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Medical image interpolation: The quest for higher quality

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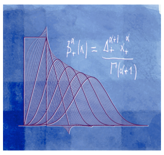

Medical Imaging Conference, Rome, October 16-22, 2004

Biomedical imaging group

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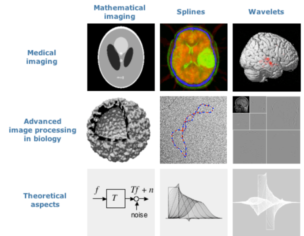
- Inter-disciplinary research strategy



OUTLINE

- Introduction
- Spline interpolation
- Splines and approximation theory
- Application example: image registration
- Interpolation in the presence of noise
- Conclusions

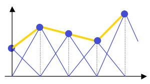
Back to the fundamentals: interpolation

- **Fundamental issue in imaging and signal processing**
  Linking the *discrete* and the *continuous*

Acquisition

Algorithm design

- **Mismatch between classical theory and practice**
  - Classical theory : Shannon's sampling theorem
  - Practice: nearest neighbor, linear interpolation
- **Limitations of Shannon's sampling theory**
  - Ideal lowpass filters do not exist
  - Incompatible with finite support signals
  - Gibbs oscillations
  - Slow decay of $\mathrm{sinc}(x)$
- **Basic problem**
  How do you interpolate a signal ?

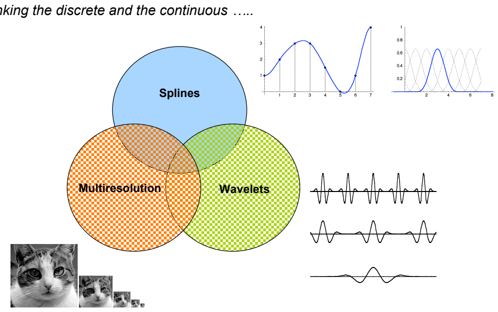


Interpolation and bio-imaging

Image processing task	Specific operation	Imaging modality
Tomographic reconstruction	• Filtered backprojection • Fourier reconstruction • Iterative techniques • 3D + time	Commercial CT (X-rays) EM PET, SPECT Dynamic CT, SPECT, PET
Sampling grid conversion	• Polar-to-cartesian coordinates • Spiral sampling • k-space sampling • Scan conversion	Ultrasound (endovascular) Spiral CT, MRI MRI
Visualization	*2D operations* • Zooming, panning, rotation • Re-sizing, scaling	All
	• Stereo imaging • Range, topography	Fundus camera OCT
	3D operations • Re-slicing • Max. intensity projection • Simulated X-ray projection	CT, MRI, MRA
	Surface/volume rendering • Iso-surface ray tracing • Gradient-based shading • Stereogram	CT MRI
Geometrical correction	• Wide-angle lenses • Projective mapping • Aspect ratio, tilt • Magnetic field distortions	Endoscopy C-Arm fluoroscopy Dental X-rays MRI
Registration	• Motion compensation • Image subtraction • Mosaicking • Correlation-averaging • Patient positioning • Retrospective comparisons • Multi-modality imaging • Stereotactic normalization • Brain warping	fMRI, fundus camera DSA Endoscopy, fundus camera, EM microscopy Surgery, radiotherapy CT/PET/MRI
Feature detection	• Contours • Ridges • Differential geometry	All
	Contour extraction • Snakes and active contours	MRI, Microscopy (cytology)

Splines: a unifying framework

Linking the discrete and the continuous …..



Splines: Bad press phenomenon

- Classical review article on interpolation, IEEE TMI, 1983
  Comparison of four interpolators:
  "The cubic B-spline provides the most smoothing."
- Classical book on *Digital Image Processing*, 1991 (2nd ed)
  About high order B-splines:
  "[out-of-band] interpolation error reduces significantly for higher order interpolation functions, but at the expense of resolution error [i.e., distortion]"

  ⋮

- Recent book on *Volume Rendering*, 1998
  "The results of scaling the original image using [cubic] B-spline interpolation are shown in Figure 5.20. You can see the blurring effects ….."

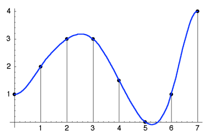
SPLINE INTERPOLATION

- Splines: definition
- B-spline basis functions
- B-spline interpolation
- Spline interpolators
- Geometric transformation of images

Splines: definition

Definition: A polynomial spline of degree n, $s(x)$, with *knots* $\ldots < x_k < x_{k+1} < \ldots$ is a function with the following two properties:

- Piecewise polynomial:
  $s(x)$ is a polynomial of degree n in each interval $[x_k, x_{k+1})$;
- Higher-order continuity:
  $s(x), s^{(1)}(x), \ldots, s^{(n-1)}(x)$ are continuous at the knots x_k.



- Effective degrees of freedom per segment:

  $n+1$ (polynomial coefficients) $-$ n (constraints) $=$ 1

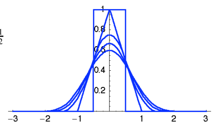
- **Cardinal splines** = unit spacing and infinite number of knots

  ⇨ The right framework for signal processing

B-spline basis functions

- B-spline of degree n

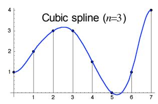
$$\beta^n(x) = \underbrace{\beta^0 * \beta^0 * \cdots * \beta^0}_{(n+1)\ \text{times}}(x), \qquad \beta^0(x) = \begin{cases} 1 & -\frac{1}{2} \le x < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

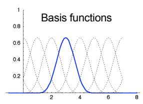


- **Theorem** (Schoenberg, 1946)

  Any cardinal spline can be represented as a linear combination of shifted B-splines:

$$s(x) = \sum_{k \in \mathbb{Z}} c(k) \beta^n(x-k)$$

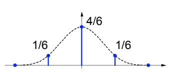




B-spline interpolation

- Discrete B-spline kernels

$$b_1^n(k) := \beta^n(x)\big|_{x=k} \quad \overset{z}{\longleftrightarrow} \quad B_1^n(z) = \sum_{k=-\lfloor n/2 \rfloor}^{\lfloor n/2 \rfloor} \beta^n(k) z^{-k}$$



- B-spline interpolation: filtering solution

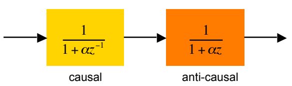
$$f(x)\big|_{x=k} = \sum_{l \in \mathbb{Z}} c(l) \beta^n(x-l)\big|_{x=k} = \left(b_1^n * c\right)(k) \qquad \Rightarrow \qquad c(k) = \left((b_1^n)^{-1} * f\right)(k)$$

- Efficient recursive solution

$$(b_1^3)^{-1}(k) \quad \overset{z}{\longleftrightarrow} \quad \frac{6}{z + 4 + z^{-1}} = \frac{-6\alpha}{(1-\alpha z)(1-\alpha z^{-1})}$$

(symmetric exponential)

⇨ Cascade of first order recursive filters



Generic C-code (splines of any degree n)

- Main recursion

```
void    ConvertToInterpolationCoefficients (
         double c[ ],  long DataLength,   double  z[ ],  long NbPoles,  double  Tolerance)
 {double Lambda = 1.0; long n, k;
   if (DataLength == 1L) return;
   for (k = 0L; k < NbPoles; k++) Lambda = Lambda * (1.0 - z[k]) * (1.0 - 1.0 / z[k]);
   for (n = 0L; n < DataLength; n++) c[n] *= Lambda;
   for (k = 0L; k < NbPoles; k++) {
      c[0] = InitialCausalCoefficient(c, DataLength, z[k], Tolerance);
      for (n = 1L; n < DataLength; n++) c[n] += z[k] * c[n - 1L];
      c[DataLength - 1L] = (z[k] / (z[k] * z[k] - 1.0))
                           * (z[k] * c[DataLength - 2L] + c[DataLength - 1L]);
      for (n = DataLength - 2L; 0 <= n; n--) c[n] = z[k] * (c[n + 1L]- c[n]); }
}
```

- Initialization

```
double  InitialCausalCoefficient (
         double  c[ ],  long   DataLength,  double  z,  double  Tolerance)
{  double Sum, zn, z2n, iz; long n, Horizon;
   Horizon = (long)ceil(log(Tolerance) / log(fabs(z)));
   if (DataLength < Horizon) Horizon = DataLength;
   zn = z; Sum = c[0];
   for (n = 1L; n < Horizon; n++) {Sum += zn * c[n]; zn *= z;}
   return(Sum);
}
```

Spline interpolation

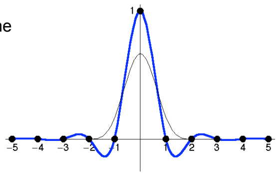
- Equivalent forms of spline representation

$$s(x) = \sum_{k\in Z} c(k)\beta^n(x-k) = \sum_{k\in Z}\left(s(k) * (b_1^n)^{-1}(k)\right)\beta^n(x-k)$$

$$= \sum_{k\in Z} s(k)\varphi^n_{\text{int}}(x-k)$$

- Cardinal (or fundamental) spline

$$\varphi^n_{\text{int}}(x) = \sum_{k\in Z}(b_1^n)^{-1}(k)\beta^n(x-k)$$



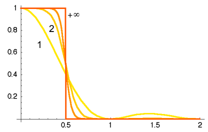
⇨ Finite cost implementation of an infinite impulse response interpolator !

Limiting behavior

- Spline interpolator

Impulse response | Frequency response

$$\varphi^n_{\text{int}}(x) \xleftrightarrow{\text{Fourier}} H^n(\omega) = \left(\frac{\sin(\omega/2)}{\omega/2}\right)^{n+1} \frac{1}{B_1^n(e^{j\omega})}$$



- Asymptotic property

The cardinal spline interpolators converge to the sinc-interpolator (ideal filter) as $n \to +\infty$:

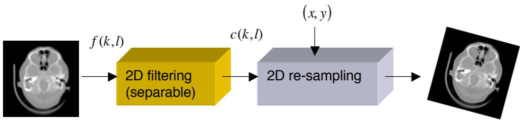
$$\lim_{n\to\infty}\varphi^n_{\text{int}}(\omega) = \text{sinc}(x), \qquad \lim_{n\to\infty} H^n(\omega) = \text{rect}(\omega/2\pi) \qquad \text{(in all } L_p\text{-norms)}$$

⇨ Includes Shannon's theory as a particular case !

Geometric transformation of images

- 2D separable model

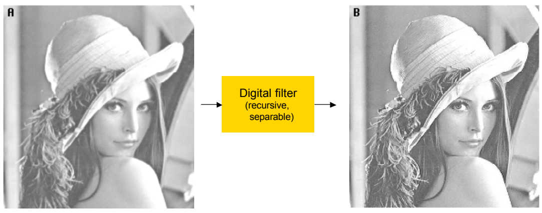
$$f(x,y) = \sum_{k=k_1}^{(k_1+K-1)} \sum_{l=l_1}^{(l_1+K-1)} c(k,l)\beta^n(x-k)\beta^n(y-l)$$



- Applications

zooming, rotation, re-sizing, re-formatting, warping

Cubic spline coefficients in 2D



Pixel values $f(k,l)$ | B-spline coefficients $c(k,l)$

Interpolation benchmark

Cumulative rotation experiment: the best algorithm wins !

Bilinear | Windowed-sinc | Cubic spline

SPLINES AND APPROXIMATION THEORY

- Order of approximation
- Quantitative L_2 approximation

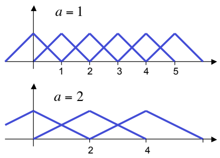
Order of approximation

- General "shift-invariant" space at scale a

$$V_a(\varphi) = \left\{ s_a(x) = \sum_{k \in Z} c(k)\varphi\left(\frac{x}{a} - k\right) : \; c \in \ell_2 \right\}$$

- Projection operator

$$\forall f \in L_2, \quad P_a f = \arg\min_{s_a \in V_a} \|f - s_a\|_{L_2} \quad \in V_a$$



- Order of approximation

DEFINITION

A scaling/generating function φ has order of approximation L iff

$$\forall f \in W_2^L, \quad \|f - P_a f\|_{L_2} \le C \cdot a^L \cdot \|f^{(L)}\|_{L_2}$$

B-splines of degree n have order of approximation $L=n+1$

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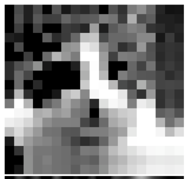
Spline reconstruction of a CAT-scan

Piecewise constant

$L = 1$



Cubic spline

$L = 4$

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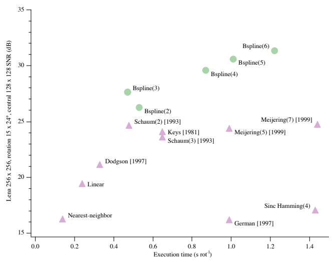
High-quality image interpolation

- B-splines & O-MOMS: best cost-performance tradeoff



Web demo

Thévenaz et al., *Handbook of Medical Image Processing*, 2000

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APPLICATION EXAMPLE

- Image Registration
  - Rigid body
  - Elastic

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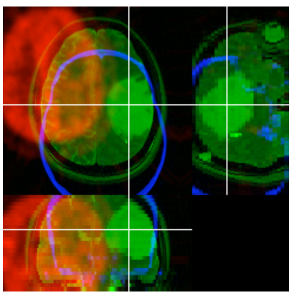
Splines: multi-modal image registration

Specificities of the approach

- Criterion: mutual-information (Collignon et al., 1996)
- Cubic spline model
  - high quality
  - sub-pixel accuracy
- Multiresolution strategy
- Marquardt-Levenberg like optimizer
  - Speed
  - Robustness



Thévenaz and Unser, *IEEE Trans. Imag Proc*, 2000

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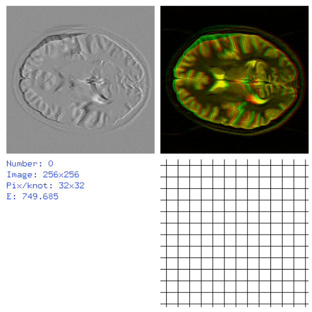
Splines: registration by elastic deformation

- Problem formulation
  - Optimization criterion:

reference image   test image

$$\min_{T \in V_a} \|f_1(\mathbf{x}) - f_2(T(\mathbf{x}))\|$$

- Principles of the method
  - Splines
    - continuous image representation
    - deformation model with adjustable size
  - Multiresolution; coarse-to-fine
    - image pyramids
    - deformation model
  - Non-linear optimization



Kybic et al., *IEEE Trans. Medical Imaging*, 2000

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INTERPOLATION OF NOISY DATA

- Tikhonov regularization
- Smoothing splines
- MMSE (or Wiener) solution

Fitting noisy data: Tikhonov regularization

- Context
  - Input data $\{f[k]\}_{k\in Z}$ corrupted by noise
  - Model : continuous-time function $s(t)$
  - Data term : $\xi_{\text{data}} = \sum_{k\in Z} |f[k]-s(k)|^2$
  - Spline energy : $\xi_L = \|D^L s\|^2_{L_2}$ (measures lack of smoothness)
- Possible formulations

A priori knowledge of noise variance — A priori knowledge of signal class

$$\min_{s(t)\in W_2^L} \xi_L \text{ subject to } \xi_{\text{data}} = C_1 \quad \text{OR} \quad \min_{s(t)\in W_2^L} \xi_{\text{data}} \text{ subject to } \xi_L \le C_2$$

⇕

$$\min_{s(t)\in W_2^L} \xi_{\text{data}} + \lambda\cdot\xi_L \quad \text{(Tikhonov regularization)}$$

Lagrange multiplier

Smoothing splines

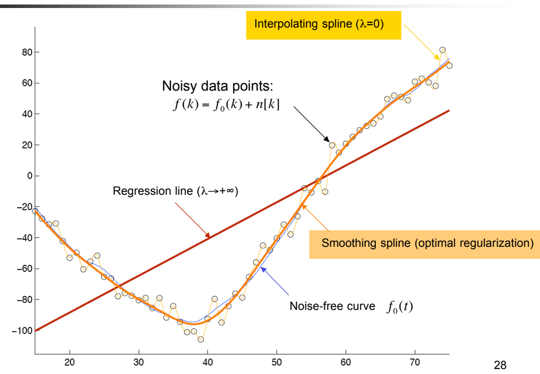
Theorem [Schoenberg, 1964]
For a given sequence $f[k]\in\ell_2$ and regularization parameter $\lambda\ge 0$, the minimizer of

$$s_\lambda(t) = \arg\min_{s\in W_2^L}\left\{\sum_{k\in Z}(f[k]-s(k))^2 + \lambda\cdot\|D^L s\|^2_{L_2}\right\}$$

is unique and is a cardinal polynomial spline of degree $n = 2L-1$.

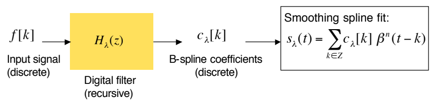
- Example : L=2
  - Spline energy : $\|D^2 s\|^2_{L_2} = \int_{-\infty}^{+\infty} |\ddot{s}(t)|^2 dt$ (bending energy)
  - Optimal solution: cubic spline (n=2×2-1=3)
  - Extreme cases:
    - λ→+∞ : best fitting line (linear regression)
    - λ→0 : cubic spline interpolant (minimum curvature solution)

Cubic smoothing spline: example



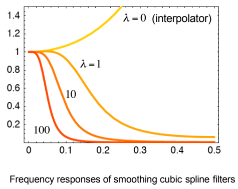
Smoothing splines: implementation

- Digital filter-based implementation



- Smoothing spline filter

$$H_\lambda(z) = \frac{1}{\sum_{k\in Z}\beta^n(k)z^{-k} + \lambda\cdot(-z+2-z^{-1})^L}$$



Frequency responses of smoothing cubic spline filters

Generalization: L*L smoothing splines

$L = D^L + a_1 D^{L-1} + \cdots + a_L I$: differential operator of order L $\overset{\mathcal{F}}{\longleftrightarrow}$ $\hat{L}(\omega) = \prod_{k=1}^{L}(j\omega - \alpha_i)$

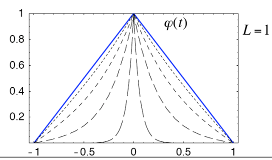
Theorem [U.-Blu, 2004]
For a given sequence $f[k]\in\ell_2$ and regularization parameter $\lambda\ge 0$, the minimizer of

$$s_\lambda(t) = \arg\min_{s\in W_2^L}\left\{\sum_{k\in Z}(f[k]-s(k))^2 + \lambda\cdot\|Ls\|^2_{L_2}\right\}$$

is the unique cardinal L*L-spline : $s_\lambda(t) = \sum_{k\in Z}(h_\lambda * f)[k]\,\varphi(t-k)$,
where $\varphi(t)$ is a corresponding exponential B-spline of order $2L$ and
where h_λ is a suitable smoothing spline filter.

- Exponential B-spline

$$\hat{\varphi}(\omega) = \frac{\prod_{k=1}^{L}|1-e^{\alpha_i - j\omega}|^2}{\prod_{k=1}^{L}|j\omega-\alpha_i|^2} \overset{\mathcal{F}^{-1}}{\longleftrightarrow}$$



Stochastic signal models

- Wide sense stationary processes

  Realization of the stochastic process: $x(t)$

  Zero mean: $E\{x(t)\}=0$

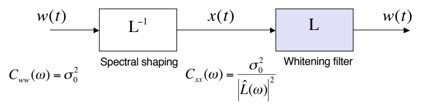
  Autocorrelation function : $E\{x(t)x(t+\tau)\}=c_{xx}(\tau)\in L_2$

  Spectral density function : $C_{xx}(\omega)=\mathcal{F}\{c_{xx}(\tau)\}\in L_2$

- Stochastic differential equation

  $L\{x(t)\}=w(t)$ (driven by white Gaussian noise)

  $c_{ww}(\tau)=\sigma_0^2\cdot\delta(\tau)$



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MMSE (or Wiener) solution

- Statistical hypotheses

  Discrete measurements (signal + noise) : $y[k]=x(k)+n[k]$

  Signal autocorrelation: $c_{xx}(\tau)$ such that $\mathrm{L}^*\mathrm{L}\{c_{xx}(\tau)\}=\sigma_0^2\cdot\delta(\tau)$

  Discrete white noise with variance σ^2 $\Rightarrow$ $c_{nn}[k]=\sigma^2\cdot\delta[k]$

- MMSE continuous-time signal estimation

  Theorem [U.-Blu, 2004]
  Under the above assumptions, the linear Minimum Mean Square Error estimator of $x(t)$ at time $t=t_0$ given the measurements $\{y[k]\}_{k\in Z}$ is $s_\lambda(t_0)$ with $\lambda=\sigma^2/\sigma_0^2$, where $s_\lambda(t)$ is $\mathrm{L}^*\mathrm{L}$ cardinal smoothing spline fit of $\{y[k]\}_{k\in Z}$, as specified previously.

  Remark: optimal over all estimators if one adds the assumption of Gaussianity

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On the optimality of splines

- Splines and continuous-time Tikhonov regularization
  - Spline interpolators are optimal: they have minimun « spline energy » (e.g., curvature) among all possible interpolants
  - Smoothing splines are optimal: they provide the best regularized fit of the input data, among all possible functions
- Splines are optimal statistical estimators
  - Smoothing spline = MMSE estimator for fractal-like ($1/\omega^L$) processes
  - Optimal regularization factor: $\lambda\propto\sigma^2$
  - Can yield optimal estimators of derivatives, etc.
  - Estimator can be fine tuned to the spectral characteristics of input signal ⇨ generalized splines

(work in progress)

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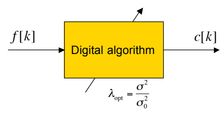
Bottom line for practical applications

1. Selection of the "optimal" spline space (e.g., cubic splines)
2. B-spline coefficients determination via an appropriate filtering algorithm (interpolation or smoothing spline)



3. Interpolation step (remains the same in all cases)

$$s(x)=\sum_{k\in Z}c[k]\varphi(x-k)$$

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CONCLUSION

- Distinctive features of splines
  - Simple to manipulate
  - Smooth and well-behaved
  - Excellent approximation properties
  - Multiresolution properties (Wavelets !)
  - Optimality properties (variational, statistical, …)
- Splines and medical imaging
  - A story of avoidance and, more recently, love….
  - Best cost/performance tradeoff
  - Many applications …..
- Unifying signal processing formulation
  - Tools: digital filters, convolution operators
  - Efficient recursive filtering solution
  - Exact calculus (differential operators, etc.)
  - Flexibility: piecewise constant to bandlimited

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Splines: the end of the tunnel

- Recent survey article on interpolation, *IEEE TMI*, 2000
  Comparison of 31 interpolation algorithms:
  "It [the cubic B-spline interpolator] produces one of the best results in terms of similarity to the original images, and of the top methods, it runs fastest."
- Addendum on spline interpolation, *IEEE TMI*, 2001
  "Therefore, high degree B-splines are preferable interpolators for numerous applications in medical imaging, particularly if high precision is required."
- Recent evaluation of interpolation, *Med. Image Anal.*, 2001
  Comparison of 126 interpolation algorithms:
  " The results show that spline interpolation is to be preferred over all other methods, both for its accuracy and its relatively low cost. "
- High-quality spline interpolation algorithms were included in the 2003 release of SPM (version 2b), a freely-available software package that is used worldwide for the statistical analysis of fMRI data.

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- Dr. Philippe Thévenaz
- Annette Unser, Artist

\+ many other researchers, and graduate students

- Software and demos at: http://bigwww.epfl.ch

Key references

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- Preprints and software can be downloaded at:
  http://bigwww.epfl.ch