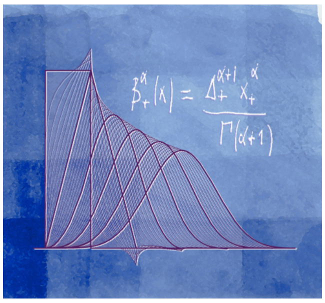
ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Ten reasons for using splines for signal & image processing

Michael Unser

Biomedical Imaging Group
EPFL, Lausanne
Switzerland



Approximation et modélisation géométrique, Paris, March 2008

OUTLINE

- Introduction
- Cardinal-spline formalism
- Ten+ good reasons for using B-splines
  - Computational
  - Theoretical
  - Conceptual
  - Practical
- Application examples in image processing

INTRODUCTION

- **Fundamental issue in signal/image processing**

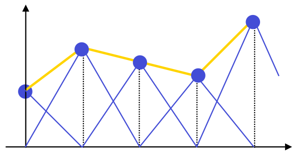
  Linking the *discrete* and the *continuous*

Acquisition

Algorithm design

- **Mismatch between theory and practice**
  - Theory : Shannon's sampling theorem
  - Practice: nearest neighbor, linear interpolation

- **Limitations of Shannon sampling theory**
  - Ideal lowpass filters do not exist
  - Incompatible with finite support signals
  - Gibbs oscillations
  - Slow decay of $\mathrm{sinc}(x)$

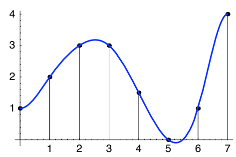


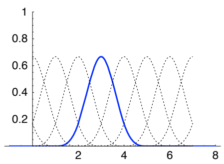
- **Basic problem**

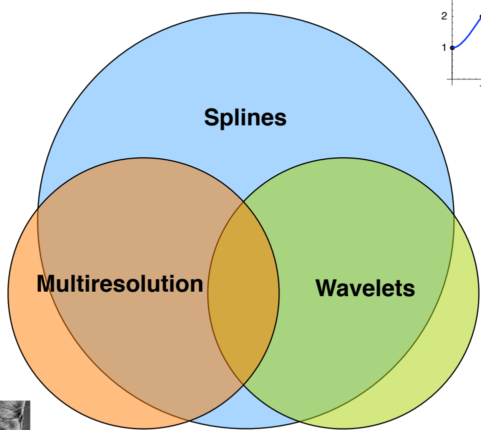
  How do you interpolate a signal efficiently ?

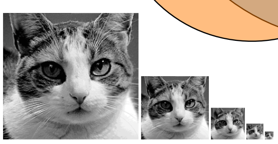
Splines: a unifying framework

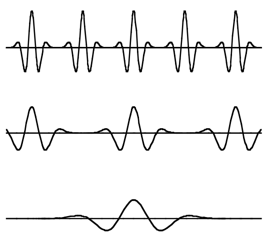
Linking the discrete and the continuous …..









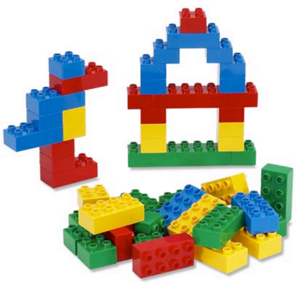


Spline interpolation and biomedical imaging

Image processing task	Specific operation	Imaging modality
Tomographic reconstruction	• Filtered backprojection • Fourier reconstruction • Iterative techniques • 3D + time	Commercial CT (X-rays) EM PET, SPECT Dynamic CT, SPECT, PET
Sampling grid conversion	• Polar-to-cartesian coordinates • Spiral sampling • k-space sampling • Scan conversion	Ultrasound (endovascular) Spiral CT, MRI MRI
Visualization	*2D operations* • Zooming, panning, rotation • Re-sizing, scaling	All
	• Stereo imaging • Range, topography	Fundus camera OCT
	3D operations • Re-slicing • Max. intensity projection • Simulated X-ray projection	CT, MRI, MRA
	Surface/volume rendering • Iso-surface ray tracing • Gradient-based shading • Stereogram	CT MRI
Geometrical correction	• Wide-angle lenses • Projective mapping • Aspect ratio, tilt • Magnetic field distortions	Endoscopy C-Arm fluoroscopy Dental X-rays MRI
Registration	• Motion compensation • Image subtraction • Mosaicking • Correlation-averaging • Patient positioning • Retrospective comparisons • Multi-modality imaging • Stereotactic normalization • Brain warping	fMRI, fundus camera DSA Endoscopy, fundus camera, EM microscopy Surgery, radiotherapy CT/PET/MRI
Feature detection	• Contours • Ridges • Differential geometry	All
	Contour extraction • Snakes and active contours	MRI, Microscopy (cytology)

CARDINAL SPLINE FORMALISM

- Distributional definition: L-splines
- Basic atoms
- Polynomial B-splines



General concept of an *L*-spline

$\mathrm{L}\{\cdot\}$: differential operator (translation-invariant)

$\delta(\boldsymbol{x}) = \prod_{i=1}^{d} \delta(x_i)$: multidimensional Dirac distribution

Definition

The continuous-domain function $s(\boldsymbol{x})$ is a ***cardinal L-spline*** iff.

$$\mathrm{L}\{s\}(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} a[\boldsymbol{k}]\delta(\boldsymbol{x}-\boldsymbol{k})$$

- Cardinality: the knots (or spline singularities) are on the (multi-)integers
  $\Rightarrow$ ideal framework for signal processing
- Generalization: includes polynomial splines as particular case ($\mathrm{L} = \frac{\mathrm{d}^N}{\mathrm{d}x^N}$)

Example: piecewise-constant splines

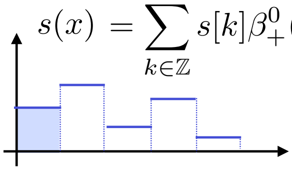
- Spline-defining operators

  Continuous-domain derivative: $\mathrm{D} = \dfrac{\mathrm{d}}{\mathrm{d}x} \quad \longleftrightarrow \quad j\omega$

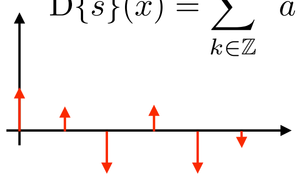
  Discrete derivative: $\Delta_+\{\cdot\} \quad \longleftrightarrow \quad 1 - e^{-j\omega}$

- Piecewise-constant or D-spline

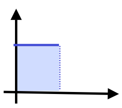
$$s(x) = \sum_{k\in\mathbb{Z}} s[k]\beta_+^0(x-k)$$



$$\mathrm{D}\{s\}(x) = \sum_{k\in\mathbb{Z}} \overbrace{a[k]}^{\Delta_+ s(k)}\,\delta(x-k)$$



- B-spline function



$$\beta_+^0(x) = \Delta_+\mathrm{D}^{-1}\{\delta\}(x) \quad \longleftrightarrow \quad \frac{1-e^{-j\omega}}{j\omega}$$

Existence of a local, shift-invariant basis?

- Space of cardinal L-splines

$$V_{\mathrm{L}} = \left\{ s(\boldsymbol{x}) : \mathrm{L}\{s\}(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} a[\boldsymbol{k}]\delta(\boldsymbol{x}-\boldsymbol{k}) \right\} \cap L_2(\mathbb{R}^d)$$

- Generalized B-spline representation

A "localized" function $\varphi(\boldsymbol{x}) \in V_{\mathrm{L}}$ is called *generalized B-spline* if it generates a Riesz basis of V_{L}; i.e., iff. there exists $(A > 0, B < \infty)$ s.t.

$$A \cdot \|c\|_{\ell_2(\mathbb{Z}^d)} \leq \left\| \sum_{\boldsymbol{k}\in\mathbb{Z}^d} c[\boldsymbol{k}]\varphi(\boldsymbol{x}-\boldsymbol{k}) \right\|_{L_2(\mathbb{R}^d)} \leq B \cdot \|c\|_{\ell_2(\mathbb{Z}^d)}$$

$$\Downarrow$$

$$V_{\mathrm{L}} = \left\{ s(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} c[\boldsymbol{k}]\varphi(\boldsymbol{x}-\boldsymbol{k}) : \boldsymbol{x}\in\mathbb{R}^d, c \in \ell_2(\mathbb{Z}^d) \right\}$$

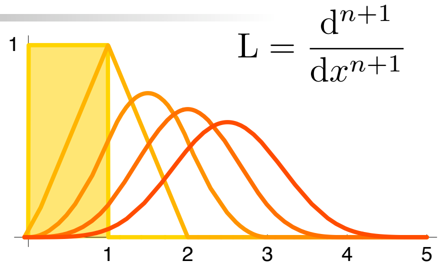
continuous-domain signal

discrete signal
(B-spline coefficients)

Polynomial B-splines

- B-spline of degree n

$$\beta_+^n(x) = \underbrace{\beta_+^0 * \beta_+^0 * \cdots * \beta_+^0}_{(n+1)\text{ times}}(x)$$



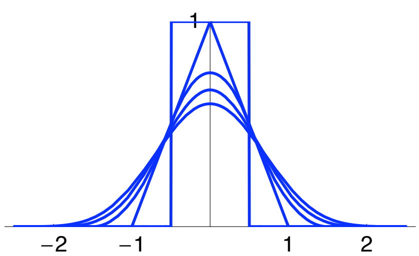
$$\mathrm{L} = \frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}$$

$$\beta_+^0(x) = \begin{cases} 1, & x \in [0,1) \\ 0, & \text{otherwise.} \end{cases}$$

- Key properties
  - Riesz basis generator for the cardinal polynomial splines
  - Shortest polynomial spline of degree n

- Symmetric B-spline

$$\beta^n(x) = \beta_+^n\left(x + \tfrac{n+1}{2}\right)$$

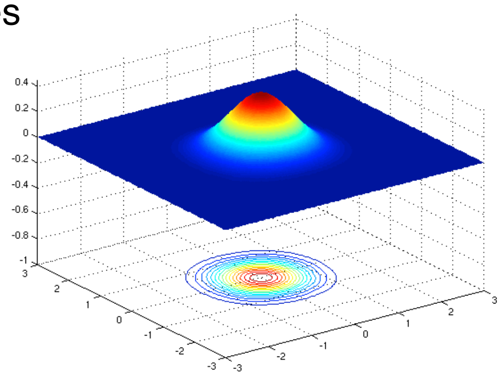


B-spline representation of images

- Symmetric, tensor-product B-splines

$$\beta^n(x_1, \cdots, x_d) = \beta^n(x_1) \times \cdots \times \beta^n(x_d)$$

$$\mathrm{L} = \frac{\partial^{dn+d}}{\partial x_1^{n+1} \cdots \partial x_d^{n+1}}$$



- Multidimensional spline function

$$s(x_1, \cdots, x_d) = \sum_{(k_1, \cdots k_d) \in \mathbb{Z}^d} c[k_1, \cdots, k_d] \; \beta^n(x_1 - k_1, \cdots, x_d - k_d)$$

continuous-space image

image array
(B-spline coefficients)

Compactly supported
basis functions

TEN REASONS FOR USING SPLINES

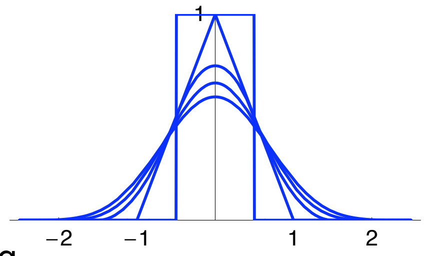
- Mathematical elegance
- Fast algorithms
- Approximation theory
- Link with *<your favorite>* theory

⋮

1. B(eautiful) basis functions

- Polynomial B-splines (centered)

$$\beta^n(x) = \frac{\Delta^{n+1} x_+^n}{(n+1)!} = (\beta^0 * \beta^{n-1})(x)$$



- Attractive properties for image processing
  - Compact support: shortest polynomial spline of degree n
  - Symmetry
  - Positivity
  - Controlled smoothness: Hölder-continuous of order n
  - Bell-shaped (optimal space-frequency localization)

$$\beta^n(x) \to \frac{1}{\sqrt{2\pi}\sigma_n} \exp(\frac{-x^2}{2\sigma_n^2}) \quad \text{with } \sigma_n = \sqrt{\tfrac{n+1}{12}}$$

Reference: (Schoenberg, 1946)

2. Fast digital-filtering algorithms

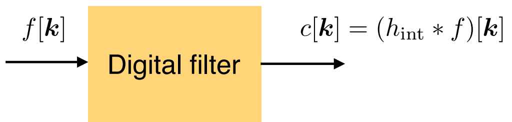
All classical spline interpolation and approximation problems can be solved efficiently using recursive digital filtering

- Interpolation problem

Given the signal samples $f[\boldsymbol{k}]$, find the B-spline coefficients $c[\boldsymbol{k}]$ such that

$$f(\boldsymbol{x})|_{\boldsymbol{x}=\boldsymbol{k}} = f[\boldsymbol{k}] = \sum_{\boldsymbol{k}_1 \in \mathbb{Z}^p} c[\boldsymbol{k}_1]\varphi(\boldsymbol{k} - \boldsymbol{k}_1)$$

$\Rightarrow$ Inverse filtering solution



with $H_{\text{int}}(\mathbf{z}) = \frac{1}{B(\mathbf{z})} = \frac{1}{\sum_{\boldsymbol{k} \in \mathbb{Z}^p} \varphi(\boldsymbol{k})\mathbf{z}^{-\boldsymbol{k}}}$

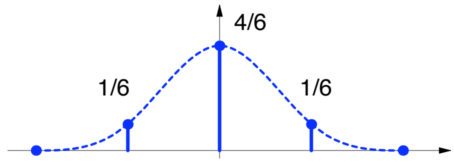
Note: $\varphi(\boldsymbol{x})$ separable $\Rightarrow$ $h_{\text{int}}[\boldsymbol{k}]$ separable

Reference: B-spline signal processing (Unser, *IEEE-SP* 1993)

Example: cubic-spline interpolation

- Cubic B-spline

$$\varphi(x) = \beta^3(x) = \begin{cases} \frac{2}{3} - \frac{1}{2}|x|^2(2-|x|), & 0 \le |x| < 1 \\ \frac{1}{6}(2-|x|)^3, & 1 \le |x| < 2 \\ 0, & \text{otherwise} \end{cases}$$



- Discrete B-spline kernel: $B(z) = \frac{z + 4 + z^{-1}}{6}$

- Interpolation filter

$$\frac{6}{z+4+z^{-1}} = \frac{(1-\alpha)^2}{(1-\alpha z)(1-\alpha z^{-1})} \quad \stackrel{z}{\longleftrightarrow} \quad h_{\text{int}}[k] = \left(\frac{1-\alpha}{1+\alpha}\right)\alpha^{|k|}$$

(symmetric exponential)

$$\alpha = -2 + \sqrt{3} = -0.171573$$

⇨ Cascade of first-order recursive filters

$$\longrightarrow \boxed{\frac{1}{1-\alpha z^{-1}}} \longrightarrow \boxed{\frac{1}{1-\alpha z}} \longrightarrow$$

causal — anti-causal

3. Simple manipulations

The polynomial spline family is closed with respect to differentiation

- Derivative operator

$$\mathrm{D}f(x) = \frac{df(x)}{dx}$$

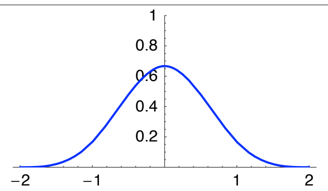
- Finite-difference operator (centered)

$$\Delta f(x) \triangleq f(x + \tfrac{1}{2}) - f(x - \tfrac{1}{2})$$

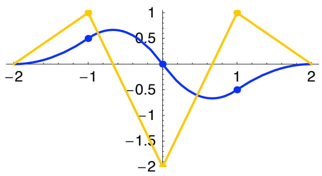
- Derivative of a B-spline (exact)

$$\mathrm{D}^m \beta^n(x) = \Delta^m \beta^{n-m}(x)$$

Discrete operator — Reduction of degree



Example: cubic B-spline



Reference: (Schoenberg, 1946)

4. Link with system theory: C-to-D converters

Exponential B-splines = the mathematical translators between continuous-time and discrete-time LSI system theories

Continuous domain
- differential equations
- circuits, analog filters
- Laplace transform:

Discrete domain
- difference equations
- digital filters
- z-transform:

zeros

$$H_{\mathrm{C}}(s) = \frac{\prod_{m=1}^{M}(s - \gamma_m)}{\prod_{n=1}^{N}(s - \alpha_n)} \qquad H_{\mathrm{D}}(z) = \frac{1}{\prod_{n=1}^{N}(z - z_n)}$$

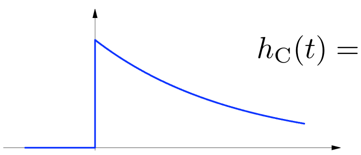
poles mapping: $z_n = e^{\alpha_n}$

Associated B-spline: $\beta_{\vec{\alpha}}(t) = \mathcal{L}^{-1}\left\{\frac{H_{\mathrm{C}}(s)}{H_{\mathrm{D}}(e^s)}\right\}(t)$

Reference: “Think analog, act digital” (Unser, *IEEE-SP* 2006)

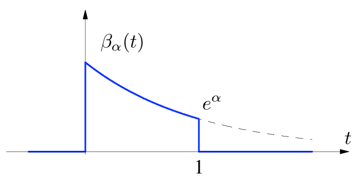
Example: 1st order system

- Continuous-time impulse response



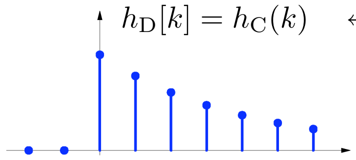
$$h_{\mathrm{C}}(t) = 1_{+}(t)\cdot e^{\alpha t} = \begin{cases} e^{\alpha t}, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad \stackrel{\mathcal{L}}{\longleftrightarrow} \quad H_{\mathrm{C}}(s) = \frac{1}{s-\alpha}$$

1st-order exponential B-spline



- Discrete-time counterpart

$$h_{\mathrm{D}}[k] = h_{\mathrm{C}}(k) \quad \stackrel{z}{\longleftrightarrow} \quad H_{\mathrm{D}}(z) = \frac{1}{z - e^{\alpha}}$$



Discrete-time signal

$$h_{\mathrm{C}}(t) = 1_{+}(t)\cdot e^{\alpha t} = \sum_{k=0}^{+\infty} e^{\alpha k}\beta_\alpha(t-k) = \sum_{k\in\mathbb{Z}} h_{\mathrm{D}}[k]\,\beta_\alpha(t-k)$$

Continuous-time signal

Compactly-supported basis functions

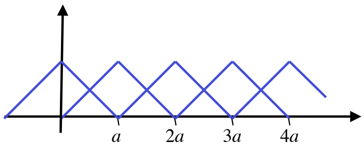
5. Best cost-performance tradeoff

Polynomial B-splines have
- maximum order of approximation for a minimum support (MOMS)
- a low asymptotic approximation constant.

This explains their superior performance in imaging applications.

- Approximation of a function at scale a

$$V_a(\varphi) = \left\{ s_a(x) = \sum_{k \in \mathbb{Z}} c[k] \varphi\left(\frac{x}{a} - k\right) : c \in \ell_2 \right\}$$



Definition: A generating function φ has order of approximation γ iff.

$$\forall f \in W_2^\gamma, \qquad \arg \min_{s_a \in V_a} \|f - s_a\|_{L_2} \leq C_\gamma \cdot a^\gamma \cdot \|f^{(\gamma)}\|_{L_2}$$

- $\beta^n(x)$ has order of approximation $\gamma = n + 1$ and $C_{\gamma,\min} = \frac{\sqrt{2\zeta(2\gamma)}}{(2\pi)^\gamma}$

Reference: (Strang-Fix, 1973; Blu-U., *IEEE-SP* 1999)

Interpolation benchmark

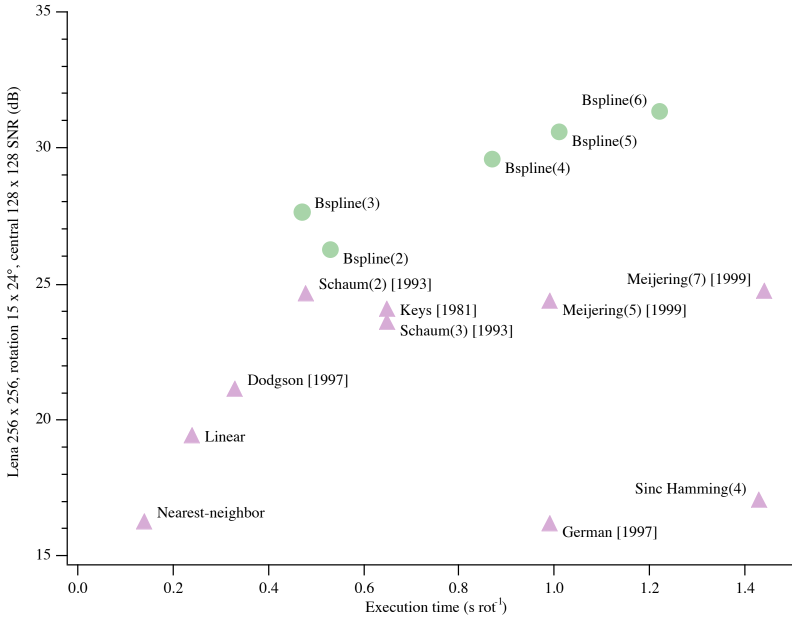
Cumulative rotation experiment: the best algorithm wins !

Bilinear

Windowed-sinc

Cubic spline

High-quality image interpolation



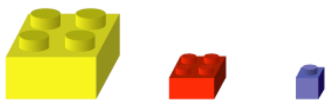
Demo

Thévenaz et al., *Handbook of Medical Image Processing*, 2000

6. Link with wavelet theory

Polynomial B-splines have remarkable dilation properties.
They play a fundamental role in wavelet theory.

- Generalized Lego™/Duplo™ relation



$$\beta_+^0(x/2) = \beta_+^0(x) + \beta_+^0(x-1)$$

B-spline dilation property: $\beta_+^n(x/2) = \sum_{k\in\mathbb{Z}} h_2^n[k]\beta_+^n(x-k)$

Binomial filter: $H_2^n(z) = \frac{1}{2^n}\sum_{k=0}^{n+1}\binom{n+1}{k} z^{-k} = \frac{1}{2^n}\left(1+z^{-1}\right)^{n+1}$

B-spline factorization theorem

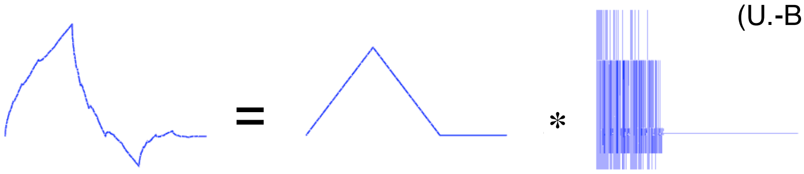
Theorem: A valid scaling function $\varphi(x)$ has order of approximation γ iff.

$$\varphi(x) = \left(\beta_+^\alpha * \varphi_0\right)(x)$$

where β_+^α with $\alpha = \gamma - 1$: regular, B-spline part

$\varphi_0 \in S'$: irregular, distributional part

(U.-Blu, *IEEE-SP*, 2003)



- Refinement filter: general case

$$H(z) = \underbrace{\left(\frac{1+z^{-1}}{2}\right)^\gamma}_{\text{spline part}} \cdot \underbrace{Q(z)}_{\text{distributional part}} \qquad \text{with} \quad |Q(e^{j\omega})| < +\infty$$

Splines: the key to wavelet theory

Sobolev smoothness
$\partial^s \varphi \in L_2$

$\Downarrow \; \gamma \geq s$

B-spline factorization: $\varphi = \beta_+^{\gamma-1} * \varphi_0$ $\Leftrightarrow$ Approximation order: $\|f - P_a f\|_{L_2} = O(a^\gamma)$ $\Leftrightarrow$ Multi-scale differentiator $\hat{\tilde{\psi}}(\omega) \propto (-j\omega)^\gamma, \; \omega \to 0$

$\Downarrow \quad \Uparrow$

general case: $N < \gamma \leq N+1$

compact support: $\gamma = N+1$ (Strang-Fix)

Polynomial reproduction
degree: $N = \lceil \gamma - 1 \rceil$

$\Updownarrow$

Vanishing moments:
$\int x^n \tilde{\psi}(x) \mathrm{d}x = 0, \; n = 0, \cdots, N$

Reference: Wavelet theory demystified (Unser-Blu, *IEEE-SP*, 2003)

7. Link with regularization theory

Spline estimators are optimal from a variational point of view.

- Smoothing-spline estimator

Discrete, noisy input:

$f[k] = s_{\text{ref}}(k) + \text{noise}$ → **Smoothing algorithm** →

Continuous-domain estimate:

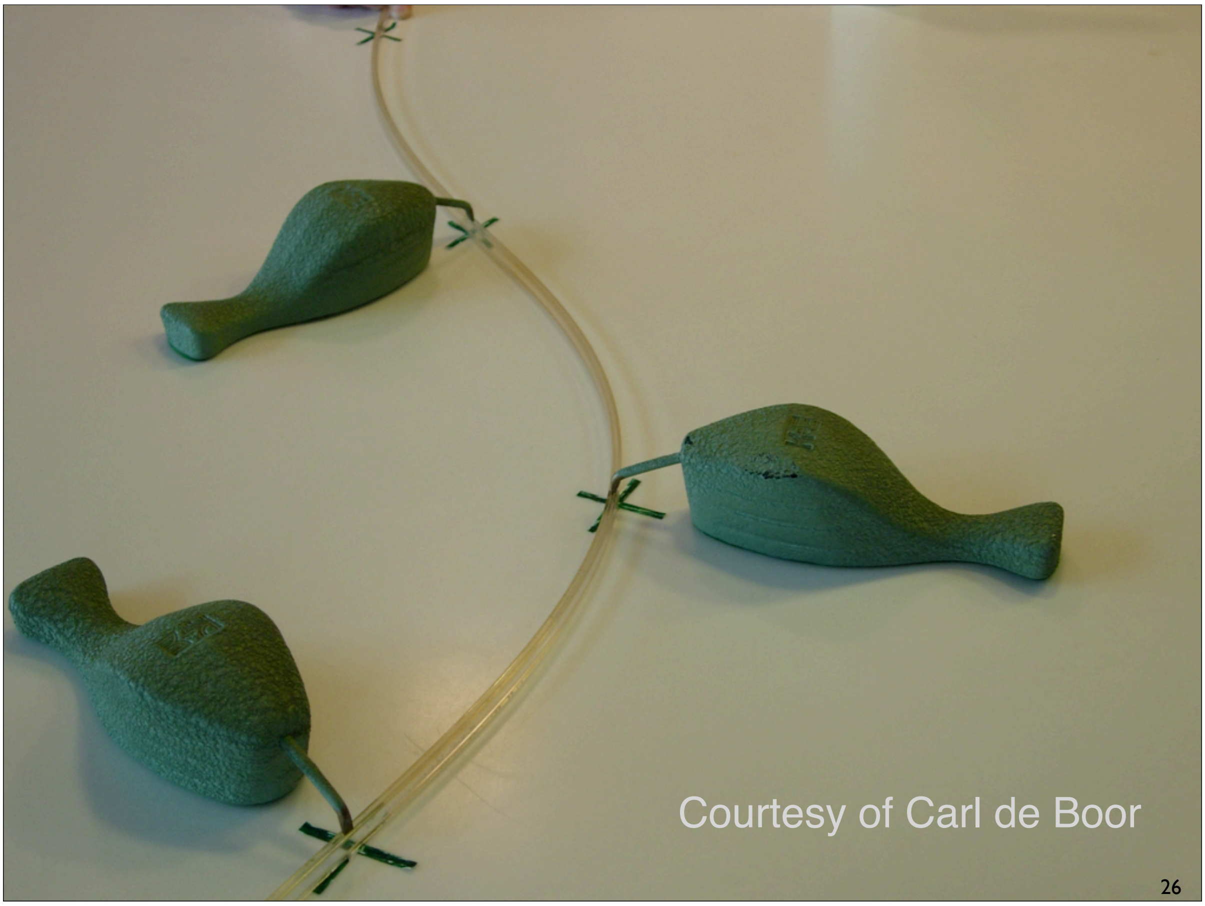
$s(x) = \sum_{k \in \mathbb{Z}} c[k] \beta^n (x - k)$

Theorem: The solution (among all functions) of the smoothing-spline problem

$$\min_{s(x)} \left\{ \sum_{k \in \mathbb{Z}} |f[k] - s(k)|^2 + \lambda \int_{-\infty}^{+\infty} |\mathrm{D}^m s(x)|^2 \mathrm{d}x \right\}$$

is a cardinal spline of degree $2m - 1$. In addition, its coefficients $c[k] = h_\lambda * f[k]$ can be obtained by suitable digital filtering of the input samples $f[k]$.

References: theory (Schoenberg, 1964), recursive filtering algorithm (Unser, 1993)



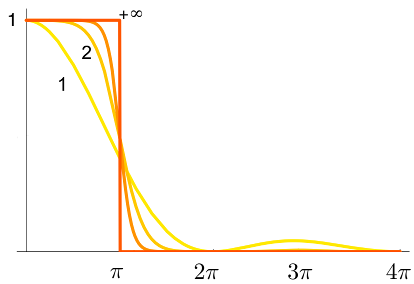
8. Link with Shannon's sampling theory

The Hilbert-space formulation of polynomial spline approximation provides an extension of Shannon's classical sampling theorem.

- Polynomial spline interpolator

Impulse response   Frequency response

$$\varphi^n_{\text{int}}(x) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad \hat{\varphi}^n_{\text{int}}(\omega) = \underbrace{\left(\frac{\sin(\omega/2)}{\omega/2}\right)^{n+1}}_{\hat{\beta}^n(\omega)} H^n_{\text{int}}(e^{j\omega})$$



- Asymptotic property

The cardinal-spline interpolators converge to the sinc interpolator (ideal filter) as the degree goes to infinity:

$$\lim_{n\to\infty} \varphi^n_{\text{int}}(x) = \text{sinc}(x), \quad \lim_{n\to\infty} \hat{\varphi}^n_{\text{int}}(\omega) = \text{rect}\left(\frac{\omega}{2\pi}\right) \qquad \text{(in all } L_p\text{-norms)}$$

References: (Schoenberg, 1973; Unser, *Proc. IEEE*, 2000)

9. Link with stochastic processes

Splines are in direct correspondence with stochastic processes (stationary or fractals) that are solution of the same partial differential equation, but with a random driving term.

Defining operator equation: $\mathrm{L}\{s(\cdot)\}(\boldsymbol{x}) = r(\boldsymbol{x})$

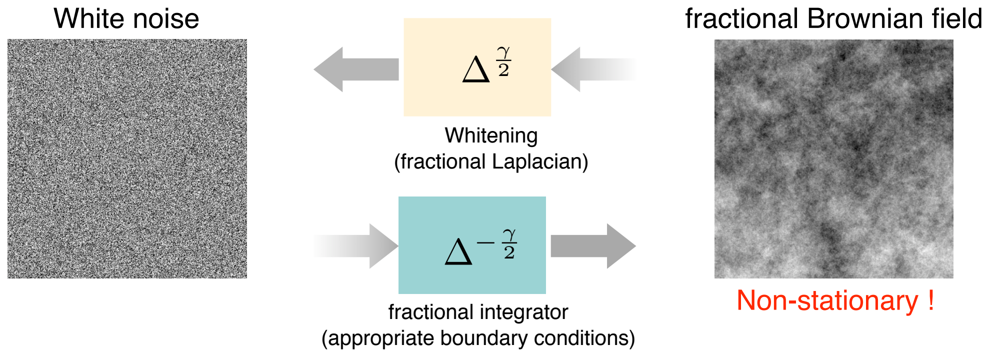
- Specific driving terms
  - $r(\boldsymbol{x}) = \delta(x)$ $\Rightarrow$ $s(\boldsymbol{x}) = \mathrm{L}^{-1}\{\delta\}(\boldsymbol{x})$: Green function
  - $r(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} a[\boldsymbol{k}]\delta(\boldsymbol{x}-\boldsymbol{k})$ $\Rightarrow$ $s(\boldsymbol{x})$: Cardinal L-spline
  - $r(\boldsymbol{x})$: white Gaussian noise $\Rightarrow$ $s(\boldsymbol{x})$: generalized stochastic process

⚠ non-empty null space of L, boundary conditions

References: stationary proc. (Unser, *IEEE-SP* 2006), fractals (Blu, *IEEE-SP* 2007)

Example: fBm and polyharmonic splines

Defining operator: $\mathrm{L} = \Delta^{\frac{\gamma}{2}} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \|\boldsymbol{\omega}\|^{\gamma}$



Formalism: Gelfand's theory of generalized stochastic processes

- Deterministic counterpart

Train of Dirac impulses: $\sum_{\boldsymbol{k}\in\mathbb{Z}^d} a[\boldsymbol{k}]\delta(\boldsymbol{x}-\boldsymbol{k})$ ⟵ $\Delta^{\frac{\gamma}{2}}$ ⟵ Polyharmonic spline (Rabut, 1992): $s(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} c[\boldsymbol{k}]\varphi_{\gamma}(\boldsymbol{x}-\boldsymbol{k})$

10. Link with estimation theory

Smoothing splines are minimum-mean-square-error estimators (e.g., hybrid Wiener filters) for a corresponding class of stochastic processes (stationary and fractal)

- Measurement model: $f[\boldsymbol{k}] = s(\boldsymbol{x})|_{\boldsymbol{x}=\boldsymbol{k}} + n[\boldsymbol{k}]$
- $s(\boldsymbol{x})$: realization of a Gaussian stationary or fractal (fBm) process s.t.

$$E\left[\mathrm{L}s(\boldsymbol{x}_1)\cdot \mathrm{L}s(\boldsymbol{x}_2)\right] = \sigma_0^2\,\delta(\boldsymbol{x}_1-\boldsymbol{x}_2) \qquad \text{(whitening operator L)}$$

- $n[\boldsymbol{k}]$: white Gaussian noise with variance σ^2
- MMSE spline estimator of signal $s(\boldsymbol{x})$:

$$E\left[s(\boldsymbol{x})|f\right] = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} (h_\lambda * f)[\boldsymbol{k}]\,\varphi_{\mathrm{L}^*\mathrm{L}}(\boldsymbol{x}-\boldsymbol{k})$$

$\varphi_{\mathrm{L}^*\mathrm{L}}(\boldsymbol{x})$: $\mathrm{L}^*\mathrm{L}$-spline generator
$h_\lambda[\boldsymbol{k}]$: smoothing spline filter
$\lambda = \sigma^2/\sigma_0^2$: regularization factor

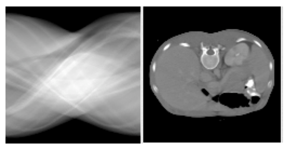
References: stationary proc. (Unser, *IEEE-SP* 2006), fBm (Blu, *IEEE-SP* 2007)

... ADDITIONAL ONES ...

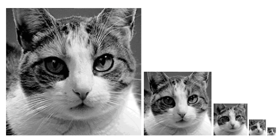
- Attractive Hilbert-space framework for continuous/discrete signal and image processing
- Splines are “π times” better than Daubechies wavelets
- Polynomial splines can be extended to fractional (and even complex) exponents
- Scale invariance and link with fractals (polynomial and fractional splines)
- Generalized (non-stationary) wavelet bases
- ...

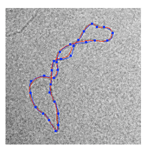
SPLINES: application examples

- Sampling and interpolation
  - Interpolation, re-sampling, grid conversion
  - Image reconstruction
  - Geometric correction

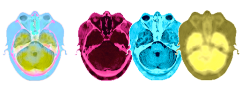


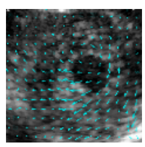
- Feature extraction
  - Contours, ridges
  - Differential geometry
  - Image pyramids
  - Shape and active contour models





- Image matching
  - Stereo
  - Image registration (multimodal, rigid body or elastic)
  - Motion analysis (optical flow)

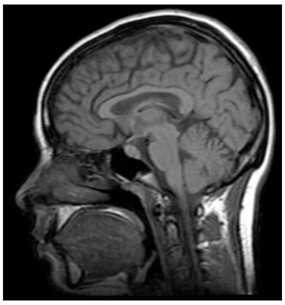


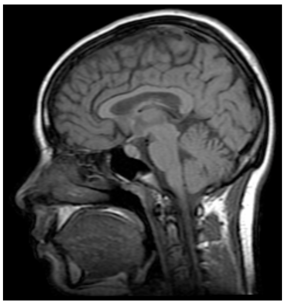


Exit

Spline approximation: LS resizing

Approximation at arbitrary scales: differential approach using splines





$$a = 1 \rightarrow 10$$

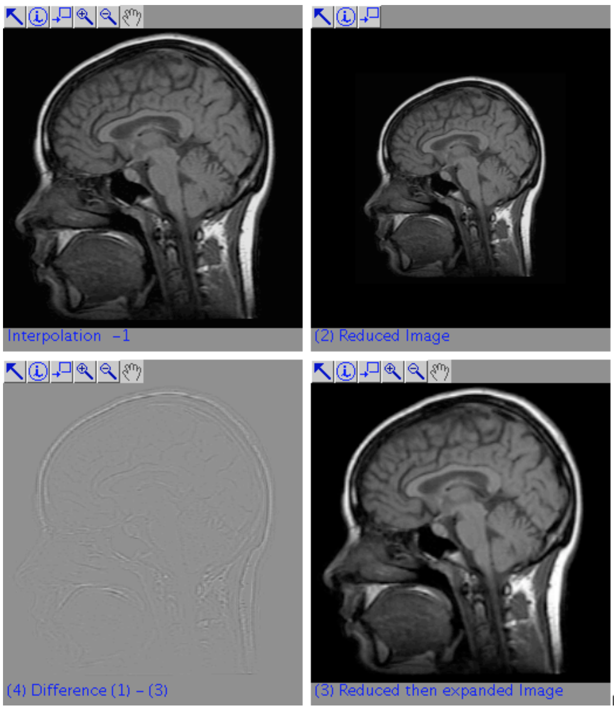
Minimum error approximation (orthogonal projection)

$$f_a(x) = \arg\min_{c_a} \|f(x) - \sum_{k\in\mathbb{Z}} c_a[k]\beta^n(x/a - k)\|^2_{L_2(\mathbb{R})}$$

Application: image resizing

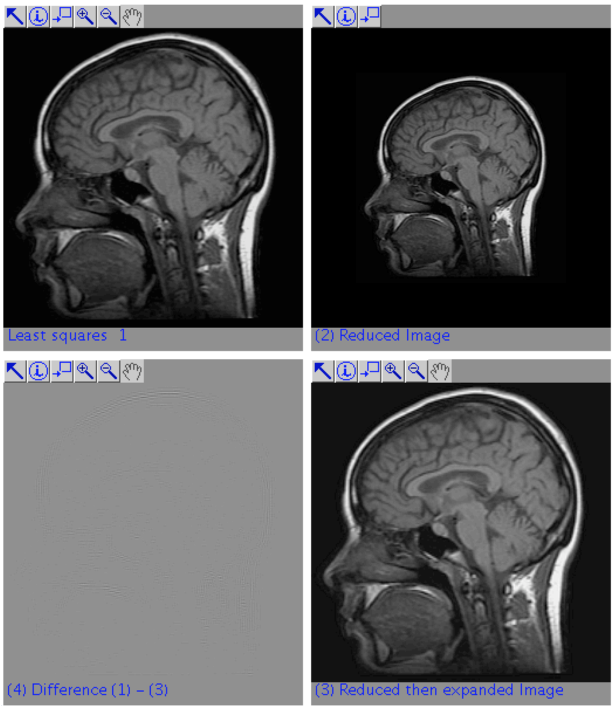
- Resizing algorithm
  - Interpolation
  - Linear splines
  - scaling= 70%

SNR=22.94 dB



Application: image resizing (LS)

- Resizing algorithm
  - Orthogonal projector
  - Linear splines
  - scaling= 70%



SNR=28.359 dB

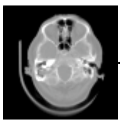
+ 5.419 dB

(Munoz et al., *IEEE Trans. Imag. Proc*, 2001)

Cubic-spline image differentials

- Convolution-based implementation

$f(k,l)$



→ 2D filtering (separable) → $c[k,l]$ → Differential mask →

$$\frac{\partial^{p+q}}{\partial x^p \partial y^q} f(k,l)$$

$$\left.\frac{\partial^{p+q}}{\partial x^p \partial y^q}\varphi(x,y)\right|_{x=k,y=l}$$

JAVA code available:
http://bigwww.epfl.ch/

- Laplacian

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} : \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -8 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

- Hessian masks

$$\frac{\partial^2}{\partial x^2} : \frac{1}{6}\begin{bmatrix} 1 & -2 & 1 \\ 4 & -8 & 4 \\ 1 & -2 & 1 \end{bmatrix} \qquad \frac{\partial^2}{\partial x \partial y} : \frac{1}{2\cdot 2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\frac{\partial^2}{\partial y^2} : \frac{1}{6}\begin{bmatrix} 1 & 4 & 1 \\ -2 & -8 & -2 \\ 1 & 4 & 1 \end{bmatrix}$$

- Gradient masks

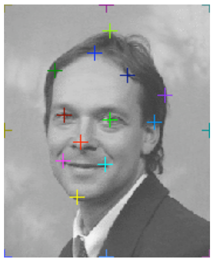
$$\frac{\partial}{\partial x} : \frac{1}{6\cdot 2}\begin{bmatrix} -1 & 0 & 1 \\ -4 & 0 & 4 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\frac{\partial}{\partial y} : \frac{1}{6\cdot 2}\begin{bmatrix} -1 & -4 & -1 \\ 0 & 0 & 0 \\ 1 & 4 & 1 \end{bmatrix}$$

Elastic registration problem

Find a diffeomorphism (warping): $\boldsymbol{x} \to \mathbf{g}(\boldsymbol{x})$ such that $f_{\mathrm{S}}(\mathbf{g}(\boldsymbol{x})) \approx f_{\mathrm{T}}(\boldsymbol{x})$

- $f_{\mathrm{S}}(\boldsymbol{x})$: source image
- $f_{\mathrm{T}}(\boldsymbol{x})$: target image (or reference)
- $\mathbf{g}(\boldsymbol{x}) = \mathbf{g}(\boldsymbol{x}|\boldsymbol{\Theta})$: parametric deformation map

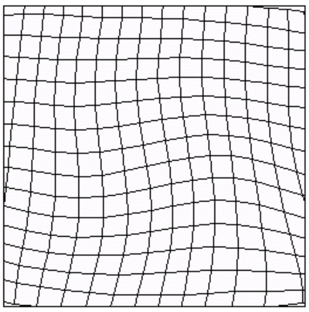


- Problem constraints
  - Similarity measure to compare images
  - Smooth deformation field (regularization)
  - Parametric model (for better efficiency)
  - Optional specification of landmarks: $\boldsymbol{x}_{\mathrm{S}}^{(n)} \to \boldsymbol{x}_{\mathrm{T}}^{(n)}$

Cubic-spline deformation map

Transformed image: $f_{\mathrm{S}}\left(\mathbf{g}(\boldsymbol{x}|\boldsymbol{\Theta}_h)\right)$

Deformation map: $$\mathbf{g}(\boldsymbol{x}|\boldsymbol{\Theta}_h) = \begin{pmatrix} g_1(\boldsymbol{x}) \\ g_2(\boldsymbol{x}) \end{pmatrix} = \sum_{\boldsymbol{k}\in\mathbb{Z}^2} \begin{pmatrix} c_1[\boldsymbol{k}] \\ c_2[\boldsymbol{k}] \end{pmatrix} \beta^3\left(\frac{\boldsymbol{x}}{h} - \boldsymbol{k}\right)$$



- Parametric model (control points)
  $\boldsymbol{\Theta}_h = (\cdots, c_1[k,l], c_2[k,l], \cdots)$
- Resolution controlled by mesh size h
- Smooth deformation (cubic splines)
- Rich variety of spatial mappings, including rigid body, affine, etc.

Registration as an optimization problem

$f_S(\boldsymbol{x}) \rightarrow f_S\left(\mathbf{g}(\boldsymbol{x}|\boldsymbol{\Theta}_{\text{opt}})\right)$ where $\boldsymbol{\Theta}_{\text{opt}} = \arg\min_{\boldsymbol{\Theta}} \{E_{\text{reg}}(f_S, f_T, \boldsymbol{\Theta})\}$

$$E_{\text{reg}}(f_S, f_T, \boldsymbol{\Theta}) = E_{\text{image}}(f_S, f_T, \boldsymbol{\Theta}) + E_{\text{rough}}(\boldsymbol{\Theta}) + E_{\text{landmark}}(\boldsymbol{\Theta})$$

- Least-squares similarity criterion

$$E_{\text{image}}(f_S, f_T, \boldsymbol{\Theta}) = \sum_{\boldsymbol{k}} |f_S(\mathbf{g}(\boldsymbol{k}|\boldsymbol{\Theta})) - f_T[\boldsymbol{k}]|^2$$

- Vector-spline roughness penalty

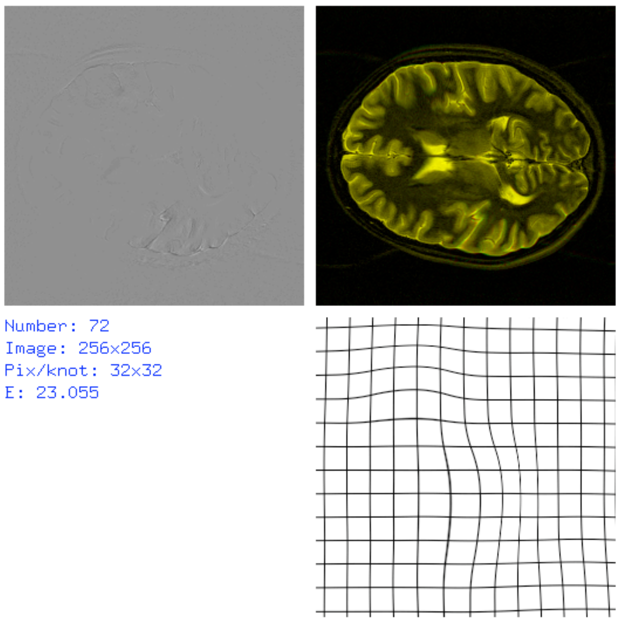
$$E_{\text{rough}}(\boldsymbol{\Theta}) = \lambda_{\text{div}} \left\| \boldsymbol{\nabla} \text{ div } \mathbf{g}(\boldsymbol{x}|\boldsymbol{\Theta}) \right\|^2_{L_2(\mathbb{R}^2)} + \lambda_{\text{rot}} \left\| \boldsymbol{\nabla} \text{ rot } \mathbf{g}(\boldsymbol{x}|\boldsymbol{\Theta}) \right\|^2_{L_2(\mathbb{R}^2)}$$

- Landmark contraints: $\boldsymbol{x}_S^{(n)} \rightarrow \boldsymbol{x}_T^{(n)}$

$$E_{\text{landmark}}(\boldsymbol{\Theta}) = \frac{\lambda}{N} \sum_{n=1}^{N} \left\| \mathbf{g}(\mathbf{x}_S^{(n)}|\boldsymbol{\Theta}) - \mathbf{x}_T^{(n)} \right\|^2$$

UnwarpJ: Implementation details

- Continuous image representation
  - cubic splines
- Consistent implementation
  - analytical derivatives
  - multilevel B-spline discretization
- Quasi-Newton optimization
  - exact gradient of criterion
- Full multiresolution strategy
  - coarse-to-fine on images
  - coarse-to-fine on deformation

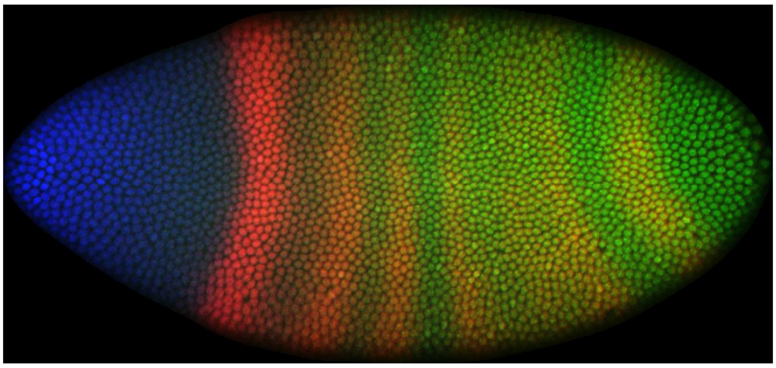


Drosophila Melanogaster embryos

Three genes, three fluorescent dyes

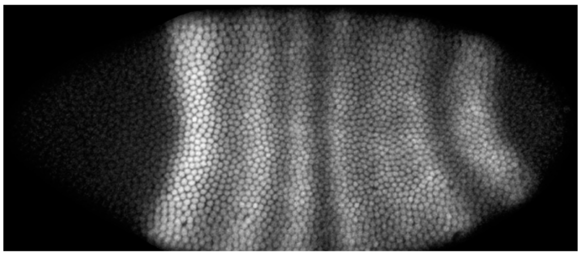
One control gene, two variable genes

Confocal scanning microscope

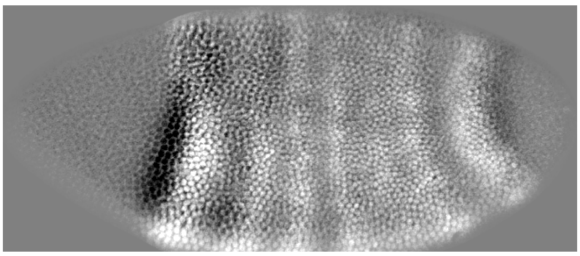


Drosophila Melanogaster embryos

Unregistered

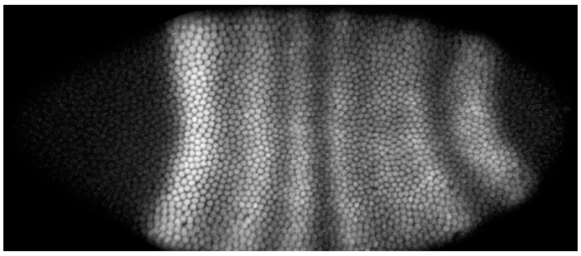


Difference

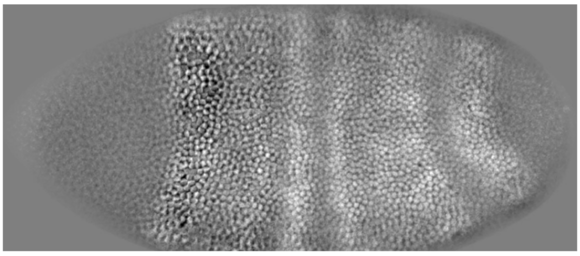


Drosophila Melanogaster embryos

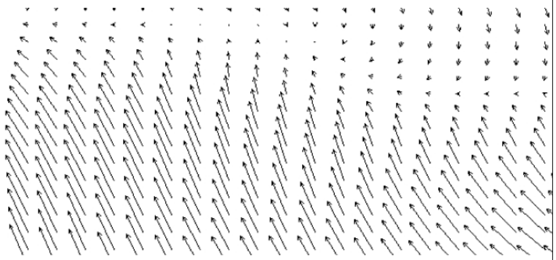
Registered



Difference



Deformation field



CONCLUSION

- B-splines are attractive computationally
  - Simple to manipulate; smooth and well-behaved
  - Fast recursive filtering algorithms ($O(1)$ per sample)
  - Multiresolution properties (pyramid, multigrid, wavelets)
- Splines: a unifying conceptual framework
  - Approximation theory
  - Link with wavelet theory
  - Signals and systems, sampling theory
  - Stochastic processes; regularization and estimation theories
- Practical Hilbert-space framework (SP counterpart of FE) for continuous/discrete image processing
  - "Think analog, act digital"
  - Toolbox: digital filters, convolution operators
  - Flexibility: piecewise-constant to bandlimited

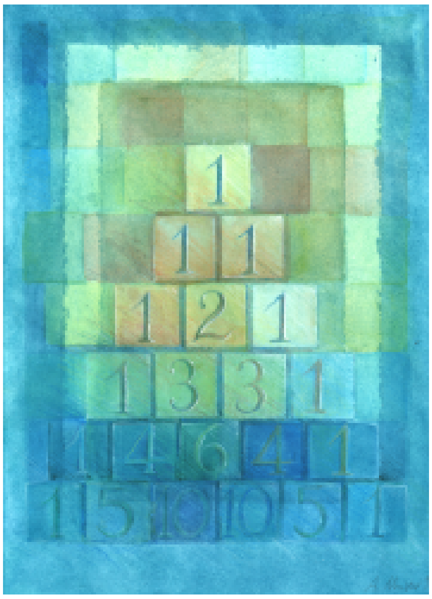
Acknowledgments

Many thanks to

- Prof. Akram Aldroubi
- Prof. Thierry Blu
- Prof. Murray Eden
- Dr. Philippe Thévenaz
- Dr. Dimitri Van De Ville

- Annette Unser, Artist

\+ many other researchers, and graduate students



Bibliography

- Selected papers
  - M. Unser, "Splines: A Perfect Fit for Signal and Image Processing," *IEEE Signal Processing Magazine*, vol. 16, no. 6, pp. 22-38, 1999.
  - P. Thévenaz, T. Blu, M. Unser, "Interpolation Revisited," *IEEE Trans. Medical Imaging*, vol. 19, no. 7, pp. 739-758, July 2000.
  - M. Unser, "Sampling—50 Years After Shannon," *Proceedings of the IEEE*, vol. 88, no. 4, pp. 569-587, April 2000.
  - M. Unser, T. Blu, "Wavelet Theory Demystified," *IEEE Trans. Signal Processing*, vol. 51, no. 2, pp. 470-483, 2003.
  - M. Unser, "Cardinal Exponential Splines: Part II—Think Analog, Act Digital," *IEEE Trans. Signal Processing*, vol. 53, no. 4, pp. 1425-1438, April 2005.

- Preprints and demos: http://bigwww.epfl.ch/