Fractional wavelets, derivatives, and Besov spaces

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Abstract

We show that a multi-dimensional scaling function of order γ (possibly fractional) can always be represented as the convolution of a polyharmonic B-spline of order γ and a distribution with a bounded Fourier transform which has neither order nor smoothness. The presence of the B-spline convolution factor explains all key wavelet properties: order of approximation, reproduction of polynomials, vanishing moments, multi-scale differentiation property, and smoothness of the basis functions. The B-spline factorization also gives new insights on the stability of wavelet bases with respect to differentiation. Specifically, we show that there is a direct correspondence between the process of moving a B-spline factor from one side to another in a pair of biorthogonal scaling functions and the exchange of fractional integrals/derivatives on their wavelet counterparts. This result yields two "eigen-relations" for fractional differential operators that map biorthogonal wavelet bases into other stable wavelet bases. This formulation provides a better understanding as to why the Sobolev/Besov norm of a signal can be measured from the ℓ_p-norm of its rescaled wavelet coefficients. Indeed, the key condition for a wavelet basis to be an unconditional basis of the Besov space $B_q^s(L_p(R^d))$ is that the *s*-order derivative of the wavelet be in L_p.

Keywords: multi-dimensional wavelets, fractional derivatives, polyharmonic splines, order of approximation, wavelet smoothness, Besov spaces

1. INTRODUCTION

Recently, we proposed a new formulation of one-dimensional wavelet theory that starts from the representation of a scaling function as the convolution of a B-spline with a tempered distribution that carries no order at all[13]. This point of view provides some new insights and facilitates the derivation of the main results of wavelet theory. Central to this formulation is the study of the properties of the (fractional) B-splines, which is easier than for other wavelets because of the availability of explicit formulas in time and frequency[12]. The key properties of the B-spline are then mapped almost mechanically to the scaling function through the convolution relation.

Our goal in this paper is to extend this factorization idea to multiple dimensions and to propose a general formulation that applies to all scaling functions and wavelets in $L_2(R^d)$. Our motivation is threefold:

First, we are interested in identifying the multi-dimensional analogs of the B-splines. The choice that we consider here are the polyharmonic B-splines[10], which are non-separable and can be extended to fractional orders. Note that these polyharmonic splines are localized versions of the Green functions of the iterated-Laplacian operators[6, 7].

Second, we want to get a more direct understanding of the interaction of wavelets with differential operators. To quote Meyer[9]: "everything takes place as if the wavelets $\psi(x/a)$ were eigenvectors of the differential operator ∂^s, with corresponding eigenvalue a^{-s}". Thus, our goal is to write down explicitly these "eigen-relations", which, to the best of our knowledge, has not been done before. Again, this is made possible by working with polyharmonic splines which can be differentiated analytically.

Third, we want to relax the classical decay conditions which are too restrictive for our purpose. In particular, they exclude fractional wavelets[12], which are precisely the type of wavelets we end up with after fractional differentiation. Meyer[9], for instance, uses the r-regularity constraint that requires the scaling function and all its derivatives up to order r to decay faster than $\|x\|^{-m}$ for any integer m. In this work, we extend the theory to slowly decreasing functions, one motivating factor being that the standard r-regularity condition would disqualify the polyharmonic B-splines which are the basis of our formulation.

2. MATHEMATICAL PRELIMINARIES

2.1 Scaling functions and wavelets

We say that a multi-dimensional scaling function $\varphi(x) \in L_2(R^d)$ is *admissible* if it satisfies the three following properties: (i) it generates a Riesz basis, (ii) it satisfies the two-scale relation

$$\varphi(x/2) = 2^d \sum_{k \in Z^d} h(k)\varphi(x-k) \quad \xleftrightarrow{\text{Fourier}} \quad \hat{\varphi}(2\omega) = H(e^{j\omega}) \cdot \hat{\varphi}(\omega) \tag{1}$$

with refinement filter $H(e^{j\omega}) = \sum_{k \in Z^d} h(k) e^{-j\omega \cdot k}$, and, (iii) it fulfills the partition of unity. These are the minimal requirements to generate a multi-resolution analysis of $L_2(R^d)$ in the sense defined by Mallat[8]. Therefore, for any given pair of admissible biorthogonal scaling functions φ and $\tilde{\varphi}$, we have the guarantee that there exists a (non-unique) set of $(2^d - 1)$ wavelets

$$\psi_m(x) = 2^d \sum_{k \in Z^d} g_m(k)\varphi(2x-k) \quad \xleftrightarrow{\text{Fourier}} \quad \hat{\psi}_m(\omega) = G_m(e^{j\omega/2}) \cdot \hat{\varphi}(\omega/2) \tag{2}$$

and their duals $\tilde{\psi}_m$, such that the functions $\left\{2^{id/2}\psi_m\left(x/2^i - k\right)\right\}_{m=1,\cdots 2^d-1,\ i \in Z,\ k \in Z^d}$ form a Riesz basis of $L_2(R^d)$. The wavelet filters are typically obtained by designing a multi-dimensional perfect-reconstruction filterbank[14]; $\tilde{H}(e^{j\omega})$ and $H(e^{j\omega})$ are the analysis and synthesis filters in the lowpass branch, while $\tilde{G}_m(e^{j\omega})$ and $G_m(e^{j\omega})$, with $m = 1, \cdots, 2^d - 1$, are their wavelet counterparts. The corresponding projector operator at scale $a = 2^i$ is specified as

$$\tilde{P}_i f = \sum_{k \in Z^p} \langle f, \tilde{\varphi}_{i,k} \rangle\ \varphi_{i,k}, \quad \text{where} \quad \varphi_{i,k}(x) = 2^{id/2}\varphi\left(x/2^i - k\right) \tag{3}$$

The wavelet transform will be of order γ—possibly, fractional—if and only if the scale truncated approximation error for smooth functions decays like the γth power of the scale; in other words, iff. $\forall f \in W_2^\gamma$, $\left\| f - \tilde{P}_i f \right\|_{L_2} = O(a^\gamma)$,with $a = 2^i$, where W_2^γ denotes the Sobolev space of order γ.

2.2 Polyharmonic B-splines

Rabut's polyharmonic B-splines of order γ provide an interesting family of scaling functions, the order of which may be fractional[10]. These functions, denoted by $\beta_\gamma(x)$, are best defined in the Fourier domain as

$$\hat{\beta}_\gamma(\omega) = \mathcal{F}\{\beta_\gamma(x)\}(\omega) = \frac{\|\sin(\omega/2)\|^\gamma}{\|\omega/2\|^\gamma} \quad \text{with} \quad \omega = (\omega_1, \cdots, \omega_d) \in R^d, \tag{4}$$

where $\sin(\omega/2) = \left(\sin(\omega_1/2), \cdots, \sin(\omega_d/2)\right)$. Despite the fact that the polyharmonic B-splines violate the rapid-decay requirements of classical wavelet theory (they typically only decay like $O(\|x\|^{-2-d})$, they generate Riesz bases and are perfectly valid scaling functions for $\gamma > d/2$ (also see Madych's chapter[5]). In addition, $\beta_\gamma(x) \in L_p(R^d)$ for $\gamma > d(1 - 1/p)$. Another important property is the convolution relation $\beta_{\gamma_1} * \beta_{\gamma_2} = \beta_{\gamma_1 + \gamma_2}$, which follows directly from the definition. In one dimension, these functions are equivalent to the symmetric fractional B-splines of degree $\alpha = \gamma - 1$ whose properties are investigated elsewhere[12].

2.3 Differential operators

In multiple dimensions, it is customary to consider an isotropic fractional differential operator ∂_*^s of order s that corresponds to the (s/2)-iterate of the Laplacian operator:

$$\partial_*^s f(x) = \Delta^{s/2} f(x) \xleftrightarrow{Fourier} \|\omega\|^s \hat{f}(\omega). \tag{5}$$

This fractional derivative is to be understood in the sense of distributions. The discrete counterpart of this operator is the finite-difference operator Δ_*^s of order s

$$\Delta_*^s \xleftrightarrow{Fourier} \|2\sin(\omega/2)\|^s = 2^s\left(\sum_{i=1}^{d}\sin(\omega_i/2)^2\right)^{s/2} \tag{6}$$

that corresponds to the (s/2)-iterate of the discrete Laplacian. Using these two operators, we can rewrite the polyharmonic B-spline as $\beta_\gamma(x) = \Delta_*^\gamma \partial_*^{-\gamma}\delta(x)$, where $\partial_*^{-\gamma}$ (fractional integral of order γ) is the convolution inverse of ∂_*^γ (fractional derivative of order γ). Next, we use the composition rules $\Delta_*^\gamma = \Delta_*^s \Delta_*^{\gamma-s}$ and $\partial_*^s \partial_*^{-\gamma} = \partial_*^{-(\gamma-s)}$ to compute the fractional derivative of a polyharmonic B-spline:

$$\partial_*^s \beta_\gamma(x) = \Delta_*^s \Delta_*^{\gamma-s} \partial_*^{-(\gamma-s)}\delta(x) = \Delta_*^s \beta_{\gamma-s}(x). \tag{7}$$

This yields an explicit differentiation formula that will play a central role in our formulation.

3. ORDER AND RELATED PROPERTIES

In conventional wavelet theory, the order is constrained to be an integer because of rather stringent decay requirements[3, 9]. We believe that these restrictions are unnecessary and that it is interesting to extend the classical constructions to fractional orders. This also requires a restatement of the classical Strang-Fix[11] conditions. If one assume that $\varphi(x)$ satisfies the Riesz basis condition—but not necessarily the two-scale relation—and has sufficient algebraic decay, then the approximation results of de Boor, DeVore and Ron[4] imply that $\varphi(x)$ will be of order γ (possibly fractional) if and only if (see also the work of Blu and Unser[1] for the equivalence in the 1d case)

$$\sum_{k\in Z^p, k\neq 0} |\hat{\varphi}(\omega + 2\pi k)|^2 = O(\|\omega\|^{2\gamma}). \tag{8}$$

We will now show that this order condition has profound implications on the properties of the corresponding wavelets. We will also see that the order will manifest itself by the presence of a polyharmonic B-spline convolution factor, which will help us get a better understanding of the whole issue of wavelet differentiation.

3.1 Wavelet manifestations of the order property

By using (8) together with the fact that the analysis wavelets $\tilde{\psi}_m$ are perpendicular to $\varphi(x-k)$, $k \in Z^d$, we can prove that the wavelets have the following frequency behavior near the origin:

$$\hat{\tilde{\psi}}_m(\omega) = O(\|\omega\|^\gamma). \tag{9}$$

In other words, they essentially behave like γth order differentiators. If we also assume that the $\tilde{\psi}_m$'s have sufficient (inverse polynomial) decay for their moments to be well-defined mathematically, the above result implies that the wavelets have the vanishing moment property

$$\int x^{n_1} \cdots x^{n_d} \tilde{\psi}_m(x)dx = 0 \quad \text{for all multi-integers } n_1 + \cdots + n_d < \gamma \tag{10}$$

which is a well-known result when the order is an integer[9]. Because of (9) and $\hat{\tilde{\varphi}}(0)=1$, we also have that $\tilde{G}_m(e^{j\omega})=\hat{\tilde{\psi}}_m(2\omega)/\hat{\tilde{\varphi}}(\omega)=O\left(\|\omega\|^{\gamma}\right)$. Assuming that the wavelet filterbank is stable (i.e., $\left|\tilde{G}_m(e^{j\omega})\right|<+\infty$, for $\omega\in R^p$), this implies that

$$\tilde{G}_m(e^{j\omega})=\|2\sin(\omega/2)\|^{\gamma}\cdot P_m(e^{j\omega}) \quad \text{with } \left|P_m(e^{j\omega})\right|<+\infty \tag{11}$$

because $0<c_0\leq\|2\sin(\omega/2)\|/\|\omega\|\leq 1,\ \forall\omega\in[-\pi,\pi]^d$. Thus, the wavelet filters must necessarily have a zero of order γ at the origin.

3.2 Order, zeros, and B-spline factorization

In a paper in preparation[2], we have established the following fractional extension of a standard result in wavelet theory:

Theorem 1. Let φ be a valid scaling function. Then, φ is of order γ if $\left|H(e^{j(\omega+\pi e_m)})\right|^2=O\left(\|\omega\|^{2\gamma}\right)$, $m=1,\cdots,2^d-1$, with $e_m=(b_1,b_2,\cdots,b_d)$, where $b_1b_2\cdots b_m$ are the digits of the binary code of m.

This theorem sets the requirement for a γth order wavelet transform: the refinement filter $H(e^{j\omega})$ must be designed to have zeros of order γ at the critical frequencies πe_m.

If we now add a slight regularity requirement on $H(e^{j\omega})$, we can derive a multidimensional extension of our previous B-spline factorization theorem[13].

Theorem 2. Let φ be a valid scaling function with $H(e^{j\omega})\in\dot{C}^h$, $h>0$. Then, φ is of order γ iff. $\varphi_\gamma(x)=\beta_\gamma(x)*\varphi_0(x)$ with $\varphi_0\in S'$, $\hat{\varphi}_0(0)=1$ and $|\hat{\varphi}_0(\omega)|<+\infty$.

The interpretation of this result is that every scaling function contains a polyharmonic B-spline convolution factor (the regular part of it) which is entirely responsible for the order, and by implication, for all wavelet properties listed in Section 3.1. The factor φ_0 (irregular part) is a singular distribution; it has no order and no smoothness at all. Its only remarkable property is that its Fourier transform is bounded on all compact subsets.

3.3 Spline factors and regularity

The B-spline factor is also fully responsible for the smoothness of the basis functions. To see why this is the case, we note we can explicitly differentiate $\varphi_\gamma=\beta_\gamma*\varphi_0$ because ∂_*^s is associative (it is a convolution operator) and because we know from (7) how to differentiate β_γ. Specifically, we get

$$\partial_*^r\varphi_\gamma=\partial_*^r\beta_\gamma*\varphi_0=\Delta_*^r\beta_{\gamma-r}*\varphi_0=\Delta_*^r\varphi_{\gamma-r}, \quad \text{with } \varphi_{\gamma-r}=\beta_{\gamma-r}*\varphi_0. \tag{12}$$

This explicit calculation leads to the following smoothness characterization theorem.

Theorem 3: If $\varphi_\gamma(x)=\beta_r*\varphi_{\gamma-r}(x)$ with $\varphi_{\gamma-r}\in L_p(R^d)$ then $\partial_*^r\varphi_\gamma\in L_p(R^d)$; i.e., φ is *r* times differentiable in the L_p-sense.

Proof: The argument uses Minkowsky's inequality and the fact that the coefficients $a_r(k)$ of the finite-difference operator Δ_*^r are in ℓ_1 for $r>0$:

$$\left\|\partial_*^r\varphi_\gamma\right\|_{L_p}\leq\sum_{k\in Z^p}|a_r(k)|\cdot\left\|\varphi_{\gamma-r}(x-k)\right\|_{L_p}=\|a_r(k)\|_{\ell_1}\cdot\left\|\varphi_{\gamma-r}\right\|_{L_p}$$

More significant is the fact that we have a converse version of the theorem for the Sobolev case $p=2$.

Theorem 4: If φ is a valid scaling function such that $\partial_*^r\varphi\in L_2(R^d)$, then $\varphi(x)=\beta_r*\varphi_0(x)$ with $\varphi_0\in L_2(R^d)$.

The proof of this result is technical and can be found elsewhere[2]. The important consequence of this result is that there cannot be any wavelet smoothness without a B-spline factor. Moreover, the theorem implies that the distribution φ_0 in Theorem 2 has no Sobolev smoothness at all; otherwise, it would be possible to pull out some more B-spline, indicating that that the order would be larger than γ.

4. WAVELETS AND DIFFERENTIATION

4.1 Biorthogonality relations

Based on the convolution property of the B-splines, we may express the scaling function φ_γ as $\varphi_\gamma = \beta_s * \varphi_{\gamma-s}$ with $\varphi_{\gamma-s} = \beta_{\gamma-s} * \varphi_0$. We also assume that $\tilde{\varphi}_{\tilde{\gamma}}$, the dual of φ_γ, is of order $\tilde{\gamma}$ so that it can be written as $\tilde{\varphi}_{\tilde{\gamma}} = \beta_{\tilde{\gamma}} * \tilde{\varphi}_0$. This allows us to manipulate the biorthogonality relation,

$$\langle \tilde{\varphi}_{\tilde{\gamma}}(x-k),\ \varphi_\gamma(x-l)\rangle = \langle \underbrace{\beta_s * \tilde{\varphi}_{\tilde{\gamma}}}_{\tilde{\varphi}_{\tilde{\gamma}+s}}(x-k),\ \varphi_{\gamma-s}(x-l)\rangle = \delta_{k-l}, \tag{13}$$

which leads to the identification of a biorthogonal pair of "order-reallocated" scaling functions $\tilde{\varphi}_{\tilde{\gamma}+s} = \beta_{\tilde{\gamma}+s} * \tilde{\varphi}_0$ and $\varphi_{\gamma-s} = \beta_{\gamma-s} * \varphi_0$. Note that the manipulation is valid and yields L_p-stable Riesz bases as long as $\varphi_{\gamma-s} \in L_p$, which also implies that $\partial_*^s \varphi_\gamma \in L_p$ (cf. Theorem 3). In a complementary fashion, we can also manipulate the wavelet biorthogonality relations

$$\langle \tilde{\psi}_m(x-k),\ \psi_m(x-l)\rangle = \langle \partial_*^{-s}\tilde{\psi}_m(x-k),\ \partial_*^s \psi_m(x-l)\rangle = \delta_{k-l}, \quad m = 1,\cdots,2^d-1, \tag{14}$$

where the symmetric fractional integration operator $\partial_*^{-s} \leftrightarrow \|\omega\|^{-s}$ is the convolution inverse of $\partial_*^s \leftrightarrow \|\omega\|^s$. Here too, we are allowed to differentiate ψ_m: we do obtain meaningful basis functions as long as φ_γ has s derivatives in the L_p-sense. Taking the fractional integral of $\tilde{\psi}_m$ for $s \le \gamma$ is also legitimate in the sense that $\partial_*^{-s}\tilde{\psi}_m \in L_p$ because of the special behavior of its Fourier transform near the origin (see (9)). We will now show that this all nicely fits together and that the "differential" wavelets $(\partial_*^{-s}\tilde{\psi}_m,\ \partial_*^s\psi_m)_{m=1,\cdots,2^d-1}$ generate a biorthogonal wavelet basis of $L_2(R^d)$.

4.2 "Differential" and "integral" wavelets

By applying the differentiation formula (12) to (2), we obtain the explicit form of the "derivative" wavelets

$$\psi_m^{(s)}(x) = \partial_*^s \psi_m(x) \ = 2^d \sum_{k\in Z} g_m^s(k)\varphi_{\gamma-s}(2x-k), \ \text{with}\ \ G_m^s(e^{j\omega}) = 2^s \|2\sin(\omega/2)\|^s G_m(e^{j\omega}). \tag{15}$$

This proves that $\psi_m^{(s)}(x)$ is indeed a wavelet with corresponding scaling function $\varphi_{\gamma-s} = \beta_{\gamma-s} * \varphi_0$. Likewise, we use (11) and express the "integral" wavelet as

$$\tilde{\psi}_m^{(-s)}(x) = \partial_*^{-s}\tilde{\psi}_m(x) = 2^d \sum_{k\in Z} \tilde{g}_m^{-s}(k)\tilde{\varphi}_{\gamma+s}(2x-k), \text{with}\ \tilde{G}_m^{-s}(e^{j\omega}) = 2^{-s}\|2\sin(\omega/2)\|^{\gamma-s} P_m(e^{j\omega}), \tag{16}$$

which also yields a valid (biorthogonal) wavelet with corresponding scaling function $\tilde{\varphi}_{\gamma+s} = \beta_{\gamma+s} * \tilde{\varphi}_0$. Finally, we use the scaling relation $\partial_*^s\{\psi(x/a)\} = a^{-s}\psi^{(s)}(x/a)$ and close the loop by deriving the "eigen-relation" for wavelet derivatives to which Meyer was alluding to:

$$\partial_*^s\{\psi_{m,i,k}\} = 2^{-is}\psi_{m,i,k}^{(s)} \quad \text{and} \quad \partial_*^{-s}\{\tilde{\psi}_{m,i,k}\} = 2^{is}\tilde{\psi}_{m,i,k}^{(-s)}. \tag{17}$$

which the short hand notation $\psi_{m,i,k}(x) = 2^{id/2}\psi_m\left(x/2^i - k\right)$. Of course, the qualifying statement is not rigorously correct—the important point is that there are basis functions with the same wavelet structure on both sides of the identity. The practical relevance of these "differential" wavelets is that they give us a direct way of gauging the fractional derivative of a signal based on its wavelet coefficients in the original basis. Specifically, we can differentiate the wavelet expansion of a signal by dividing its wavelet coefficients by 2^{is}, which yields a representation in the modified wavelet basis $\{\psi^{(s)}_{m,i,k}\}$. Since $\{\psi^{(s)}_{m,i,k}\}$ is also an unconditional wavelet basis of $L_2(R^d)$, it follows that the l_2-norm of the rescaled wavelet coefficients will be equivalent to the Sobolev norm of the signal. The Besov case is analogous with l_p-norms being used instead; the argument there is more involved and relies on some Riesz-type L_p-norm equivalences [9]. Also note that the wavelets that have just been specified are fractional ones, which, in itself, may serve as an a posteriori justification for our extended formulation.

5. CONCLUSION

The main contribution of this paper has been to show that a valid multi-dimensional scaling function can always be expressed as the convolution product of a polyharmonic B-spline (the regular part of it) and a distribution with bounded Fourier transform. The B-spline factor carries all the order of approximation. As such, it is entirely responsible for the key mathematical features of the transform: reproduction of polynomials, vanishing moments, multi-differentiation, and smoothness of the basis functions.

An advantage of this new formulation is that it makes the issue of wavelet differentiation much more transparent. By analogy with the Fourier transform for which (isotropic) fractional differentiation corresponds to a multiplication by $\|\omega\|^s$, this operation corresponds to a division of the wavelet coefficients by a^s, which is consistent with the observation that the scale is inversely proportional to the frequency. The fact that we have obtained explicit wavelet differentiation formulas is also interesting from a practical point of view, because this yields an exact algorithm for reconstructing the fractional derivative of a signal from its wavelet expansion, or, equivalently, for computing the wavelet transform of the derivative of a signal.

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