})|^p \mathrm{d}\boldsymbol{x} \right)^{1/p} \quad ; \quad \|c\|_{\ell_p(\mathbb{Z}^d)} := \left(\sum_{\boldsymbol{k}\in\mathbb{Z}^d} |c[\boldsymbol{k}]|^p \right)^{1/p},$$

with $1 \leq p < \infty$. When $p = \infty$, these norms should be adjusted as

$$\|f\|_{L_\infty(\mathbb{R}^d)} := \operatorname*{ess\,sup}_{\boldsymbol{x}\in\mathbb{R}^d} |f(\boldsymbol{x})|; \quad \|c\|_{\ell_\infty(\mathbb{Z}^d)} := \max_{\boldsymbol{k}\in\mathbb{Z}^d} |c[\boldsymbol{k}]|.$$

The spaces $L_p(\mathbb{R}^d)$ and $\ell_p(\mathbb{Z}^d)$ consist of functions and sequences, respectively, whose p-norms are finite. Note that, in order for the p-norm to be positive definite, each element of $L_p(\mathbb{R}^d)$ is considered as an equivalent class of functions that are equal *almost everywhere*. Most of the time we just write $f = g$ for two functions in the same equivalent class, but when extra care is needed, we switch to the notation $f \overset{\text{a.e.}}{=} g$. Equipped with these p-norms, the spaces $L_p(\mathbb{R}^d)$ and $\ell_p(\mathbb{Z}^d)$ become Banach spaces for all $p \geq 1$. The scalar product is defined as

$$\langle f, g\rangle := \int_{\mathbb{R}^d} f(\boldsymbol{x})g(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \quad \text{for } f \in L_p(\mathbb{R}^d), g \in L_{p'}(\mathbb{R}^d);$$

$$\langle a, b\rangle := \sum_{\boldsymbol{k}\in\mathbb{Z}^d} a[\boldsymbol{k}]b[\boldsymbol{k}], \quad \text{for } a \in \ell_p(\mathbb{Z}^d), b \in \ell_{p'}(\mathbb{Z}^d).$$

The notation $\langle\cdot,\cdot\rangle$ is also used for the action of a distribution on a test function. The reason why we did not put any complex conjugation in the above definition of the scalar product is to make it compatible with the linearity of distributions. We also adopt other standard notations used in Schwartz' distribution theory [40]. In particular, the Fourier transform of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ is defined as

$$\langle \mathcal{F}f, \varphi\rangle = \left\langle \hat{f}, \varphi \right\rangle := \langle f, \hat{\varphi}\rangle, \quad \forall\varphi \in \mathcal{S}(\mathbb{R}^d),$$

where

$$\hat{\varphi}(\boldsymbol{\xi}) := \int_{\mathbb{R}^d} \varphi(\boldsymbol{x})\mathrm{e}^{-2\pi\mathrm{j}\langle\boldsymbol{\xi},\boldsymbol{x}\rangle}\mathrm{d}\boldsymbol{x}.$$

The inverse Fourier-transform operator is denoted by $\mathcal{F}^{-1}$. The set of continuous functions is denoted by $\mathcal{C}(\mathbb{R}^d)$.

The non-decaying signals in this paper are modeled as elements of weighted L_p spaces. Their growth is controlled by some decaying weighting function, various types of which being extensively discussed in [41]. We assume implicitly throughout the paper that any weighting function is real-valued, positive, symmetric, and continuous.

Definition 1 *(Submultiplicative weights).* A weighting function $w : \mathbb{R}^d \to \mathbb{R}$ is called (weakly) submultiplicative if it is positive, symmetric, continuous, and there exists a constant C_w such that

$$w(\boldsymbol{x} + \boldsymbol{y}) \leq C_w\, w(\boldsymbol{x})w(\boldsymbol{y}), \quad \forall\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d. \tag{1}$$

From the above definition, the submultiplicativity of w implies $w(\boldsymbol{x}) \leq C_w\, w(\boldsymbol{x})w(\boldsymbol{0})$, or $w(\boldsymbol{0}) \geq 1/C_w$. On the other hand, $w(\boldsymbol{0}) = w(\boldsymbol{x} - \boldsymbol{x}) \leq C_w\, w(\boldsymbol{x})w(-\boldsymbol{x}) = C_w\,(w(\boldsymbol{x}))^2$ and, thus, $w(\boldsymbol{x}) \geq \sqrt{w(\boldsymbol{0})/C_w} \geq 1/C_w$, for all $\boldsymbol{x}$. This means that every submultiplicative weighting function is lower-bounded. An important example of submultiplicative weights is $w(\boldsymbol{x}) = (1 + \|\boldsymbol{x}\|^2)^{\alpha/2}$ for some $\alpha \geq 0$. For this weight, we can choose C_w to be $C_\alpha = 2^{\alpha/2}$.

Definition 2 *(Weighted L_p and ℓ_p spaces).* For $p \geq 1$ and a weighting function w, a function f is in $L_{p,w}(\mathbb{R}^d)$ if (fw) is in $L_p(\mathbb{R}^d)$; a sequence c is in $\ell_{p,w}(\mathbb{Z}^d)$ if $\{c[\boldsymbol{k}]w(\boldsymbol{k})\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ is in $\ell_p(\mathbb{Z}^d)$. The corresponding weighted norms are defined as

$$\|f\|_{L_{p,w}(\mathbb{R}^d)} := \|fw\|_{L_p(\mathbb{R}^d)}; \qquad \|c\|_{\ell_{p,w}(\mathbb{Z}^d)} := \|c\,w[\cdot]\|_{\ell_p(\mathbb{Z}^d)}.$$

Note that, if w is submultiplicative, then $L_{p,w}(\mathbb{R}^d) \subset L_p(\mathbb{R}^d)$ and $\ell_{p,w}(\mathbb{Z}^d) \subset \ell_p(\mathbb{Z}^d)$ because w is lower-bounded.

Definition 3 *(Weighted hybrid-norm spaces).* For $p, q \geq 1$, the hybrid-norm space $W_{p,q}(\mathbb{R}^d)$ includes all functions $f : \mathbb{R}^d \to \mathbb{C}$ whose norm defined as

$$\|f\|_{W_{p,q}(\mathbb{R}^d)} := \left(\int_{[0,1]^d} \left(\sum_{\boldsymbol{k}\in\mathbb{Z}^d} |f(\boldsymbol{x}+\boldsymbol{k})|^p \right)^{q/p} \mathrm{d}\boldsymbol{x} \right)^{1/q}$$

is finite, with usual adjustments when p or q is infinity. The weighted hybrid-norm space $W_{p,q,w}(\mathbb{R}^d)$ with respect to a weighting function w is defined according to the following weighted norm:

$$\|f\|_{W_{p,q,w}(\mathbb{R}^d)} := \|fw\|_{W_{p,q}(\mathbb{R}^d)}.$$

These hybrid- (mixed-) norm spaces were mentioned in [28,29]. They are similar to Wiener amalgam spaces $\widetilde{W}_{p,q}(\mathbb{R}^d)$ [30–32], where the discrete and continuous norms are mixed in reverse order, like

$$\|f\|_{\widetilde{W}_{p,q}(\mathbb{R}^d)} := \left(\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \left(\int_{[0,1]^d} |f(\boldsymbol{x}+\boldsymbol{k})|^q \, \mathrm{d}\boldsymbol{x} \right)^{p/q} \right)^{1/p}.$$

Deeper results regarding Wiener amalgam spaces and their generalizations can be found in [33–38]. We note some obvious inclusions of hybrid-norm spaces: $W_{p,q_2,w}(\mathbb{R}^d) \subset W_{p,q_1,w}(\mathbb{R}^d)$ when $1 \leq q_1 \leq q_2 \leq \infty$; and $W_{p_1,q,w}(\mathbb{R}^d) \subset W_{p_2,q,w}(\mathbb{R}^d)$ when $1 \leq p_1 \leq p_2 \leq \infty$. Also, it is easy to see that $W_{p,p,w}(\mathbb{R}^d) = L_{p,w}(\mathbb{R}^d)$.

3. Sampling in non-decaying shift-invariant spaces

In this section, we consider the sampling and reconstruction of non-decaying signals in some *weighted* L_p space by projecting them into a *shift-invariant* space spanned by the integer shifts of a generator. This generator function can be thought of as a reconstruction (synthesis) filter, whereas its *biorthogonal partner* plays the role of a sampling (analysis) filter. In order for the filtering and projection to make sense, we need some Riesz-type bounds for the shift-invariant space of non-decaying signals. Throughout this section, we fix $p \geq 1$ and assume that w is some submultiplicative weighting function. The *non-decaying* shift-invariant space associated with the kernel (generator) φ is defined as

$$V_{p,1/w}(\varphi) := \left\{ f = \sum_{k\in\mathbb{Z}^d} c[\boldsymbol{k}]\varphi(\cdot - \boldsymbol{k}) : c \in \ell_{p,1/w}(\mathbb{Z}^d) \right\}.$$

The decaying weight $1/w$ controls the growing rate of the signals living in $V_{p,1/w}(\varphi)$. When $w = 1$, we omit the subscript $1/w$ and simply write $V_p(\varphi)$. As a special case, $V_2(\mathrm{sinc})$ is nothing but the space of bandlimited signals in Shannon's sampling theory. A biorthogonal kernel $\tilde{\varphi}$ of φ is defined through the relation

$$\langle\tilde{\varphi}, \varphi(\cdot - \boldsymbol{k})\rangle = \delta[\boldsymbol{k}], \quad \forall \boldsymbol{k} \in \mathbb{Z}^d.$$

We shall show that, if the kernels φ and $\tilde{\varphi}$ are in appropriate hybrid-norm spaces, then the result of sampling a signal in $L_{p,1/w}(\mathbb{R}^d)$ using the kernel $\tilde{\varphi}$ followed by a reconstruction using the kernel φ is simply a projection onto $V_{p,1/w}(\varphi)$. The projection becomes orthogonal when $\tilde{\varphi}$ is also an element of $V_{p,1/w}(\varphi)$. In this case, we call it a *dual* kernel of φ and use the notation φ_{dual} instead of $\tilde{\varphi}$. It is well-known [2,7] that such a dual kernel exists and is unique if $\{\varphi(\cdot - \boldsymbol{k})\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ is a Riesz basis for $V_2(\varphi)$ or, equivalently, if $\sum_{\boldsymbol{k}\in\mathbb{Z}^d} |\hat{\varphi}(\boldsymbol{\xi} + \boldsymbol{k})|^2$ is bounded from above and below for almost every $\boldsymbol{\xi}$. The dual kernel φ_{dual} is then determined in the Fourier domain by

$$\widehat{\varphi_{\text{dual}}}(\boldsymbol{\xi}) = \frac{\hat{\varphi}(\boldsymbol{\xi})}{\sum_{\boldsymbol{k}\in\mathbb{Z}^d} |\hat{\varphi}(\boldsymbol{\xi} + \boldsymbol{k})|^2}.$$

We shall also show that, if the generator φ is in the hybrid-norm space $W_{1,q,w}(\mathbb{R}^d)$ for $q := \max(p, p')$ and if $\{\varphi(\cdot - \boldsymbol{k})\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ is a Riesz basis for $V_2(\varphi)$, then $V_{p,1/w}(\varphi)$ is a closed subspace of $L_{p,1/w}(\mathbb{R}^d)$ and $\{\varphi(\cdot - \boldsymbol{k})\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ is an unconditional basis for $V_{p,1/w}(\varphi)$. We stress that Aldroubi and Gröchenig in [10] obtained a similar result but only under the condition that φ lies in the stricter amalgam space $\widetilde{W}_{1,\infty,w}(\mathbb{R}^d)$. By Minkowski's inequality, it is not hard to see that $\widetilde{W}_{1,\infty,w}(\mathbb{R}^d) \subset \widetilde{W}_{1,q,w}(\mathbb{R}^d) \subset W_{1,q,w}(\mathbb{R}^d)$. This condition on the generator, however, is too strong for our present purpose because it does not allow us to control p. By operating on hybrid-norm spaces instead of Wiener amalgams, our main contribution in this section is to relax the admissibility condition on the generator of the shift-invariant space so that it now depends on p. This relaxation is required to prove the boundedness of the ideal sampling in Section 4.

Since we do prefilter the signal before taking its samples, it is important to make sure that the filtered signal is continuous. It is well-known that the convolution of an L_p function with an $L_{p'}$ function is continuous. Proposition 1 gives a weighted version of this fact.

Proposition 1. *Let $h \in L_{p,w}(\mathbb{R}^d)$ and $f \in L_{p',1/w}(\mathbb{R}^d)$, where $p \geq 1$ and w is a submultiplicative weighting function. Then, the convolution $g = h * f$ is continuous and included in $L_{\infty,1/w}(\mathbb{R}^d)$.*

Proof. See Section 5. □

Next, we present a series of Riesz-type bounds that relate the weighted discrete norm of the samples to the weighted continuous norms of the original and interpolated signals. We first show these bounds for a submultiplicative (growing) weight w and then use duality to obtain similar bounds for the decaying weight $1/w$.

Proposition 2. *Let $p \geq 1$ and let w be a submultiplicative weighting function. If $\varphi \in W_{1,p,w}(\mathbb{R}^d)$ and $c \in \ell_{p,w}(\mathbb{Z}^d)$, then the function $f = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} c[\boldsymbol{k}]\varphi(\cdot - \boldsymbol{k})$ belongs to $L_{p,w}(\mathbb{R}^d)$ and we have that*

$$\|f\|_{L_{p,w}(\mathbb{R}^d)} \leq C_w \|c\|_{\ell_{p,w}(\mathbb{Z}^d)} \|\varphi\|_{W_{1,p,w}(\mathbb{R}^d)},$$

where C_w is the constant given in (1).

Proof. See Section 5. □

Proposition 3. *Let $p \geq 1$ and let w be a submultiplicative weighting function. If $f \in L_{p,w}(\mathbb{R}^d)$ and $\varphi \in W_{1,p',w}(\mathbb{R}^d)$, then $f * \varphi$ is a well-defined continuous function. Its sampled sequence $\{c[\boldsymbol{k}] := (f * \varphi)(\boldsymbol{k})\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ belongs to $\ell_{p,w}(\mathbb{Z}^d)$ and the bound*

$$\|c\|_{\ell_{p,w}(\mathbb{Z}^d)} \leq C_w \, \|f\|_{L_{p,w}(\mathbb{R}^d)} \, \|\varphi\|_{W_{1,p',w}(\mathbb{R}^d)}$$

holds, where C_w is the constant given in (1).

Proof. See Section 5. □

Propositions 2 and 3 are required to develop our theory. They are an extension of some earlier results in [10] for which the generator φ has to be strictly in the amalgam space $\widetilde{W}_{1,\infty,w}(\mathbb{R}^d)$. This extension actually constitutes the technical part of the proofs. We now state the counterparts of these results for the decaying weighting function $1/w$. Based on the comment in [10] that $1/w$ is a *moderate* weight, part of the following results for $\varphi \in \widetilde{W}_{1,\infty,w}(\mathbb{R}^d)$ can be also be deduced from [10, Lemmas 2.9 & 2.10], although the authors did not specifically mention the possibility of sampling non-decaying signals. Our reason for using the weight $1/w$ is to explicitly indicate the sampling of non-decaying signals.

Proposition 4. *Let $p \geq 1$ and let w be a submultiplicative weighting function. If $\varphi \in W_{1,p,w}(\mathbb{R}^d)$ and $c \in \ell_{p,1/w}(\mathbb{Z}^d)$, then the function $f = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} c[\boldsymbol{k}]\varphi(\cdot - \boldsymbol{k})$ belongs to $L_{p,1/w}(\mathbb{R}^d)$ and we have that*

$$\|f\|_{L_{p,1/w}(\mathbb{R}^d)} \leq C_w \, \|c\|_{\ell_{p,1/w}(\mathbb{Z}^d)} \, \|\varphi\|_{W_{1,p,w}(\mathbb{R}^d)} \,,$$

where C_w is the constant given in (1).

Proof. By duality, we can express the $\frac{1}{w}$-weighted norm of f as

$$\|f\|_{L_{p,1/w}(\mathbb{R}^d)} = \sup_{\|g\|_{L_{p',w}(\mathbb{R}^d)}=1} \langle f, g\rangle \,.$$

Using Hölder's inequality and Proposition 3, we obtain

$$\begin{aligned}
\|f\|_{L_{p,1/w}(\mathbb{R}^d)} &= \sup_{\|g\|_{L_{p',w}(\mathbb{R}^d)}=1} \left\langle \sum_{\boldsymbol{k}\in\mathbb{Z}^d} c[\boldsymbol{k}]\varphi(\cdot - \boldsymbol{k}), g \right\rangle \\
&= \sup_{\|g\|_{L_{p',w}(\mathbb{R}^d)}=1} \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \frac{c[\boldsymbol{k}]}{w(\boldsymbol{k})} \cdot w(\boldsymbol{k}) \, \langle \varphi(\cdot - \boldsymbol{k}), g\rangle \\
&\leq \sup_{\|g\|_{L_{p',w}(\mathbb{R}^d)}=1} \|c\|_{\ell_{p,1/w}(\mathbb{Z}^d)} \cdot \|(g * \varphi^\vee)[\cdot]\|_{\ell_{p',w}(\mathbb{Z}^d)} \\
&\leq \|c\|_{\ell_{p,1/w}(\mathbb{Z}^d)} \sup_{\|g\|_{L_{p',w}(\mathbb{R}^d)}=1} C_w \, \|g\|_{L_{p',w}(\mathbb{R}^d)} \, \|\varphi^\vee\|_{W_{1,p,w}(\mathbb{R}^d)} \\
&= C_w \, \|c\|_{\ell_{p,1/w}(\mathbb{Z}^d)} \, \|\varphi\|_{W_{1,p,w}(\mathbb{R}^d)} \,,
\end{aligned}$$

which is the desired bound. □

Proposition 5. *Let $p \geq 1$ and let w be a submultiplicative weighting function. If $f \in L_{p,1/w}(\mathbb{R}^d)$ and $\varphi \in W_{1,p',w}(\mathbb{R}^d)$, then $f * \varphi$ is a well-defined continuous function. Its sampled sequence $\{c[\boldsymbol{k}] := (f * \varphi)(\boldsymbol{k})\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ belongs to $\ell_{p,1/w}(\mathbb{Z}^d)$ and the bound*

$$\|c\|_{\ell_{p,1/w}(\mathbb{Z}^d)} \leq C_w \, \|f\|_{L_{p,1/w}(\mathbb{R}^d)} \, \|\varphi\|_{W_{1,p',w}(\mathbb{R}^d)}$$

holds, where C_w is the constant given in (1).

Proof. As $\varphi \in W_{1,p',w}(\mathbb{R}^d) \subset W_{p',p',w}(\mathbb{R}^d) = L_{p,w}(\mathbb{R}^d)$, the continuity of $f * \varphi$ is due to Proposition 1. Similar to the proof of Proposition 4, we use the duality, Hölder's inequality, and Proposition 2 to obtain the desired bound as

$$\begin{aligned}
\|c\|_{\ell_{p,1/w}(\mathbb{Z}^d)} &= \sup_{\|b\|_{\ell_{p',w}(\mathbb{Z}^d)}=1} \langle b, c\rangle \\
&= \sup_{\|b\|_{\ell_{p',w}(\mathbb{Z}^d)}=1} \sum_{\boldsymbol{k}\in\mathbb{Z}^d} b[\boldsymbol{k}] \int_{\mathbb{R}^d} f(\boldsymbol{x})\varphi(\boldsymbol{k}-\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\
&= \sup_{\|b\|_{\ell_{p',w}(\mathbb{Z}^d)}=1} \int_{\mathbb{R}^d} \frac{f(\boldsymbol{x})}{w(\boldsymbol{x})} \cdot w(\boldsymbol{x}) \sum_{\boldsymbol{k}\in\mathbb{Z}^d} b[\boldsymbol{k}]\varphi(\boldsymbol{k}-\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\
&\le \|f\|_{L_{p,1/w}(\mathbb{R}^d)} \sup_{\|b\|_{\ell_{p',w}(\mathbb{Z}^d)}=1} \left\| \sum_{\boldsymbol{k}\in\mathbb{Z}^d} b[\boldsymbol{k}]\varphi^\vee(\cdot-\boldsymbol{k}) \right\|_{L_{p',w}(\mathbb{R}^d)} \\
&\le \|f\|_{L_{p,1/w}(\mathbb{R}^d)} \sup_{\|b\|_{\ell_{p',w}(\mathbb{Z}^d)}=1} C_w \|b\|_{\ell_{p',w}(\mathbb{Z}^d)} \|\varphi^\vee\|_{W_{1,p',w}(\mathbb{R}^d)} \\
&= C_w \|f\|_{L_{p,1/w}(\mathbb{R}^d)} \|\varphi\|_{W_{1,p',w}(\mathbb{R}^d)}. \qquad \square
\end{aligned}$$

Theorem 1 captures both the sampling and the reconstruction of signals in $L_{p,1/w}(\mathbb{R}^d)$.

Theorem 1. *Let $\varphi \in W_{1,p,w}(\mathbb{R}^d)$ and $\tilde{\varphi} \in W_{1,p',w}(\mathbb{R}^d)$ be two biorthogonal functions such that $\langle \varphi, \tilde{\varphi}(\cdot-\boldsymbol{k})\rangle = \delta[\boldsymbol{k}]$, $\forall \boldsymbol{k} \in \mathbb{Z}^d$. We also assume that the weighting function w is submultiplicative. Then, the linear operator*

$$P_\varphi f := \sum_{k\in\mathbb{Z}^d} \langle f, \tilde{\varphi}(\cdot-\boldsymbol{k})\rangle\, \varphi(\cdot-\boldsymbol{k})$$

is a projector that continuously maps $L_{p,1/w}(\mathbb{R}^d)$ into the subspace $V_{p,1/w}(\varphi) \subset L_{p,1/w}(\mathbb{R}^d)$.

Proof. The fact that P_φ is a projector (i.e., $P_\varphi^2 = P_\varphi$) simply follows from the biorthogonality of φ and $\tilde{\varphi}$. In particular, for all $f \in L_{p,1/w}(\mathbb{R}^d)$, we have that

$$\begin{aligned}
P_\varphi^2 f &= \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \langle P_\varphi f, \tilde{\varphi}(\cdot-\boldsymbol{k})\rangle\, \varphi(\cdot-\boldsymbol{k}) \\
&= \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \left\langle \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} \langle f, \tilde{\varphi}(\cdot-\boldsymbol{\ell})\rangle\, \varphi(\cdot-\boldsymbol{\ell}), \tilde{\varphi}(\cdot-\boldsymbol{k}) \right\rangle \varphi(\cdot-\boldsymbol{k}) \\
&= \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} \langle f, \tilde{\varphi}(\cdot-\boldsymbol{\ell})\rangle\, \delta[\boldsymbol{k}-\boldsymbol{\ell}]\varphi(\cdot-\boldsymbol{k}) \\
&= \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \langle f, \tilde{\varphi}(\cdot-\boldsymbol{k})\rangle\, \varphi(\cdot-\boldsymbol{k}) \\
&= P_\varphi f.
\end{aligned}$$

To establish the continuity, we only need to show that P_φ is bounded in the $L_{p,1/w}$-norm. We note that the expansion coefficients of $P_\varphi f$ are given by $c[\boldsymbol{k}] = \langle f, \tilde{\varphi}(\cdot-\boldsymbol{k})\rangle = (f * \tilde{\varphi}^\vee)(\boldsymbol{k})$, which allows us to invoke Proposition 5 to deduce that

$$\|c\|_{\ell_{p,1/w}(\mathbb{Z}^d)} \le C_w \|\tilde{\varphi}\|_{W_{1,p',w}(\mathbb{R}^d)} \|f\|_{L_{p,1/w}(\mathbb{R}^d)}.$$

Using this bound and Proposition 4, we obtain

$$\|P_\varphi f\|_{L_{p,1/w}(\mathbb{R}^d)} \le C_w \|c\|_{\ell_{p,1/w}(\mathbb{Z}^d)} \|\varphi\|_{W_{1,p,w}(\mathbb{R}^d)} \le \underbrace{C_w^2 \|\varphi\|_{W_{1,p,w}(\mathbb{R}^d)} \|\tilde{\varphi}\|_{W_{1,p',w}(\mathbb{R}^d)}}_{\text{constant}} \|f\|_{L_{p,1/w}(\mathbb{R}^d)},$$

which shows the boundedness of the operator P_φ. □

Before presenting Theorem 2, which is the central result of this section, we need two complementary results.

Lemma 1. *Let $p, q \ge 1$ and let w be a submultiplicative weighting function. If $\varphi \in W_{p,q,w(\mathbb{R}^d)}$ and $c \in \ell_{1,w}(\mathbb{Z}^d)$, then the function $f = \sum_{\boldsymbol{k} \in \mathbb{Z}^d} c[\boldsymbol{k}]\varphi(\cdot - \boldsymbol{k})$ also belongs to $W_{p,q,w}(\mathbb{R}^d)$ and we have that*

$$\|f\|_{W_{p,q,w}(\mathbb{R}^d)} \le C_w \|c\|_{\ell_{1,w}(\mathbb{Z}^d)} \|\varphi\|_{W_{p,q,w}(\mathbb{R}^d)},$$

where C_w is the constant given in (1).

Proof. See Section 5. □

Proposition 6. *Let $q \ge 2$ and let w be a submultiplicative weighting function that additionally satisfies the Gelfand–Raikov–Shilov (GRS) condition*

$$\lim_{n\to\infty} w(n\boldsymbol{k})^{1/n} = 1, \quad \forall \boldsymbol{k} \in \mathbb{Z}^d. \tag{2}$$

Suppose that the generator φ is in $W_{1,q,w}(\mathbb{R}^d)$ and that $\{\varphi(\cdot - \boldsymbol{k})\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ is a Riesz basis for $V_2(\varphi)$. Then, the dual generator φ_{dual} is in $W_{1,q,w}(\mathbb{R}^d)$ as well.

Proof. See Section 5. □

Theorem 2. *Let $p \ge 1$, $q = \max(p, p')$, and let w be a submultiplicative weighting function satisfying the GRS condition. Assume that $\varphi \in W_{1,q,w}$ and that $\{\varphi(\cdot - \boldsymbol{k})\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ is a Riesz basis for $V_2(\varphi)$. Then, there exist constants $C_{w,p,\varphi} < \infty$ and $\widetilde{C}_{w,p,\varphi} > 0$ such that*

$$\widetilde{C}_{w,p,\varphi} \|c\|_{\ell_{p,1/w}(\mathbb{Z}^d)} \le \left\| \sum_{\boldsymbol{k}\in\mathbb{Z}^d} c[\boldsymbol{k}]\varphi(\cdot - \boldsymbol{k}) \right\|_{L_{p,1/w}(\mathbb{R}^d)} \le C_{w,p,\varphi} \|c\|_{\ell_{p,1/w}(\mathbb{Z}^d)}, \quad \forall c \in \ell_{p,1/w}(\mathbb{Z}^d). \tag{3}$$

This norm equivalence implies that $V_{p,1/w}(\varphi)$ is a closed subspace of $L_{p,1/w}(\mathbb{R}^d)$ and that $\{\varphi(\cdot - \boldsymbol{k})\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ is an unconditional basis for $V_{p,1/w}(\varphi)$.

Proof. Let $f := \sum_{\boldsymbol{k}\in\mathbb{Z}^d} c[\boldsymbol{k}]\varphi(\cdot - \boldsymbol{k})$. The right-hand-side inequality of (3) can be obtained by invoking Proposition 4

$$\begin{aligned} \|f\|_{L_{p,1/w}(\mathbb{R}^d)} &\le C_w \|c\|_{\ell_{p,1/w}(\mathbb{Z}^d)} \|\varphi\|_{W_{1,p,w}(\mathbb{R}^d)} \\ &\le C_w \|c\|_{\ell_{p,1/w}(\mathbb{Z}^d)} \|\varphi\|_{W_{1,q,w}(\mathbb{R}^d)} \qquad (4) \\ &= C_{w,p,\varphi} \|c\|_{\ell_{p,1/w}(\mathbb{Z}^d)}, \end{aligned}$$

where (4) is a consequence of Hölder's inequality and the fact that $q \ge p$, and where the constant $C_{w,p,\varphi}$ is equal to $C_w \|\varphi\|_{W_{1,q,w}(\mathbb{R}^d)}$.

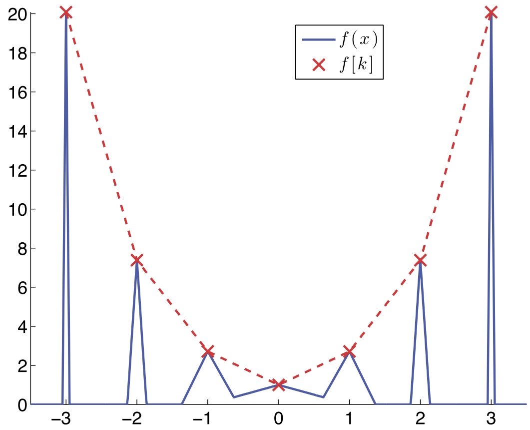

Fig. 1. Illustration of a continuous function $f(x) \in L_1(\mathbb{R})$ whose sampled sequence $f[k]$ grows exponentially.

The left-hand-side inequality can be obtained by noting that $c[\boldsymbol{k}] = \langle f, \varphi_{\text{dual}}(\cdot - \boldsymbol{k})\rangle = (f * \varphi^{\vee}_{\text{dual}})(\boldsymbol{k}), \forall \boldsymbol{k} \in \mathbb{Z}^d$. Since $q = \max(p, p') \geq 2$ and from Proposition 6, it must be that $\varphi_{\text{dual}} \in W_{1,q,w}(\mathbb{R}^d)$, and so $\varphi^{\vee}_{\text{dual}} \in W_{1,q,w}(\mathbb{R}^d)$. This allows us to invoke Proposition 5 to get

$$\begin{aligned}\|c\|_{\ell_{p,1/w}(\mathbb{Z}^d)} &\leq C_w \ \|f\|_{L_{p,1/w}(\mathbb{R}^d)} \ \|\varphi^{\vee}_{\text{dual}}\|_{W_{1,p',w}(\mathbb{R}^d)} = C_w \ \|f\|_{L_{p,1/w}(\mathbb{R}^d)} \ \|\varphi_{\text{dual}}\|_{W_{1,p',w}(\mathbb{R}^d)} \\ &\leq C_w \ \|f\|_{L_{p,1/w}(\mathbb{R}^d)} \ \|\varphi_{\text{dual}}\|_{W_{1,q,w}(\mathbb{R}^d)} \qquad (5) \\ &= \widetilde{C}_{w,p,\varphi} \ \|f\|_{L_{p,1/w}(\mathbb{R}^d)}\,,\end{aligned}$$

where (5) is a consequence of Hölder's inequality and the fact that $q \geq p'$, and where the constant $\widetilde{C}_{w,p,\varphi}$ is equal to $C_w \ \|\varphi_{\text{dual}}\|_{W_{1,q,w}(\mathbb{R}^d)}$. $\square$

4. Ideal sampling

4.1. Sampling in weighted Sobolev spaces

We have considered so far the sampling of growing signals with a decaying sampling kernel in some weighted hybrid-norm space. In the case of ideal sampling, the kernel is just a Dirac impulse so that we need to restrict the signals to a subspace of weighted-L_p with some order of smoothness; namely, a weighted Sobolev space. (As illustrated in Fig. 1, if no smoothness is imposed on the analog signal, its samples might blow up exponentially even though the signal is continuous and absolutely integrable.) In the most basic sense, a Sobolev space L_p^k of order k, for a natural number k, consists of functions whose derivatives up to order k are all included in L_p. The concept can be extended for weighted-L_p spaces with a fractional order s by means of the Fourier transform. As we are dealing with non-decaying functions, their Fourier transforms should be treated in the generalized sense of distributions.

Definition 4 *(Weighted Sobolev spaces)*. For $s \in \mathbb{R}$, $1 < p < \infty$, and for a weighting function w, the weighted Sobolev space $L^s_{p,w}(\mathbb{R}^d)$ is defined as

$$L^s_{p,w}(\mathbb{R}^d) := \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \mathcal{F}^{-1}\left\{(1 + \|\cdot\|^2)^{\frac{s}{2}} \hat{f}\right\} \in L_{p,w}(\mathbb{R}^d)\right\}.$$

This space is equipped with the weighted Sobolev norm

$$\|f\|_{L^s_{p,w}(\mathbb{R}^d)} := \left\|\mathcal{F}^{-1}\left\{(1 + \|\cdot\|^2)^{\frac{s}{2}} \hat{f}\right\}\right\|_{L_{p,w}(\mathbb{R}^d)}.$$

In Definition 4, because $(1+\|\cdot\|^2)^{\frac{s}{2}}$ is a smooth and slowly increasing function for every s, the multiplication $(1+\|\cdot\|^2)^{\frac{s}{2}}\hat{f}$ is a well-defined tempered distribution. An important observation is that any distribution $f \in L^s_{p,w}(\mathbb{R}^d)$ can be written as the convolution $f = f_s * \varphi_s$, where $f_s := \mathcal{F}^{-1}\left\{(1+\|\cdot\|^2)^{\frac{s}{2}}\hat{f}\right\} \in L_{p,w}(\mathbb{R}^d)$, and $\varphi_s := \mathcal{F}^{-1}\left\{(1+\|\cdot\|^2)^{-\frac{s}{2}}\right\}$. In the literature, the function φ_s when $s > 0$ is often referred to as *Bessel potential kernel*. The following properties of this kernel will be useful later. First, it is easy to see that φ_s is a real and symmetric function. Second, it is well-known from Sobolev space theory (see for example [42, Proposition 6.1.5]) that φ_s is a positive function that decays exponentially outside a neighborhood of the origin. More precisely, there exists a constant C_s such that

$$\varphi_s(\boldsymbol{x}) \le C_s\, \mathrm{e}^{-\pi\|\boldsymbol{x}\|}, \quad \forall\, \|\boldsymbol{x}\| > \frac{1}{\pi}. \tag{6}$$

Third, it is also known that $\varphi_s \in L_{p'}(\mathbb{R}^d)$ whenever $s > d/p$ (cf. [42, page 14]). Proposition 7 gives a stronger property of φ_s.

Proposition 7. *Let $1 < p < \infty$, $s > d/p$, and $w(\boldsymbol{x}) = (1+\|\boldsymbol{x}\|^2)^{\alpha/2}$ for some $\alpha \ge 0$. Then, the Bessel potential kernel φ_s is an element of the weighted hybrid-norm space $W_{1,p',w}(\mathbb{R}^d)$.*

Proof. We first show that $\varphi_s \in L_{p',w}(\mathbb{R}^d)$. Indeed, from the listed properties of φ_s we have that

$$\begin{aligned}\|\varphi_s\|^{p'}_{L_{p',w}(\mathbb{R}^d)} &= \int\limits_{\|\boldsymbol{x}\|\le\frac{1}{\pi}} |w(\boldsymbol{x})|^{p'}|\varphi_s(\boldsymbol{x})|^{p'}\mathrm{d}\boldsymbol{x} + \int\limits_{\|\boldsymbol{x}\|\ge\frac{1}{\pi}} |w(\boldsymbol{x})|^{p'}|\varphi_s(\boldsymbol{x})|^{p'}\mathrm{d}\boldsymbol{x} \\ &\le \|\varphi_s\|^{p'}_{L_{p'}(\mathbb{R}^d)} \sup_{\|\boldsymbol{x}\|\le\frac{1}{\pi}} |w(\boldsymbol{x})|^{p'} + C_s \int\limits_{\|\boldsymbol{x}\|\ge\frac{1}{\pi}} (1+\|\boldsymbol{x}\|^2)^{\frac{p'\alpha}{2}}\,\mathrm{e}^{-p'\pi\|\boldsymbol{x}\|}\mathrm{d}\boldsymbol{x} < \infty.\end{aligned}$$

Next, we show that $\varphi_s \in W_{1,p',w}(\mathbb{R}^d)$ or, equivalently, that $\left\|\sum_{\boldsymbol{k}\in\mathbb{Z}^d}\varphi_{s,w}(\cdot+\boldsymbol{k})\right\|_{L_{p'}(\mathbb{T}^d)}$ is finite, where $\mathbb{T} := [0,1]$ and $\varphi_{s,w} := w\varphi_s$. In order to achieve that, we break this norm in two parts using Minkowski's inequality

$$\left\|\sum_{\boldsymbol{k}\in\mathbb{Z}^d}\varphi_{s,w}(\cdot+\boldsymbol{k})\right\|_{L_{p'}(\mathbb{T}^d)} \le \underbrace{\left\|\sum_{\|\boldsymbol{k}\|\le\sqrt{d}+\frac{1}{\pi}}\varphi_{s,w}(\cdot+\boldsymbol{k})\right\|_{L_{p'}(\mathbb{T}^d)}}_{A} + \underbrace{\left\|\sum_{\|\boldsymbol{k}\|>\sqrt{d}+\frac{1}{\pi}}\varphi_{s,w}(\cdot+\boldsymbol{k})\right\|_{L_{p'}(\mathbb{T}^d)}}_{B}, \tag{7}$$

and then show that each of the two terms in the RHS of (7) is finite. On one hand, the term A can be bounded since

$$\begin{aligned}A &\le \sum_{\|\boldsymbol{k}\|\le\sqrt{d}+\frac{1}{\pi}} \|\varphi_{s,w}(\cdot+\boldsymbol{k})\|_{L_{p'}(\mathbb{R}^d)} \\ &= \#\left\{\boldsymbol{k}\in\mathbb{Z}^d : \|\boldsymbol{k}\| \le \sqrt{d}+\frac{1}{\pi}\right\}\cdot\|\varphi_s\|_{L_{p',w}(\mathbb{R}^d)}\end{aligned}$$

is finite because $\varphi_s \in L_{p',w}(\mathbb{R}^d)$ and the set $\{\boldsymbol{k}\in\mathbb{Z}^d : \|\boldsymbol{k}\| \le \sqrt{d}+1/\pi\}$ has finite elements. In this estimate, we have extended the integrating region from $\mathbb{T}^d$ to $\mathbb{R}^d$. On the other hand, the term B can be bounded as

$$B = \left(\int\limits_{[0,1]^d}\left(\sum_{\|\boldsymbol{k}\|>\sqrt{d}+\frac{1}{\pi}}\varphi_{s,w}(\boldsymbol{x}+\boldsymbol{k})\right)^{p'}\mathrm{d}\boldsymbol{x}\right)^{1/p'} \tag{8}$$

$$\le C_s \left(\int\limits_{[0,1]^d} \left(\sum_{\|\boldsymbol{k}\| > \sqrt{d} + \frac{1}{\pi}} w(\boldsymbol{x}+\boldsymbol{k}) \mathrm{e}^{-\pi\|\boldsymbol{x}+\boldsymbol{k}\|} \right)^{p'} \mathrm{d}\boldsymbol{x} \right)^{1/p'} \tag{9}$$

$$\le C_{s,\alpha} \left(\int\limits_{[0,1]^d} (1+\|\boldsymbol{x}\|^2)^{\frac{p'\alpha}{2}} \mathrm{e}^{p'\pi\|\boldsymbol{x}\|} \, \mathrm{d}\boldsymbol{x} \right)^{1/p'} \cdot \sum_{\boldsymbol{k}\in\mathbb{Z}^d} (1+\|\boldsymbol{k}\|^2)^{\frac{\alpha}{2}} \mathrm{e}^{-\pi\|\boldsymbol{k}\|} \tag{10}$$

$$\le C_{s,\alpha} \sup_{\boldsymbol{x}\in[0,1]^d} (1+\|\boldsymbol{x}\|^2)^{\frac{\alpha}{2}} \mathrm{e}^{\pi\|\boldsymbol{x}\|} \cdot \sum_{\boldsymbol{k}\in\mathbb{Z}^d} (1+\|\boldsymbol{k}\|^2)^{\frac{\alpha}{2}} \mathrm{e}^{-\pi\|\boldsymbol{k}\|}$$

$$= C_{s,\alpha} (1+d)^{\frac{\alpha}{2}} \mathrm{e}^{\pi\sqrt{d}} \cdot \sum_{\boldsymbol{k}\in\mathbb{Z}^d} (1+\|\boldsymbol{k}\|^2)^{\frac{\alpha}{2}} \mathrm{e}^{-\pi\|\boldsymbol{k}\|} \tag{11}$$

$$< +\infty,$$

where (9) follows from (6); (10) follows from the submultiplicativity of w and from $\|\boldsymbol{x}+\boldsymbol{k}\| \ge \|\boldsymbol{k}\| - \|\boldsymbol{x}\|$. Finally, the sum in (11) is finite because the polynomial growth $(1+\|\boldsymbol{k}\|^2)^{\alpha/2}$ will be dominated by the exponential decay $\mathrm{e}^{-\pi\|\boldsymbol{k}\|}$ when $\|\boldsymbol{k}\|$ is large enough. The proof is completed. □

Note that $L^s_{p,w}(\mathbb{R}^d)$ is identical to $L_{p,w}(\mathbb{R}^d)$ when $s = 0$. Is it true that every element of $L^s_{p,w}(\mathbb{R}^d)$ is an $L_{p,w}(\mathbb{R}^d)$ function? Proposition 8 gives an affirmative answer to that question when $s > 0$ and the weighting function w is decaying polynomially.

Proposition 8. *If $1 < p < \infty$, $s > 0$, and $w(\boldsymbol{x}) = (1+\|\boldsymbol{x}\|^2)^{\alpha/2}$ for some $\alpha \ge 0$, then $L^s_{p,1/w} \subset L_{p,1/w}$.*

Proof. As a special case of Proposition 7, we have that $\varphi_s \in W_{1,1,w}(\mathbb{R}^d) = L_{1,w}(\mathbb{R}^d)$, for all $s > 0$. Let f be an element of $L^s_{p,1/w}(\mathbb{R}^d)$. By duality we can write

$$\|f\|_{L_{p,1/w}(\mathbb{R}^d)} = \sup_{\|g\|_{L_{p',w}(\mathbb{R}^d)}=1} \langle f, g \rangle .$$

Using the submultiplicativity of w, Hölder's, and Young's inequalities, we have that

$$\begin{aligned}
\|f\|_{L_{p,1/w}(\mathbb{R}^d)} &= \sup_{\|g\|_{L_{p',w}(\mathbb{R}^d)}=1} \langle f_s * \varphi_s, g \rangle \\
&= \sup_{\|g\|_{L_{p',w}(\mathbb{R}^d)}=1} \left\langle \frac{f_s}{w}, w \cdot (g * \varphi_s) \right\rangle \\
&\le \sup_{\|g\|_{L_{p',w}(\mathbb{R}^d)}=1} \|f_s\|_{L_{p,1/w}(\mathbb{R}^d)} \, C_\alpha \, \|g\|_{L_{p',w}(\mathbb{R}^d)} \, \|\varphi_s\|_{L_{1,w}(\mathbb{R}^d)} \\
&= C_\alpha \, \|f_s\|_{L_{p,1/w}(\mathbb{R}^d)} \, \|\varphi_s\|_{L_{1,w}(\mathbb{R}^d)} < \infty,
\end{aligned}$$

which implies that $f \in L_{p,1/w}(\mathbb{R}^d)$, completing the proof. □

We are now ready to state the main result of this section about the boundedness of the sampling operator for non-decaying signals of sufficient smoothness in the L_p sense. While L_p functions are equivalent classes, we select the unique member that is continuous and can therefore be safely sampled.

Theorem 3. *Let $1 < p < \infty$, $s > d/p$, and $w(\boldsymbol{x}) = (1+\|\boldsymbol{x}\|^2)^{\alpha/2}$ for some $\alpha \ge 0$. Then, the sampling operator $f \mapsto f|_{\mathbb{Z}^d}$ is bounded from $L^s_{p,1/w}(\mathbb{R}^d) \cap \mathcal{C}(\mathbb{R}^d)$ to $\ell_{p,1/w}(\mathbb{Z}^d)$, i.e., there exists a constant $C_{\alpha,s,p}$ depending only on α, s, and p, such that*

$$\|f[\cdot]\|_{\ell_{p,1/w}(\mathbb{Z}^d)} \leq C_{\alpha,s,p} \|f\|_{L^s_{p,1/w}(\mathbb{R}^d)}, \quad \forall f \in L^s_{p,1/w}(\mathbb{R}^d) \cap \mathcal{C}(\mathbb{R}^d). \tag{12}$$

Proof. From Proposition 7, we know that $f \overset{\text{a.e.}}{=} f_s * \varphi_s$, where $f_s \in L_{p,1/w}(\mathbb{R}^d)$ and $\varphi_s \in W_{1,p',w}(\mathbb{R}^d)$. Since $W_{1,p',w}(\mathbb{R}^d) \subset W_{p',p',w}(\mathbb{R}^d) = L_{p',w}(\mathbb{R}^d)$, it follows from Proposition 1 that the convolution $f_s * \varphi_s$ is continuous everywhere. Combining this with the fact that $f \overset{\text{a.e.}}{=} f_s * \varphi_s$ and f is continuous, we deduce that $f = f_s * \varphi_s$ everywhere.

Finally, it is safe to write $f(\boldsymbol{k}) = (f_s * \varphi_s)(\boldsymbol{k})$, for all $\boldsymbol{k} \in \mathbb{Z}^d$, and invoke Proposition 5 to get

$$\begin{aligned} \|f[\cdot]\|_{\ell_{p,1/w}(\mathbb{Z}^d)} &\leq C_\alpha \|\varphi_s\|_{W_{1,p',w}(\mathbb{R}^d)} \|f_s\|_{L_{p,1/w}(\mathbb{R}^d)} \\ &= \underbrace{C_\alpha \|\varphi_s\|_{W_{1,p',w}(\mathbb{R}^d)}}_{C_{\alpha,s,p}} \|f\|_{L^s_{p,1/w}(\mathbb{R}^d)}, \end{aligned}$$

which shows that (12) is true, completing the proof. □

4.2. Spline interpolation

From Section 4.1, a non-decaying signal can be stably sampled if its weak derivatives up to order $d/p + \varepsilon$ are bounded by a polynomial (in the L_p sense). Now, consider the spline interpolation of such a signal $f(\boldsymbol{x})$ from its samples $\{f(\boldsymbol{k})\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ using an interpolant φ_{int}. The interpolated signal is given by

$$f_{\text{int}}(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} f(\boldsymbol{k})\varphi_{\text{int}}(\boldsymbol{x} - \boldsymbol{k}). \tag{13}$$

To make sure that $f_{\text{int}}(\boldsymbol{k}) = f(\boldsymbol{k})$, for all $\boldsymbol{k} \in \mathbb{Z}^d$, the function φ_{int} has to satisfy the interpolation condition

$$\varphi_{\text{int}}(\boldsymbol{k}) = \delta[\boldsymbol{k}], \quad \forall \boldsymbol{k} \in \mathbb{Z}^d.$$

Unfortunately, this condition significantly limits the choice of an interpolating kernel with desirable properties (such as localization, smoothness, etc.). Therefore, it is preferable to perform the interpolation on the filtered version of the samples of the signal using a better kernel such as B-splines [4–6]. More precisely, the interpolated signal can be alternatively obtained by

$$f_{\text{int}}(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} c[\boldsymbol{k}]\varphi(\boldsymbol{x} - \boldsymbol{k}), \tag{14}$$

where φ is the chosen kernel (such as a B-spline), and the coefficients $\{c[\boldsymbol{k}]\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ are obtained by filtering the samples $\{f[\boldsymbol{k}]\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ with a digital interpolation filter $h[\cdot]$ whose discrete-domain Fourier transform is given by

$$\hat{h}(\boldsymbol{\xi}) := \frac{1}{\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \varphi(\boldsymbol{k})\mathrm{e}^{-2\pi\mathrm{j}\langle\boldsymbol{\xi},\boldsymbol{k}\rangle}}. \tag{15}$$

In (15), we assumed that $\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \varphi(\boldsymbol{k})\mathrm{e}^{-2\pi\mathrm{j}\langle\boldsymbol{\xi},\boldsymbol{k}\rangle} \neq 0$, for almost all $\boldsymbol{\xi} \in \mathbb{R}^d$. The interpolant φ_{int} and the kernel φ are then related by

$$\varphi_{\text{int}} = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} h[\boldsymbol{k}]\varphi(\cdot - \boldsymbol{k}). \tag{16}$$

The bound of Proposition 9 provides a safeguard for the spline interpolation of a non-decaying signal, given that the original signal lies in some weighted Sobolev space with a large-enough order of smoothness and the interpolating kernel behaves nicely.

Proposition 9. *Let $1 < p < \infty$, $s > d/p$, and $w(\boldsymbol{x}) = (1 + \|\boldsymbol{x}\|^2)^{\alpha/2}$ for some $\alpha \geq 0$. Assume that $\varphi \in W_{1,p,w}(\mathbb{R}^d)$, $\varphi[\cdot] \in \ell_{1,w}(\mathbb{Z}^d)$, and that $\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \varphi(\boldsymbol{k})\mathrm{e}^{-2\pi \mathrm{j}\langle\boldsymbol{\xi},\boldsymbol{k}\rangle}$ is nonzero for almost all $\boldsymbol{\xi}$. If $f \in L^s_{p,1/w}(\mathbb{R}^d) \cap \mathcal{C}(\mathbb{R}^d)$, then the interpolated function f_{int} given in (14) is included in $L_{p,1/w}(\mathbb{R}^d)$, and we have that*

$$C_{\alpha,p,\varphi} \, \|f_{\mathrm{int}}\|_{L_{p,1/w}(\mathbb{R}^d)} \leq \|f[\cdot]\|_{\ell_{p,1/w}(\mathbb{Z}^d)} \leq C_{\alpha,s,p} \, \|f\|_{L^s_{p,1/w}(\mathbb{R}^d)} . \tag{17}$$

Proof. Since the right-hand-side inequality follows directly from Theorem 3, we only need to show the left-hand-side inequality. Note that the weight $w(\boldsymbol{x}) = (1 + \|\boldsymbol{x}\|^2)^{\alpha/2}$ is submultiplicative and satisfies the GRS condition specified in (2). Since $\varphi[\cdot] \in \ell_{1,w}(\mathbb{Z}^d)$ and $\hat{h}(\boldsymbol{\xi}) = 1/\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \varphi(\boldsymbol{k})\mathrm{e}^{-2\pi \mathrm{j}\langle\boldsymbol{\xi},\boldsymbol{k}\rangle}$, it follows from the weighted version of Wiener's lemma [41, Theorem 6.2] that the interpolation filter h is a sequence in $\ell_{1,w}(\mathbb{Z}^d)$. Combining this with (16) and Lemma 1, we deduce that the interpolant φ_{int} also belongs to $W_{1,p,w}(\mathbb{R}^d)$. It then follows from (13) and Proposition 4 that

$$\|f_{\mathrm{int}}\|_{L_{p,1/w}(\mathbb{R}^d)} \leq \underbrace{C_\alpha \, \|\varphi_{\mathrm{int}}\|_{W_{1,p,w}(\mathbb{R}^d)}}_{1/C_{\alpha,p,\varphi}} \|f[\cdot]\|_{\ell_{p,1/w}(\mathbb{Z}^d)} ,$$

which proves the left-hand-side inequality. □

5. Proofs

5.1. Proof of Proposition 1

We first show that g is a well-defined function in $L_{\infty,1/w}(\mathbb{R}^d)$ using Hölder's inequality and the submultiplicativity of w. In particular, for all $\boldsymbol{x} \in \mathbb{R}^d$, we have that

$$\begin{aligned} \frac{1}{w(\boldsymbol{x})}|(h * f)(\boldsymbol{x})| &= \left| \int_{\mathbb{R}^d} w(\boldsymbol{y})h(\boldsymbol{y}) \frac{f(\boldsymbol{x}-\boldsymbol{y})}{w(\boldsymbol{x})w(-\boldsymbol{y})} \,\mathrm{d}\boldsymbol{y} \right| \\ &\leq C_w \left| \int_{\mathbb{R}^d} w(\boldsymbol{y})h(\boldsymbol{y}) \frac{f(\boldsymbol{x}-\boldsymbol{y})}{w(\boldsymbol{x}-\boldsymbol{y})} \,\mathrm{d}\boldsymbol{y} \right| \\ &\leq C_w \|wh\|_{L_p(\mathbb{R}^d)} \, \|f/w\|_{L_{p'}(\mathbb{R}^d)} \\ &= C_w \|h\|_{L_{p,w}(\mathbb{R}^d)} \, \|f\|_{L_{p',1/w}(\mathbb{R}^d)}. \end{aligned} \tag{18}$$

To prove the continuity of g, we first recall a classical result in functional analysis: $\mathcal{C}_{\mathrm{c}}(\mathbb{R}^d)$, the set of continuous and compactly supported functions, is dense in $L_p(\mathbb{R}^d)$. Since $\mathcal{C}_{\mathrm{c}}(\mathbb{R}^d) \subset L_{p,w}(\mathbb{R}^d) \subset L_p(\mathbb{R}^d)$, it follows that $\mathcal{C}_{\mathrm{c}}(\mathbb{R}^d)$ is also dense in $L_{p,w}(\mathbb{R}^d)$. Hence, it suffices to show that g is continuous for $h \in \mathcal{C}_{\mathrm{c}}(\mathbb{R}^d)$. This assumption allows us to pick an interval $[-a, a]^d$ that includes the supports of both h and $h(\cdot - \boldsymbol{x}_0)$ for $\boldsymbol{x}_0$ sufficiently small. By the shift-invariance of convolution and the bound (18) we have that, for any $\boldsymbol{x} \in \mathbb{R}^d$,

$$\begin{aligned} |g(\boldsymbol{x}) - g(\boldsymbol{x}-\boldsymbol{x}_0)| &= |((h - h(\cdot - \boldsymbol{x}_0)) * f)(\boldsymbol{x})| \\ &\leq C_w \, w(\boldsymbol{x})\|f\|_{L_{p',1/w}(\mathbb{R}^d)} \, \|h - h(\cdot - \boldsymbol{x}_0)\|_{L_{p,w}(\mathbb{R}^d)} \\ &\leq C_w \, w(\boldsymbol{x})\|f\|_{L_{p',1/w}(\mathbb{R}^d)} \, (2a)^{\frac{d}{p}} \sup_{\boldsymbol{y}\in[-a,a]} w(\boldsymbol{y}) \sup_{\boldsymbol{y}\in[-a,a]} |h(\boldsymbol{y}) - h(\boldsymbol{y}-\boldsymbol{x}_0)| \\ &\leq C_{w,\boldsymbol{x},f,a} \sup_{\boldsymbol{y}\in[-a,a]} |h(\boldsymbol{y}) - h(\boldsymbol{y}-\boldsymbol{x}_0)|, \end{aligned}$$

where the constant $C_{w,\boldsymbol{x},f,a}$ is a notational aggregate of the factors that depend neither on h nor on $\boldsymbol{x}_0$. Since h is uniformly continuous, the RHS converges to zero as $\|\boldsymbol{x}_0\| \to 0$, which proves the continuity of g at the point $\boldsymbol{x}$.

5.2. Proof of Proposition 2

By breaking the integral over $\mathbb{R}^d$ in parts over cubes of size 1 in each dimension, we get

$$\|f\|_{L_{p,w}(\mathbb{R}^d)} = \left(\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \int_{[0,1]^d} |w(\boldsymbol{x}+\boldsymbol{k}) f(\boldsymbol{x}+\boldsymbol{k})|^p \,\mathrm{d}\boldsymbol{x} \right)^{1/p}$$

$$\leq C_w \left(\int_{[0,1]^d} \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \left| \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} w(\boldsymbol{\ell})|c[\boldsymbol{\ell}]| \cdot w(\boldsymbol{x}+\boldsymbol{k}-\boldsymbol{\ell})|\varphi(\boldsymbol{x}+\boldsymbol{k}-\boldsymbol{\ell})| \right|^p \mathrm{d}\boldsymbol{x} \right)^{1/p} \tag{19}$$

$$= C_w \left(\int_{[0,1]^d} \|c_w * \varphi_{w,\boldsymbol{x}}\|^p_{\ell_p(\mathbb{Z}^d)} \,\mathrm{d}\boldsymbol{x} \right)^{1/p} \tag{20}$$

$$\leq C_w \left(\int_{[0,1]^d} \|c_w\|^p_{\ell_p(\mathbb{Z}^d)} \cdot \|\varphi_{w,\boldsymbol{x}}\|^p_{\ell_1(\mathbb{Z}^d)} \,\mathrm{d}\boldsymbol{x} \right)^{1/p} \tag{21}$$

$$= C_w \, \|c\|_{\ell_{p,w}(\mathbb{Z}^d)} \, \|\varphi\|_{W_{1,p,w}(\mathbb{R}^d)} \,,$$

where (19) is justified by the Fubini–Tonelli theorem and the submultiplicativity of w; the convolution in (20) was obtained by putting $c_w := w|c|$ and $\varphi_{w,\boldsymbol{x}}[\cdot] := w(\cdot+\boldsymbol{x})|\varphi(\cdot+\boldsymbol{x})|$; and, finally, (21) is a consequence of Young's inequality for discrete convolutions.

5.3. Proof of Proposition 3

Since $f \in L_{p,w}(\mathbb{R}^d) \subset L_p(\mathbb{R}^d) \subset L_{p,1/w}(\mathbb{R}^d)$ and $\varphi \in W_{1,p',w}(\mathbb{R}^d) \subset W_{p',p',w}(\mathbb{R}^d) = L_{p',w}(\mathbb{R}^d)$, it follows from Proposition 1 that the convolution $f * \varphi$ is a continuous function. Thus, its samples on the integer multigrid are well-defined. Next, we bound the discrete norm of these samples as

$$\|c\|_{\ell_{p,w}(\mathbb{Z}^d)} = \left(\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \left| w(\boldsymbol{k}) \int_{\mathbb{R}^d} f(\boldsymbol{x})\varphi(\boldsymbol{k}-\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \right|^p \right)^{1/p}$$

$$\leq C_w \left(\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \left| \int_{\mathbb{R}^d} f_w(\boldsymbol{x}) \cdot \varphi_w(\boldsymbol{k}-\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \right|^p \right)^{1/p} \tag{22}$$

$$= C_w \left(\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \left| \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} \int_{[0,1]^d} f_w(\boldsymbol{x}+\boldsymbol{\ell}) \cdot \varphi_w(\boldsymbol{k}-(\boldsymbol{x}+\boldsymbol{\ell})) \,\mathrm{d}\boldsymbol{x} \right|^p \right)^{1/p}$$

$$= C_w \left(\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \left| \int_{[0,1]^d} \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} f_w(\boldsymbol{x}+\boldsymbol{\ell}) \cdot \varphi_w(\boldsymbol{k}-\boldsymbol{x}-\boldsymbol{\ell}) \,\mathrm{d}\boldsymbol{x} \right|^p \right)^{1/p}$$

$$\leq C_w \int\limits_{[0,1]^d} \left(\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \left| \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} f_w(\boldsymbol{x}+\boldsymbol{\ell}) \cdot \varphi_w(\boldsymbol{k}-\boldsymbol{x}-\boldsymbol{\ell}) \right|^p \right)^{1/p} \mathrm{d}\boldsymbol{x} \tag{23}$$

$$= C_w \int\limits_{[0,1]^d} \|f_{w,\boldsymbol{x}} * \varphi_{w,-\boldsymbol{x}}\|_{\ell_p(\mathbb{Z}^d)} \,\mathrm{d}\boldsymbol{x} \tag{24}$$

$$\leq C_w \int\limits_{[0,1]^d} \|f_{w,\boldsymbol{x}}\|_{\ell_p(\mathbb{Z}^d)} \cdot \|\varphi_{w,-\boldsymbol{x}}\|_{\ell_1(\mathbb{Z}^d)} \,\mathrm{d}\boldsymbol{x}, \tag{25}$$

where (22) follows from the submultiplicativity of w and from putting $f_w := w|f|$ and $\varphi_w := w|\varphi|$; (23) follows from Minkowski's inequality for integrals; in (24) we defined the two sequences $f_{w,\boldsymbol{x}}[\cdot] := f_w(\cdot+\boldsymbol{x})$ and $\varphi_{w,-\boldsymbol{x}}[\cdot] := \varphi_w(\cdot-\boldsymbol{x})$ for each $\boldsymbol{x}\in[0,1]^d$; and, finally, (25) follows from Young's inequality.

We then proceed by applying Hölder's inequality to the RHS of (25) to get

$$\begin{aligned}
\|c\|_{\ell_{p,w}(\mathbb{Z}^d)} &\leq C_w \left(\int\limits_{[0,1]^d} \|f_{w,\boldsymbol{x}}\|^p_{\ell_p(\mathbb{Z}^d)} \,\mathrm{d}\boldsymbol{x} \right)^{1/p} \cdot \left(\int\limits_{[0,1]^d} \|\varphi_{w,-\boldsymbol{x}}\|^{p'}_{\ell_1(\mathbb{Z}^d)} \,\mathrm{d}\boldsymbol{x} \right)^{1/p'} \\
&= C_w \left(\int\limits_{[0,1]^d} \sum_{\boldsymbol{k}\in\mathbb{Z}^d} |f_w(\boldsymbol{x}+\boldsymbol{k})|^p \,\mathrm{d}\boldsymbol{x} \right)^{1/p} \cdot \left(\int\limits_{[0,1]^d} \left(\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \varphi_w(\boldsymbol{k}-\boldsymbol{x}) \right)^{p'} \mathrm{d}\boldsymbol{x} \right)^{1/p'} \\
&= C_w \left(\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \int\limits_{[0,1]^d} |f_w(\boldsymbol{x}+\boldsymbol{k})|^p \,\mathrm{d}\boldsymbol{x} \right)^{1/p} \cdot \left(\int\limits_{[-1,0]^d} \left(\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \varphi_w(\boldsymbol{x}+\boldsymbol{k}) \right)^{p'} \mathrm{d}\boldsymbol{x} \right)^{1/p'} \\
&= C_w \left(\int\limits_{\mathbb{R}^d} |f(\boldsymbol{x})w(\boldsymbol{x})|^p \,\mathrm{d}\boldsymbol{x} \right)^{1/p} \cdot \left(\int\limits_{[0,1]^d} \left(\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \varphi_w(\boldsymbol{x}+\boldsymbol{k}) \right)^{p'} \mathrm{d}\boldsymbol{x} \right)^{1/p'} \\
&= C_w \,\|f\|_{L_{p,w}(\mathbb{R}^d)} \,\|\varphi\|_{W_{1,p',w}(\mathbb{R}^d)},
\end{aligned}$$

thus completing the proof.

5.4. Proof of Lemma 1

As usual, let us put $c_w := w|c|$ and $\varphi_{w,\boldsymbol{x}}[\cdot] := w(\cdot+\boldsymbol{x})|\varphi(\cdot+\boldsymbol{x})|$. By using the submultiplicative property of w and Young's inequality, we get the estimate

$$\begin{aligned}
\|f\|_{W_{p,q,w}(\mathbb{R}^d)} &= \left(\int\limits_{[0,1]^d} \left(\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \left| w(\boldsymbol{x}+\boldsymbol{k}) \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} c[\boldsymbol{\ell}]\varphi(\boldsymbol{x}+\boldsymbol{k}-\boldsymbol{\ell}) \right|^p \right)^{q/p} \mathrm{d}\boldsymbol{x} \right)^{1/q} \\
&\leq C_w \left(\int\limits_{[0,1]^d} \left(\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \left| \sum_{\boldsymbol{\ell}\in\mathbb{Z}^d} c_w[\boldsymbol{\ell}]\varphi_{w,\boldsymbol{x}}[\boldsymbol{k}-\boldsymbol{\ell}] \right|^p \right)^{q/p} \mathrm{d}\boldsymbol{x} \right)^{1/q} \\
&= C_w \left(\int\limits_{[0,1]^d} \left(\|c_w * \varphi_{w,\boldsymbol{x}}\|_{\ell_p(\mathbb{Z}^d)} \right)^q \mathrm{d}\boldsymbol{x} \right)^{1/q}
\end{aligned}$$

$$\le C_w \left(\int\limits_{[0,1]^d} \left(\|c_w\|_{\ell_1(\mathbb{Z}^d)} \cdot \|\varphi_{w,\boldsymbol{x}}\|_{\ell_p(\mathbb{Z}^d)} \right)^q \mathrm{d}\boldsymbol{x} \right)^{1/q}$$

$$= C_w \ \|c_w\|_{\ell_1(\mathbb{Z}^d)} \left(\int\limits_{[0,1]^d} \left(\sum_{\boldsymbol{k}\in\mathbb{Z}^d} |w(\boldsymbol{x}+\boldsymbol{k})\varphi(\boldsymbol{x}+\boldsymbol{k})|^p \right)^{q/p} \mathrm{d}\boldsymbol{x} \right)^{1/q}$$

$$= C_w \ \|c\|_{\ell_{1,w}(\mathbb{Z}^d)} \|\varphi\|_{W_{p,q,w}(\mathbb{R}^d)},$$

which is the desired bound.

5.5. *Proof of Proposition 6*

Let us define the autocorrelation sequence of φ as

$$a[\boldsymbol{k}] := \int\limits_{\mathbb{R}^d} \varphi(\boldsymbol{x})\overline{\varphi(\boldsymbol{x}-\boldsymbol{k})}\,\mathrm{d}\boldsymbol{x}, \quad \text{for } \boldsymbol{k} \in \mathbb{Z}^d. \tag{26}$$

We first want to show that $a \in \ell_{1,w}(\mathbb{Z}^d)$. Let us put $\psi(\boldsymbol{x}) := \overline{\varphi(-\boldsymbol{x})}$. Then, similar to the bound (25) in the proof of Proposition 3, we can get

$$\begin{aligned}
\|a\|_{\ell_{1,w}(\mathbb{Z}^d)} &= \|(\varphi * \psi)[\cdot]\|_{\ell_{1,w}(\mathbb{Z}^d)} \\
&\le C_w \int\limits_{[0,1]^d} \|\varphi_{w,\boldsymbol{x}}\|_{\ell_1(\mathbb{Z}^d)} \cdot \|\psi_{w,-\boldsymbol{x}}\|_{\ell_1(\mathbb{Z}^d)} \,\mathrm{d}\boldsymbol{x} &(27)\\
&= C_w \int\limits_{[0,1]^d} 1 \cdot \|\varphi_{w,\boldsymbol{x}}\|^2_{\ell_1(\mathbb{Z}^d)} \,\mathrm{d}\boldsymbol{x} &(28)\\
&\le C_w \left(\int\limits_{[0,1]^d} \|\varphi_{w,\boldsymbol{x}}\|^q_{\ell_1(\mathbb{Z}^d)} \,\mathrm{d}\boldsymbol{x} \right)^{2/q} &(29)\\
&= C_w \cdot \|\varphi\|^2_{W_{1,q,w}(\mathbb{R}^d)} < \infty,
\end{aligned}$$

where we adopted in (27) again the notation $f_{w,\boldsymbol{x}}[\cdot] := w(\cdot+\boldsymbol{x})|f(\cdot+\boldsymbol{x})|$; (28) is because $\|\psi_{w,-\boldsymbol{x}}\|_{\ell_1(\mathbb{Z}^d)} = \|\varphi_{w,\boldsymbol{x}}\|_{\ell_1(\mathbb{Z}^d)}$; (29) follows from Hölder's inequality and the fact that $q \ge 2$. So, we have proved that $a \in \ell_{1,w}(\mathbb{Z}^d)$.

Next, from the hypothesis that $\{\varphi(\cdot-\boldsymbol{k})\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ is a Riesz basis for $V_2(\varphi)$, the dual generator φ_{dual} is uniquely determined by the expansion

$$\varphi_{\mathrm{dual}} = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} b[\boldsymbol{k}]\varphi(\cdot-\boldsymbol{k}), \tag{30}$$

where the coefficient sequence b is given in the Fourier domain by

$$\hat{b}(\boldsymbol{\xi}) = \frac{1}{\sum_{\boldsymbol{k}\in\mathbb{Z}^d} |\hat{\varphi}(\boldsymbol{\xi}+\boldsymbol{k})|^2} = \frac{1}{\hat{a}(\boldsymbol{\xi})}.$$

Here, $\hat{a}$ and $\hat{b}$ are the Fourier series associated with the sequences of coefficients a and b, respectively. Because $a \in \ell_{1,w}(\mathbb{Z}^d)$ and w satisfies the GRS condition, we now invoke the weighted version of Wiener's lemma (see [41, Theorem 6.2]) to deduce that $b \in \ell_{1,w}(\mathbb{Z}^d)$. Finally, from (30) and Lemma 1, we have that

$$\|\varphi_{\mathrm{dual}}\|_{W_{1,q,w}(\mathbb{R}^d)} \le C_w \|b\|_{\ell_{1,w}(\mathbb{Z}^d)} \|\varphi\|_{W_{1,q,w}(\mathbb{R}^d)} < \infty.$$

In conclusion, the dual kernel φ_{dual} belongs to $W_{1,q,w}(\mathbb{R}^d)$, too.

References

[1] A.J. Jerri, The Shannon sampling theorem—its various extensions and applications: a tutorial review, Proc. IEEE 65 (11) (1977) 1565–1596.
[2] M. Unser, Sampling—50 years after Shannon, Proc. IEEE 88 (4) (2000) 569–587.
[3] C. Shannon, Communication in the presence of noise, Proc. IRE 37 (1) (1949) 10–21.
[4] M. Unser, A. Aldroubi, M. Eden, B-spline signal processing: part I—theory, IEEE Trans. Signal Process. 41 (2) (1993) 821–833.
[5] M. Unser, A. Aldroubi, M. Eden, B-spline signal processing: part II—efficiency design and applications, IEEE Trans. Signal Process. 41 (2) (1993) 834–848.
[6] M. Unser, Splines: a perfect fit for signal and image processing, IEEE Signal Process. Mag. 16 (6) (1999) 22–38.
[7] A. Aldroubi, M. Unser, Sampling procedures in function spaces and asymptotic equivalence with Shannon's sampling theory, Numer. Funct. Anal. Optim. 15 (1) (1994) 1–21.
[8] A. Aldroubi, H.G. Feichtinger, Exact iterative reconstruction algorithm for multivariate irregularly sampled functions in spline-like spaces: The L^p-theory, Proc. Amer. Math. Soc. 126 (9) (1998) 2677–2686.
[9] A. Aldroubi, K. Gröchenig, Beurling–Landau-type theorems for non-uniform sampling in shift invariant spline spaces, J. Fourier Anal. Appl. 6 (1) (2000) 93–103.
[10] A. Aldroubi, K. Gröchenig, Nonuniform sampling and reconstruction in shift-invariant spaces, SIAM Rev. 43 (4) (2001) 585–620.
[11] A. Aldroubi, Non-uniform weighted average sampling and reconstruction in shift-invariant and wavelet spaces, Appl. Comput. Harmon. Anal. 13 (2) (2002) 151–161.
[12] M.Z. Nashed, Q. Sun, Sampling and reconstruction of signals in a reproducing kernel subspace of $L_p(\mathbb{R}^d)$, J. Funct. Anal. 258 (7) (2010) 2422–2452.
[13] M.Z. Nashed, Q. Sun, J. Xian, Convolution sampling and reconstruction of signals in a reproducing kernel subspace, Proc. Amer. Math. Soc. 141 (6) (2013) 1995–2007.
[14] M.Z. Nashed, G.G. Walter, General sampling theorems for functions in reproducing kernel Hilbert spaces, Math. Control Signals Systems 4 (4) (1991) 363–390.
[15] C.V.M. van der Mee, M.Z. Nashed, S. Seatzu, Sampling expansions and interpolation in unitarily translation invariant reproducing kernel Hilbert spaces, Adv. Comput. Math. 19 (4) (2003) 355–372.
[16] D. Han, M.Z. Nashed, Q. Sun, Sampling expansions in reproducing kernel Hilbert and Banach spaces, Numer. Funct. Anal. Optim. 30 (9–10) (2009) 971–987.
[17] A. Aldroubi, Q. Sun, W.-S. Tang, Convolution, average sampling, and a Calderon resolution of the identity for shift-invariant spaces, J. Fourier Anal. Appl. 11 (2) (2005) 215–244.
[18] A. Aldroubi, Q. Sun, W.-S. Tang, Non-uniform average sampling and reconstruction in multiply generated shift-invariant spaces, Constr. Approx. 20 (2) (2004) 173–189.
[19] H.G. Feichtinger, K. Gröchenig, Iterative reconstruction of multivariate band-limited functions from irregular sampling values, SIAM J. Math. Anal. 23 (1) (1992) 244–261.
[20] H.G. Feichtinger, K. Gröchenig, Irregular sampling theorems and series expansions of band-limited functions, J. Math. Anal. Appl. 167 (2) (1992) 530–556.
[21] E. Pfaffelhuber, Sampling series for band-limited generalized functions, IEEE Trans. Inform. Theory 17 (6) (1971) 650–654.
[22] G.G. Walter, Sampling bandlimited functions of polynomial growth, SIAM J. Math. Anal. 19 (5) (1988) 1198–1203.
[23] G.G. Walter, Nonuniform sampling of bandlimited functions of polynomial growth, SIAM J. Math. Anal. 23 (4) (1992) 995–1003.
[24] A.I. Zayed, A.G. García, Nonuniform sampling of bandlimited signals with polynomial growth on the real axis, IEEE Trans. Inform. Theory 43 (5) (1997) 1717–1721.
[25] M. Unser, P.D. Tafti, An Introduction to Sparse Stochastic Processes, Cambridge University Press, 2014.
[26] M. Unser, P.D. Tafti, Q. Sun, A unified formulation of Gaussian versus sparse stochastic processes—part I: continuous-domain theory, IEEE Trans. Inform. Theory 60 (3) (2014) 1945–1962.
[27] M. Unser, P.D. Tafti, A. Amini, H. Kirshner, A unified formulation of Gaussian versus sparse stochastic processes—part II: discrete-domain theory, IEEE Trans. Inform. Theory 60 (5) (2014) 3036–3051.
[28] G. Zimmermann, Projective multiresolution analysis and generalized sampling, Ph.D. thesis, University of Maryland, College Park, MD, Dec. 1994.
[29] J.J. Benedetto, G. Zimmermann, Sampling multipliers and the Poisson summation formula, J. Fourier Anal. Appl. 3 (5) (1997) 505–523.
[30] N. Wiener, On the representation of functions by trigonometric integrals, Math. Z. 24 (1) (1926) 575–616.
[31] N. Wiener, Tauberian theorems, Ann. Math. 33 (1) (1932) 1–100.

[32] N. Wiener, The Fourier Integral and Certain of Its Applications, MIT Press, Cambridge, 1933.
[33] H.G. Feichtinger, Banach convolution algebras of Wiener type, in: Proc. Conf. Functions, Series, Operators, Budapest, Hungary, 1980, pp. 509–524.
[34] H.G. Feichtinger, Generalized amalgams, with applications to Fourier transform, Canad. J. Math. 42 (3) (1990) 395–409.
[35] H.G. Feichtinger, New results on regular and irregular sampling based on Wiener amalgams, in: K. Jarosz (Ed.), Proc. Conf. Function Spaces, in: Lecture Notes in Pure and Appl. Math., vol. 136, Dekker, New York, 1991, pp. 107–121.
[36] H.G. Feichtinger, Wiener amalgams over Euclidean spaces and some of their applications, in: K. Jarosz (Ed.), Proc. Conf. Function Spaces, in: Lecture Notes in Pure and Appl. Math., vol. 136, Dekker, New York, 1991, pp. 123–137.
[37] J.J.F. Fournier, J. Steward, Amalgams of L^p and ℓ^q, Bull. Amer. Math. Soc. 13 (1985) 1–21.
[38] C. Heil, An introduction to weighted Wiener amalgams, in: M. Krishna, R. Radha, S. Thangavelu (Eds.), Wavelets and Their Applications, Allied Publishers, New Delhi, 2003, pp. 183–216.
[39] H.Q. Nguyen, M. Unser, Generalized Poisson summation formula for tempered distributions, in: Proc. 11th Int. Conf. Sampling Theory and Applications, SampTA'15, Washington, DC, USA, 2015, pp. 1–5.
[40] L. Schwartz, Théorie des Distributions, Hermann, Paris, France, 1966.
[41] K. Gröchenig, Weight functions in time-frequency analysis, arXiv:math/0611174, 2006.
[42] L. Grafakos, Modern Fourier Analysis, 2nd edition, Springer, 2008.