Medical image interpolation: The quest for higher quality

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Medical Imaging Conference, Rome, October 16-22, 2004

OUTLINE
- Introduction
- Spline interpolation
- Splines and approximation theory
- Application example: image registration
- Interpolation in the presence of noise
- Conclusions

Back to the fundamentals: interpolation
- Fundamental issue in imaging and signal processing
  - Linking the discrete and the continuous
- Mismatch between classical theory and practice
  - Classical theory: Shannon’s sampling theorem
  - Practice: nearest neighbor, linear interpolation
- Limitations of Shannon’s sampling theory
  - Ideal lowpass filters do not exist
  - Incompatible with finite support signals
  - Gibbs oscillations
  - Slow decay of sinc(x)
- Basic problem
  - How do you interpolate a signal?

Interpolation and bio-imaging

Splines: a unifying framework
- Linking the discrete and the continuous...
Splines: Bad press phenomenon

- Classical review article on interpolation, IEEE TMI, 1983
- Comparison of four interpolators:
  "The cubic B-spline provides the most smoothing."
  About high order B-splines:
  "Out-of-band interpolation error reduces significantly for higher order interpolation functions, but at the expense of resolution error [i.e., distortion]."
- Recent book on Volume Rendering, 1998
  "The results of scaling the original image using [cubic] B-spline interpolation are shown in Figure 5.20. You can see the blurring effects ......."

Splines: definition

Definition: A polynomial spline of degree $s$, $s(x)$, with knots $\ldots < x_i < x_{i+1} < \ldots$ is a function with the following two properties:

- Piecewise polynomial: $s(x)$ is a polynomial of degree $n$ in each interval $[x_i, x_{i+1}]$;
- Higher-order continuity: $s(x_i), s'(x_i), \ldots, s^{(n-1)}(x_i)$ are continuous at the knots $x_i$.

- Effective degrees of freedom per segment:
  
  \[
  \text{Cardinal splines} = \text{unit spacing and infinite number of knots} \quad \Rightarrow \quad \text{The right framework for signal processing}
  \]

B-spline interpolation

- Discrete B-spline kernels
  \[f_n'(k) = f_n(k) \quad \text{with} \quad f_n(k) = \sum_{i} \beta_{i,n} f_i(k) \quad \text{for} \quad k \in Z\]
- B-spline interpolation: filtering solution
  \[f(k) = \sum_{n} c_n f_n(k) \rightarrow c_k = \left(\sum_{n} f_n(k)\right)_{k} \quad \text{for} \quad k \in Z\]
- Efficient recursive solution
  \[f_n'(k) = \sum_{l} \frac{\beta_{l,n}}{d} \quad \text{for} \quad d = 4 \times (n+1) \quad \text{and} \quad \beta_{l,n} = \text{symmetric exponential}\]

\[\text{Cascade of first order recursive filters}\]

Splines: basis functions

- B-spline of degree $n$:
  \[f_n(k) = f_n(k) \quad \text{for} \quad k \in Z\]

\[\text{Theorem (Schoenberg, 1946)}\]

\[\text{Any cardinal spline can be represented as a linear combination of shifted B-splines:}\]

\[x(s) = \sum_{n} c_n f_n(s - k)\]

Generic C-code (splines of any degree $n$)

- Main recursion

```c
void GenericCcode(int Degree, int NbPoles, double Tolerance) {
    double Lambda = 1.0; long n, k;
    nbpoles = NbPoles; double  z[3],  long NbPoles,  double  Tolerance)
            for (n = DataLength - 2L; 0 <= n; n--) c[n] = z[k] * (c[n + 1L] - c[n]); }
    for (n = 1L; n < Horizon; n++) {Sum += c[n] = z[k] * (c[n + 1L] - c[n]); }
    if (Horizon < Horizon) Horizon = Horizon;
    return(Sum);
}
```

- Initialization

```c
double  z[3],  long NbPoles,  double  Tolerance) {
            for (n = DataLength - 2L; 0 <= n; n--) c[n] = z[k] * (c[n + 1L] - c[n]); }
    for (n = 1L; n < Horizon; n++) {Sum += c[n] = z[k] * (c[n + 1L] - c[n]); }
    if (Horizon < Horizon) Horizon = Horizon;
    return(Sum);
}
```
Spline interpolation

- Equivalent forms of spline representation

\[
x(n) = \sum_{k \in \mathbb{Z}} x(k) \phi_n(x - k) = \sum_{k \in \mathbb{Z}} (x(k) - x(k - 1)) \phi_n(x - k)
\]

- Cardinal (or fundamental) spline

\[
\phi_n(x) = \sum_{k \in \mathbb{Z}} \phi_n(x - k)
\]

Finite cost implementation of an infinite impulse response interpolator

Limiting behavior

- Spline interpolator

\[
\phi_n(x) = \frac{\sin(\frac{x}{2})}{\frac{x}{2}} + B_1 \delta_1(x)
\]

- Asymptotic property

\[
\lim_{n \to \infty} \phi_n(x) = \text{sinc}(x), \quad \lim_{n \to \infty} H_n(x) = \text{rect}\left(\frac{x}{2\pi}\right)
\]

The cardinal spline interpolator converges to the sinc-interpolator (ideal filter) as \( n \to \infty \):

- Includes Shannon’s theory as a particular case!

Geometric transformation of images

- 2D separable model

\[
f(k, l) = \sum_{k=0}^{K-1} \sum_{l=0}^{L-1} c(k, l) \phi_n(x - k) \phi_n(y - l)
\]

- Applications

  - zooming, rotation, re-sizing, re-formatting, warping

Cubic spline coefficients in 2D

- Digital filter (recursive, separable)

- Pixel values \( f(k, l) \)

- B-spline coefficients \( c(k, l) \)

Interpolation benchmark

Cumulative rotation experiment: the best algorithm wins!

- Bilinear
- Windowed-sinc
- Cubic spline

SPLINES AND APPROXIMATION THEORY

- Order of approximation
- Quantitative \( L_2 \) approximation
Order of approximation

- General “shift-invariant” space at scale \( a \):
  \[ V_0(g) = \{ f \} : f = \sum_{i} \psi_i \tau_i(x) \]  
- Projection operator
  \[ \forall f \in L^2, \quad P_a f = \arg\min_{s \in V_a} f - s \in L^2 \subseteq V_a \]  
- Order of approximation
  \[ A \text{ scaling/generating function } \psi \text{ has order of approximation } L \iff \forall f \in W^L, \quad f - P_a f \in L^2 \leq C \cdot a^L \cdot f(L^2) \]  

DEFINITION

B-splines of degree \( n \) have order of approximation \( L_n = n + 1 \)

Spline reconstruction of a CAT-scan

- Piecewise constant
- Cubic spline
  \[ L = 4 \]  

High-quality image interpolation

- B-splines & O-MOMS: best cost-performance tradeoff

APPLICATION EXAMPLE

- Image Registration
  - Rigid body
  - Elastic

Splines: multi-modal image registration

Specificities of the approach
- Criterion: mutual-information (Colignon et al., 1996)
- Cubic spline model
  - High quality
  - Sub-pixel accuracy
- Multiresolution strategy
- Marquardt-Levenberg like optimizer
  - Speed
  - Robustness

Splines: registration by elastic deformation

- Problem formulation
  - Optimization criterion: reference image vs. test image
- Minimize \( \| f(x) - f(x') \| \)

- Principles of the method
  - Splines
    - Continuous image representation
    - Deformation model with adjustable size
  - Multiresolution, coarse-to-fine
  - Image pyramids
  - Deformation model
  - Non-linear optimization


Kybic et al., IEEE Trans. Medical Imaging, 2000
INTERPOLATION OF NOISY DATA

- Tikhoonov regularization
- Smoothing splines
- MMSE (or Wiener) solution

Fitting noisy data: Tikhonov regularization

- Context
  - Input data \( \{ y(n) \}_{n=1}^{N} \) corrupted by noise
  - Model: continuous-time function \( x(t) \)
  - Data term: \( z_n = \sum_{k=-\infty}^{\infty} y(n-k) \)
  - Spline energy: \( E_z = \sum_{n} |z(n)|^2 \) (measures lack of smoothness)

- Possible formulations
  - Minimize \( \sum_{n} |z(n)|^2 + \lambda \sum_{n} |\dot{z}(n)|^2 \)
  - Subject to \( \sum_{n} z_n = C \)

Smoothing splines

- Theorem [Schoenberg, 1944]
  For a given sequence \( f(k) \in \mathbb{Z} \) and regularization parameter \( \lambda \neq 0 \), the minimizer of
  \( \sum_{k=-\infty}^{\infty} (f(k) - \dot{x}(k))^2 + \lambda \sum_{k=-\infty}^{\infty} |\ddot{x}(k)|^2 \)
  is unique and is a cardinal polynomial spline of degree \( n = 2\lambda - 1 \).

- Example: \( \lambda = 2 \)
  - Spline energy \( \sum_{k=-\infty}^{\infty} (f(k) - \dot{x}(k))^2 \) (bending energy)
  - Optimal solution: cubic spline \((n=2; x=1=3)\)
  - Extreme cases:
    - \( \lambda \to \infty \) (least fitting line (linear regression))
    - \( \lambda \to 0 \) (cubic spline interpolator (minimum curvature solution))

- Possible formulations
  - Minimize \( \sum_{k=-\infty}^{\infty} (f(k) - \dot{x}(k))^2 + \lambda \sum_{k=-\infty}^{\infty} |\ddot{x}(k)|^2 \)
  - Subject to \( \sum_{k=-\infty}^{\infty} z_k = C \)

- Generalization: \( \sum_{k} \lambda_k = 2 \lambda \) smoothing splines
  - \( L = \sum_{k} \lambda_k = 2 \lambda \) : differential operator of order \( 2\lambda \)
  - \( \sum_{k=-\infty}^{\infty} (f(k) - \dot{x}(k))^2 + \lambda \sum_{k=-\infty}^{\infty} |\ddot{x}(k)|^2 \)
  is the unique cardinal \( L \)-spline: \( \dot{x}(k) = \sum_{n=-\infty}^{\infty} (f(n) \psi(n-k)) \)
  where \( \psi(n) \) is a corresponding exponential B-spline of order \( 2\lambda \) and where \( \lambda \) is a suitable smoothing spline filter.
**Stochastic signal models**

- Wide sense stationary processes
  - Realization of the stochastic process: \( x(t) \)
  - Zero mean: \( E[x(t)] = 0 \)
  - Autocorrelation function: \( R_x(t) = E[x(t)x(t+\tau)] \in L_2 \)
  - Spectral density function: \( S_x(f) = F[R_x(t)] \in L_2 \)

- Stochastic differential equation
  - \( L_x = w(t) \)
  - \( \dot{x}_\tau(t) = \alpha_s^2 \delta(t) \)

**MMSE (or Wiener) solution**

- Statistical hypotheses
  - Discrete measurements (signal + noise): \( y(k) = x(k) + n(k) \)
  - Signal autocorrelation: \( c_x(\tau) \) such that \( L_x c_x(\tau) = c_x(\tau) \alpha_s^2 \delta(\tau) \)
  - Discrete white noise with variance \( \sigma_n^2 \rightarrow c_n(\tau) = \sigma_n^2 \delta(\tau) \)

- MMSE continuous-time signal estimation

**Bottom line for practical applications**

1. Selection of the "optimal" spline space (e.g., cubic splines)
2. B-spline coefficients determination via an appropriate filtering algorithm (interpolation or smoothing spline)
3. Interpolation step (remains the same in all cases)

**On the optimality of splines**

- Splines and continuous-time Tikhonov regularization
  - Spline interpolators are optimal: they have minimum \( s \) spline energy (e.g., curvature) among all possible interpolants
  - Smoothing splines are optimal: they provide the best regularized fit of the input data, among all possible functions

- Splines are optimal statistical estimators
  - Smoothing spline = MMSE estimator for fractal-like (1/\( f^\alpha \)) processes
  - Optimal regularization factor: \( \lambda \propto \sigma_s^2 \)
  - Can yield optimal estimators of derivatives, etc.
  - Estimator can be fine tuned to the spectral characteristics of input signal \( s \) generalized splines (work in progress)

**CONCLUSION**

- Distinctive features of splines
  - Simple to manipulate
  - Smooth and well-behaved
  - Excellent approximation properties
  - Multiresolution properties (Wavelets)
  - Optimality properties (variational, statistical, ...)

- Splines and medical imaging
  - A story of avoidance and, more recently, love....
  - Best cost/performance tradeoff
  - Many applications ....

- Unifying signal processing formulation
  - Tools: digital filters, convolution operators
  - Efficient recursive filtering solution
  - Exact calculus (differential operators, etc.)
  - Flexibility: piecewise constant to bandlimited
Acknowledgments

Many thanks to

- Dr. Thierry Blu
- Prof. Akram Aldroubi
- Prof. Murray Eden
- Dr. Philippe Thévenaz
- Annette Unser, Artist
- many other researchers, and graduate students

Software and demos at: [http://bigwww.epfl.ch](http://bigwww.epfl.ch)

Key references

- **Spline basics**

- **Splines and approximation theory**

- **Comparison of interpolators**

- **Interpolation of noisy data**

Preprints and software can be downloaded at:

[http://bigwww.epfl.ch](http://bigwww.epfl.ch)