Polynomial Spline Signal Approximations: Filter Design and Asymptotic Equivalence with Shannon's Sampling Theorem

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Abstract—**The least-squares polynomial spline approximation of a signal $g(t) \in \mathbb{L}_2(\mathbb{R})$ is obtained by projecting $g(t)$ on $\mathbb{S}^n(\mathbb{R})$ (the space of polynomial splines of order n). We show that this process can be linked to the classical problem of cardinal spline interpolation [1] by first convolving $g(t)$ with a B-spline of order n. More specifically, the coefficients of the B-spline interpolation of order $2n+1$ of the sampled filtered sequence are identical to the coefficients of the least-squares approximation of $g(t)$ of order n. We then show that this approximation can be obtained from a succession of three basic operations: prefiltering, sampling, and postfiltering, which confirms the parallel with the classical sampling/reconstruction procedure for bandlimited signals. We determine the frequency responses of these filters for three equivalent spline representations using alternative sets of shift-invariant basis functions of $\mathbb{S}^n(\mathbb{R})$: the standard expansion in terms of B-spline coefficients, a representation in terms of sampled signal values, and a representation using orthogonal basis functions. For the two latter cases, we prove that the frequency response of these filters converge to the ideal lowpass filter pointwise and in all L_p-norms with $1 \leq p \leq \infty$ as the order of the spline tends to infinity, which establishes the asymptotic equivalence with Shannon's sampling theorem.**

Index Terms—**Interpolation, B-splines, polynomial splines, spline filters, sampling theorem, least-squares approximation, asymptotic convergence.**

I. Introduction

The Whittaker–Shannon sampling theorem states that every signal function $\tilde{g}(t)$ ($t \in \mathbb{R}$) that is bandlimited to $f \in [-\frac{1}{2}T, \frac{1}{2}T]$, can be completely reconstructed from its samples $h(kT)$ where ($k \in \mathbb{Z}$) [2]. An important implication of this result is the standard sampling/reconstruction paradigm for the sampled representation of a signal $g(t) \in \mathbb{L}_2(\mathbb{R})$ that involves three distinct processing steps. Without loss of generality, we assume that $T = 1$ to simplify the notation. First, the function needs to be bandlimited through the use of a so-called anti-aliasing filter (ideal lowpass filter):

$$\tilde{g}(t) = \int_{-\infty}^{+\infty} \operatorname{sinc}(\tau) g(t-\tau)\, d\tau = \operatorname{sinc} * g(t), \qquad t \in \mathbb{R}, \tag{1.1}$$

where sinc $(t) := \sin(\pi t)/\pi t$ and the asterisk denotes the convolution operator. Second, $\tilde{g}(t)$ is sampled by multiplication with a delta sampling function (denoted by $i_\delta(t)$):

$$\tilde{g}_\delta(t) = g(t) \cdot i_\delta(t) = \sum_{k=\infty}^{+\infty} \tilde{g}(k)\delta(t-k), \qquad t \in \mathbb{R}. \tag{1.2}$$

Finally, the bandlimited approximation is reconstructed without any loss using an ideal interpolation filter:

$$\tilde{g}(t) = \operatorname{sinc} * \tilde{g}_\delta(t), \qquad t \in \mathbb{R}. \tag{1.3}$$

The whole process (as well as (1.1) taken on its own) can be interpreted as a projection of $g(t)$ onto the subspace of functions that are bandlimited to $f \in [-\frac{1}{2}, \frac{1}{2}]$. As a consequence, the quadratic error between $g(t)$ and its approximation $\tilde{g}(t)$ is minimized given the reconstruction algorithm described by (1.3).

For numerical computations, the ideal interpolation formulae (1.3) is not practical due to the slow rate of decay of the interpolation kernel. An attractive alternative is a reconstruction by polynomial spline interpolation, which is conveniently described as [3]

$$g^n(t) = \sum_{k=\infty}^{+\infty} c(k)\beta^n(t-k) = \beta^n * c_\delta(t), \qquad t \in \mathbb{R}, \tag{1.4}$$

where $c(k)$ are the B-spline coefficients and where $\beta^n(t)$ is the central B-spline of order n defined as

$$\beta^n(t) := \sum_{j=0}^{n+1} \frac{(-1)^j}{n!} \binom{n+1}{j} \left(t + \frac{n+1}{2} - j\right)_+^n, \qquad t \in \mathbb{R}, \tag{1.5}$$

where $(x)_+ = \max\{0, x\}$. The B-spline function $\beta^n(t)$ can also be constructed by repeated convolution of a B-spline of order 0:

$$\beta^n(t) = \underbrace{\beta^0 * \beta^0 * \cdots * \beta^0}_{(n+1) \text{ times}}(t), \tag{1.6}$$

where $\beta^0(x)$ is the indicator function in the interval $[-\frac{1}{2}, \frac{1}{2})$. The main difficulty in this interpolation procedure is to determine the B-spline coefficients $c(k)$ that interpolate a

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given sequence of discrete values $g(k) \in l_2(\mathbb{Z})$ (except for the trivial cases of zero- and first-order interpolation). This problem is commonly referred to as the cardinal spline interpolation problem and has been extensively studied by Schoenberg [1], [3]. More recently, we have established the link between this process and conventional digital filtering theory and described fast inversion and reconstruction algorithms using recursive filters [4]. We have also shown that the spline interpolator of order n approaches an ideal sinc interpolator as n goes to infinity [5].

In the case of B-spline interpolation, Hummel has shown that the use of an appropriate prefilter different from the one used in (1.1) prior to sampling can lead to a reduced mean square error after reconstruction [6]; this author derived weighting functions for zero-order, linear and cubic spline interpolation. The purpose of this paper is to extend those results by providing explicit filter formulas for splines of any order and to stress the parallel between this process and classical sampling theory (1.1)–(1.3). More important, we will show in Section IV that the optimal prefilters converge (pointwise and in all L_p-norms) to an ideal lowpass filter as n tends to infinity, thus establishing the asymptotic equivalence with Shannon's sampling theorem.

Least-squares spline approximation is also closely related to the wavelet representation which is an orthogonal multiresolution decomposition of a signal [7]–[9]. This formulation suggests the use of an alternative representation using orthogonal basis functions that is also considered in this paper. Interesting enough, we will show that the asymptotic convergence to an ideal lowpass filter also holds for the corresponding convolution operators.

A. Notations and Definitions

$\mathbb{L}_2(\mathbb{R})$ denotes the space of measurable, square-integrable, functions $g(t)$, $t \in \mathbb{R}$. The inner product of two functions $g(t) \in \mathbb{L}_2(\mathbb{R})$ and $h(t) \in \mathbb{L}_2(\mathbb{R})$ is

$$(g(t), h(t)) = \int_{-\infty}^{+\infty} g(t)h(t)\, dt.$$

The Fourier transform of any function $g(t) \in \mathbb{L}_2(\mathbb{R})$ is represented by a roman capital letter:

$$G(f) = \int_{-\infty}^{+\infty} g(t)e^{-j2\pi ft}\, dt.$$

$l_2(\mathbb{Z})$ is the vector space of square-summable sequences:

$$l_2(\mathbb{Z}) = \left\{u(k), (k \in \mathbb{Z}): \sum_{k=-\infty}^{+\infty} |u(k)|^2 < +\infty\right\}.$$

$\mathbb{S}^n(\mathbb{R})$ is the subset of functions of $\mathbb{L}_2(\mathbb{R})$ that are of class $\mathbb{C}^{n-1}$ (i.e., continuous functions with continuous derivatives up to order $n-1$) and are equal to a polynomial of degree n on each interval $[k, k+1]$, $k \in \mathbb{Z}$, when n is odd, and $[k - 1/2, k + 1/2]$, $k \in \mathbb{Z}$ when n is even. $\mathbb{S}^n(\mathbb{R})$ is a closed vector space and is defined by (1.4) where $c(k) \in l_2(\mathbb{Z})$ [1, Theorem 12, p. 199].

II. Least-Squares B-Spline Approximation

In this section, we provide a short derivation that establishes a formal link between the least-squares B-spline approximation of a function $g(t) \in \mathbb{L}_2(\mathbb{R})$ and the classical problem of cardinal B-spline interpolation of a discrete sequence $a(k) \in l_2(\mathbb{Z})$. This result suggests a simple computational procedure for the determination of the B-spline coefficients.

Since $\mathbb{S}^n(\mathbb{R})$ is a closed subspace of $\mathbb{L}_2(\mathbb{R})$, we know that the minimum L_2-norm approximation of $g(t)$ can be found by projecting this function into $\mathbb{S}^n(\mathbb{R})$. Consequently, the residual error $[g(t) - g^n(t)]$ is orthogonal to $\mathbb{S}^n(\mathbb{R})$, which implies that

$$(g(t) - g^n(t), \beta^n(t-k)) = 0, \qquad \forall k \in \mathbb{Z}. \quad (2.1)$$

Using (1.4), this expression is rewritten as

$$(g(t), \beta^n(t-k)) = \left(\sum_{l=-\infty}^{+\infty} c(l)\beta^n(t-l), \beta^n(t-k)\right), \qquad \forall k \in \mathbb{Z},$$

and is also equivalent to

$$(g(t), \beta^n(t-k)) = \sum_{l=-\infty}^{+\infty} c(l)(\beta^n(t-l), \beta^n(t-k)), \qquad \forall k \in \mathbb{Z}. \quad (2.2)$$

We now use the well-known property that $\beta^n * \beta^n(t) = \beta^{2n+1}(t)$ (which follows directly from (1.6)) and define the discrete sequences

$$a(k) = (\beta^n * g)(k) = (g(t), \beta^n(t-k)), \qquad \forall k \in \mathbb{Z}, \quad (2.3)$$

$$b^{2n+1}(k) := \beta^{2n+1}(k) = (\beta^n(t), \beta^n(t-k)), \qquad \forall k \in \mathbb{Z}. \quad (2.4)$$

Since $g(t) \in \mathbb{L}_2(\mathbb{R})$, it is not difficult to show that $a(k) \in l_2(\mathbb{Z})$ by using the fact that the support of $\beta^n(t)$ is compact. Substituting these quantities in (2.2), we find that the sequence of least squares B-spline coefficients $c(k)$ must satisfy

$$\sum_{l=-\infty}^{+\infty} c(l) b^{2n+1}(k-l) = b^{2n+1} * c(k) = a(k), \qquad \forall k \in \mathbb{Z}. \quad (2.5)$$

Interestingly enough, the inversion of this difference equation is equivalent to the problem of finding the B-spline coefficients of order $2n+1$ that provide an exact interpolation of $a(k)$. From the existence theorems of Schoenberg [1], we know that the solution of (2.5) exists and is unique. Specifically, we can write

$$c(k) = (b^{2n+1})^{-1} * a(k), \qquad \forall k \in \mathbb{Z}, \quad (2.6)$$

where $(b^{2n+1})^{-1}$ denotes the convolution inverse of the sampled B-spline kernel b^{2n+1}. This solution can be computed efficiently using the recursive algorithm described in [4].

In summary, the least-squares B-spline approximation of a signal $g(t)$ can be carried out in three simple steps: 1) a convolution of the signal with a continuous B-spline kernel $\beta^n(t)$, 2) a sampling to provide the discrete sequence $a(k) \in l_2(\mathbb{Z})$, and 3) a digital filtering using a so-called direct B-spline filter of order $2n + 1$ (cf. (2.6)).

From (1.6), we obtain the Fourier transform of a B-spline of order n:

$$B^n(f) = \left[\text{sinc}(f)\right]^{n+1}. \tag{2.7}$$

The Fourier transform of $(b^{2n+1})^{-1}$ is the inverse of the Fourier transform of b^{2n+1} and is given by (cf. [4])

$$S^{2n+1}(f) = \frac{1}{b^{2n+1}(0) + 2\sum_{k=1}^{n+1} b^{2n+1}(k)\cos(2\pi kf)} = \frac{1}{\sum_{k=-\infty}^{+\infty} \left[\text{sinc}(f-k)\right]^{2n+2}}. \tag{2.8}$$

The representation of the denominator of (2.8) in terms of sampled B-spline coefficients or an infinite sum of shifted sinc functions (Poisson's summation formula) are equivalent. In the reminder of the paper, we will use the latter which is slightly more compact, although it is clear that the former is better suited for making graphs or performing numerical computations. We note that the filter coefficients in the central equality of (2.8) can either be determined directly from (1.5), or computed recursively as described in [5].

III. Equivalent B-Spline Representations

In order to emphasize the similarity between B-spline approximation and the conventional sampling/reconstruction procedure described by (1.1)–(1.3), we have chosen to reformulate the former algorithm as a succession of three basic operations: 1) prefiltering, 2) sampling, and 3) postfiltering for interpolation (cf. Fig. 1). In particular, we discuss three equivalent approaches using different sets of shift-invariant basis functions of $\mathbb{S}^n(\mathbb{R})$.

A. B-Spline Coefficients

The standard B-spline representation of a signal uses shifted B-spline basis functions (cf. (1.4) and (1.5)). It is not difficult to show that the algorithm described previously is equivalent to the following procedure

$$c_\delta(t) = (\mathring{\beta}^n * g(t)).i_\delta(t), \qquad t \in \mathbb{R}, \tag{3.1}$$

in which convolutions (2.3) and (2.6) have been combined in a single prefilter:

$$\mathring{\beta}(t) = \sum_{k=-\infty}^{+\infty} (b^{2n+1})^{-1}(k)\beta^n(t-k). \tag{3.2}$$

We refer to this latter function as the dual spline of order n; in fact, it can be verified that $\mathring{\beta}(t)$ satisfies the biorthogonality condition

$$\forall k, l \in \mathbb{Z}, \quad (\beta^n(x-k), \mathring{\beta}^n(x-l)) = \begin{cases} 1, & \text{if } k = l \\ 0, & \text{otherwise}. \end{cases} \tag{3.3}$$

From (3.2), it follows immediately that the Fourier transform of $\mathring{\beta}(t)$ is

$$\mathring{\beta}^n(f) = B^n(f)S^{2n+1}(f) = \frac{\left[\text{sinc}(f)\right]^{n+1}}{\sum_{k=-\infty}^{+\infty}\left[\text{sinc}(f-k)\right]^{2n+2}}, \qquad f \in \mathbb{R}. \tag{3.4}$$

The corresponding reconstruction algorithm is (1.4) and is in all points similar to (1.3). The frequency responses of these operators for $n = 1$ and 3 are shown in Fig. 1(a).

B. Sampled Signal Representation

Another useful representation of the function $g^n(t) \in \mathbb{S}^n(\mathbb{R})$ that approximates $g(t)$ is an expansion in which the coefficients are the sampled values of the interpolating function. This representation is given by

$$g^n(t) = \sum_{k=-\infty}^{+\infty} g^n(k)\eta^n(t-k) = \eta^n * g_\delta^n(t), \qquad t \in \mathbb{R}, \tag{3.5}$$

where $\eta^n(t)$ is the so-called cardinal (or fundamental) spline of order n [5]. An explicit formula for $\eta^n(t)$ is

$$\eta(t) = \sum_{k=-\infty}^{+\infty} (b^n)^{-1}(k)\beta^n(t-k), \tag{3.6}$$

where $(b^n)^{-1}$ is the impulse response of the direct B-spline filter of order n [4]. The cardinal splines together with sinc (t) share the fundamental interpolation property

$$\forall k \in \mathbb{Z}, \quad \eta^n(k) = \begin{cases} 1, & \text{if } k = 0, \\ 0, & \text{if } k \neq 0, \end{cases} \tag{3.7}$$

which can be directly linked to (3.6) (i.e., $\eta(k) = (b^n)^{-1} * b^n(k) = \delta_0(k)$, where $\delta_0(k)$ is the unit impulse at the origin). Hence, the Fourier transform of $\eta^n(t)$ is given by

$$H^n(f) = B^n(f)S^n(f) = \frac{\left[\text{sinc}(f)\right]^{n+1}}{\sum_{k=-\infty}^{+\infty}\left[\text{sinc}(f-k)\right]^{n+1}}. \tag{3.8}$$

The corresponding prefiltering algorithm is obtained by substituting the B-spline coefficients given by (3.1) in (1.4) and resampling the reconstructed signal. An equivalent representation is

$$g_\delta^n(t) = (\mathring{\eta}^n * g(t)).i_\delta(t), \qquad t \in \mathbb{R}. \tag{3.9}$$

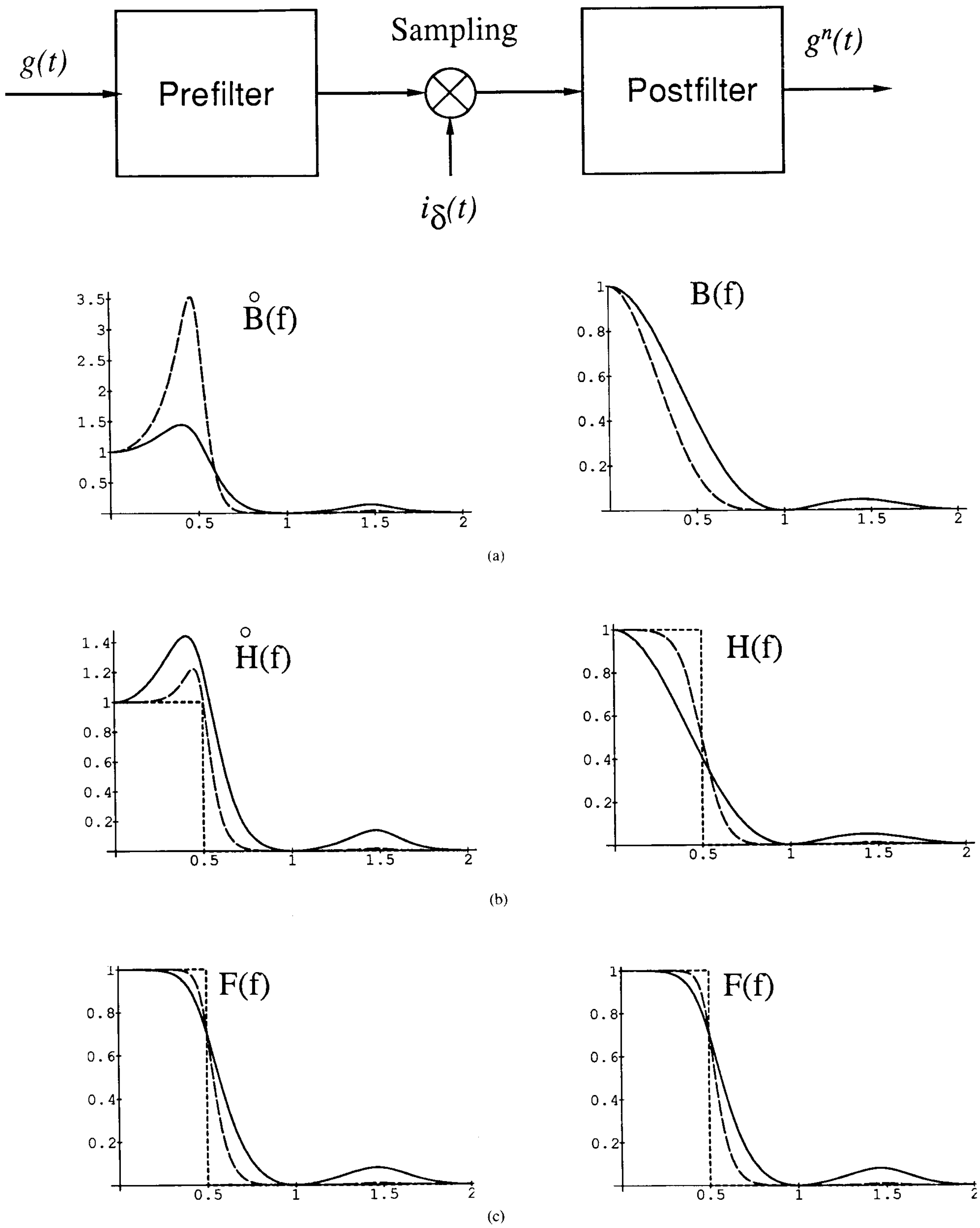

Fig. 1. Three equivalent procedures for the least-squares B-spline approximation of a signal with the frequency responses of the corresponding filters for $n = 1$, 3 and $+\infty$. $n = 1$: —— (piecewise linear), $n = 3$: --- (cubic spline), and $n \to +\infty$: ······ (bandlimited approximation). (a) B-spline coefficients. (b) Sampled signal values. (c) Orthogonal basis functions.

where $\mathring{\eta}^n(t)$ is the convolution of $\mathring{\beta}^n(t)$ with a sampled B-spline kernel. Thus, the frequency response of this operator is

$$\mathring{H}^n(f) = \frac{\mathring{B}^n(f)}{S^n(f)} = \frac{[\mathrm{sinc}(f)]^{n+1} \sum_{k=-\infty}^{+\infty} [\mathrm{sinc}(f-k)]^{n+1}}{\sum_{k=-\infty}^{+\infty} [\mathrm{sinc}(f-k)]^{2n+2}}. \tag{3.10}$$

This process is illustrated in Fig. 1(b) for splines of order 1 and 3. The dotted lines correspond to the limit of these transfer functions as n goes to infinity. The issue of the convergence of both $\mathring{\eta}^n(t)$ and $\eta^n(t)$ to an ideal lowpass filter is treated in Section IV.

C. Orthogonal Expansion

Alternatively, $g^n(t) \in \mathbb{S}^n(\mathbb{R})$ may be represented in terms of basis functions that are orthogonal in addition to being shifted replicates of one another. These functions have been described by Mallat within the more general framework of the wavelet decomposition [8] and can be linked to the wavelet functions for polynomial splines derived by Lemarié [9]. The corresponding orthogonal representation of $g^n(t)$ is

$$g^n(t) = \sum_{k=-\infty}^{+\infty} d(k)\phi^n(t-k) = \phi^n * d_\delta(t), \tag{3.11}$$

where the basis functions of order n are symmetrical and satisfy the orthogonality condition:

$$\forall k \in \mathbb{Z}, \quad (\phi^n(t), \phi^n(t-k)) = \begin{cases} 1, & \text{if } k = 0, \\ 0, & \text{if } k \neq 0. \end{cases} \tag{3.12}$$

Due to this property, the expansion coefficients are obtained by simple inner product. An equivalent formulation is

$$d_\delta(t) = (\phi^n * g(t)).i_\delta(t). \tag{3.13}$$

It can be shown that the Fourier transform of $\phi^n(t)$ is given by

$$F^n(f) = \sqrt{\mathring{B}^n(f) B^n(f)} = \frac{[\mathrm{sinc}(f)]^{n+1}}{\sqrt{\sum_{k=-\infty}^{+\infty} [\mathrm{sinc}(f-k)]^{2n+2}}}, \tag{3.14}$$

which is an alternative but equivalent form of (56) and (57) in [8]. The corresponding procedure is illustrated in Fig. 1(c). As in the previous case, $F^n(f)$ converges to the ideal lowpass filter as n goes to infinity (cf. Theorem 3).

We note that (3.14) can be obtained in a straightforward manner by recognizing the fact that alternative sets of shift invariant basis functions can be generated by convolution of a basic B-spline kernel:

$$\phi^n(t) = u_\delta * \beta^n(t), \tag{3.15}$$

where $u_\delta(t)$ is the sampled representation of an invertible shift invariant operator $u(k) \in \mathbb{l}_2(\mathbb{Z})$. The corresponding inner product can be expressed as a discrete convolution:

$$(\phi^n(t), \phi^n(t-k)) = u' * u * b^{2n+1}(k), \tag{3.16}$$

where $u'(k) = u(-k)$, and is equivalent to (3.12), if and only if $u' * u = (b^{2n+1})^{-1}$, which implies that the Fourier transform of the discrete sequence $(u' * u)(k)$ must be equal to (2.8).

D. Comments

Fig. 1 illustrates the similarity between polynomial spline approximation and the conventional sampling/reconstruction procedure for bandlimited signals. Not too surprisingly, all prefilters attenuate frequencies above $1/(2T)$, which is consistent with the requirement for an antialiasing filter in conventional sampling theory. We note, however, that in the first case the prefilter tends to emphasize high frequencies in the bandpass region to compensate for the smoothing effect of the B-spline postfiltering. The same effect, but to a lesser extent, is also observed in the second case, although it tends to vanish for higher order splines. The response is the closest to an ideal bandpass filter in the third case. If one looks at the reconstruction part of the algorithm only, the parallel with the conventional approach holds only for the second scheme, which is the only one that precisely uses a polynomial spline interpolator as postfilter.

The important point that we want to emphasize here is that the choice of a given interpolation algorithm should predetermine the selection of the prefilter. In fact, both prefiltering and interpolation algorithms are specified by the underlying signal space. The main advantage of a consistent design that takes into account these considerations is to minimize the loss of information occurring during discretization. For instance, it turns out that the use of an ideal anti-aliasing filter is not necessarily optimal, unless it is used in conjunction with a sinc interpolator, as specified by the sampling theorem.

The equivalent of Shannon's sampling theorem for polynomial splines is that a function $g^n(t) \in \mathbb{S}^n(\mathbb{R})$ can be recovered without any loss from its samples. This result simply follows from the properties of projection operators. However, there are two important distinctions to be made. First, unlike in the case of bandlimited functions, there is no shift invariance since $g^n(t-\tau)$, with $0 < \tau < 1$, is in general not an element of $\mathbb{S}^n(\mathbb{R})$. Second, the output of the prefilter described by (3.5) and (3.6) will in general not be a function of $\mathbb{S}^n(\mathbb{R})$, although its samples will coincide with those of $g^n(t)$. In other words, the prefilter alone is not a projection operator, in contrast with the ideal lowpass filter in (1.1) that computes a bandlimited approximation of $g(t)$. In Section IV, we will show that these differences vanish as the order of the splines tends to infinity.

Despite the use of different filters, the three approaches that have been considered are globally equivalent. The products of the pre- and postfilter frequency responses are identical in all cases. These various schemes are associated with different product decompositions of $S^{2n+1}(f) = X_p(f)Y_p(f)$, where $X_p(f)$ and $Y_p(f)$ are periodic functions of f. The factors $X_p(f)$ and $Y_p(f)$ can be interpreted as the frequency responses of sampled (or discrete) filtering modules that can be applied before or after sampling without any modification of the global system response.

The process by which a signal is represented in terms of the coefficients of either one of the spline expansions previously described is a particular form of discretization. Each representation has some specific advantages and the selection of the most appropriate one depends on the kind of signal processing to be performed. For instance, the representation in term of B-spline basis functions (Section III-A) is computationally the most attractive if one uses the procedure described in Section II. It is also well suited for signal analyses involving operations such as differentiation, integration, searching for extrema, etc..., which are best expressed in terms of B-spline coefficients [10], [11]. The representation in term of sampled signal values (section III-B) is the most appropriate for conventional digital signal processing. It is very similar to the classical sampling/reconstruction approach in the sense that the reconstruction is obtained by interpolation. We also note that this representation can be obtained from the previous one by simple convolution with a sampled B-spline kernel, which is a finite impulse response (FIR) filter of length $\lfloor n/2 \rfloor + 1$. Finally, the representation in term of orthogonal basis functions (Section III-C) has the advantage of using identical pre- and postfilters. It is particularly well suited for multiresolution analysis due to the orthogonality of the basis functions, and has specifically been designed for that purpose [7]. However, in contrast with approaches A and B, the corresponding IIR filters cannot be implemented recursively because of the square root in the denominator of their transfer function. In practice, this means that these filters must be approximated by FIR operators, which may result in some approximation error or in an increase in computation.

IV. Convergence to an Ideal Lowpass Filter

By using the fact that sinc $(f-k) = (-1)^k \sin(\pi f)/\pi(f-k)$, the transfer functions of the filters discussed in the preceding section are rewritten as

$$H^n(f) = \frac{1}{1+U^n(f)}, \qquad (4.1)$$

$$\mathring{H}^n(f) = \frac{1+U^n(f)}{1+U^{2n+1}(f)}, \qquad (4.2)$$

$$F^n(f) = \frac{1}{\sqrt{1+U^{2n+1}(f)}}, \qquad (4.3)$$

where the function $U^n(f)$ is defined as

$$U^n(f) = \begin{cases} \sum_{k=1}^{+\infty} \left(\frac{1}{(k/f+1)^{n+1}} + \frac{1}{(k/f-1)^{n+1}} \right), & n \text{ odd}, \\ \sum_{k=1}^{+\infty} (-1)^k \left(\frac{1}{(k/f+1)^{n+1}} - \frac{1}{(k/f-1)^{n+1}} \right), & n \text{ even}. \end{cases} \qquad (4.4)$$

It will be shown here that these operators approach an ideal lowpass filter as n goes to infinity. This results establishes the asymptotic correspondence between polynomial spline interpolation and the well-known sampling theorem. For this purpose, we start with the following Lemmas.

Lemma 1: The function $|U^n(f)|$ is bounded from above:

$$|U^n(f)| < \frac{4}{(1/f-1)^{n+1}}, \qquad \forall f \in (0,1), \quad (4.5)$$

and

$$|U^n(f)| < 6\left(\frac{f}{\gamma}\right)^{n+1}, \qquad \forall f \in (1,+\infty), \quad (4.6)$$

with $0 \le \gamma = \min\{f - \lfloor f \rfloor, 1-(f-\lfloor f \rfloor)\} \le 1/2$, where $\lfloor f \rfloor$ denotes the "floor" function that performs the integer truncation of f.

Lemma 2: The function $U^{2n+1}(f)$ is bounded from below:

$$\left(\frac{f}{\gamma}\right)^{2n+2} < U^{2n+1}(f), \qquad \forall f \in (1/2,+\infty), \quad (4.7)$$

where $0 \le \gamma \le 1/2$ is defined as in Lemma 1. These inequalities are then used to prove the following theorems.

Theorem 1: The Fourier transforms of the optimal prefilter $\mathring{H}^n(f)$ for polynomial spline interpolation converge pointwise to an ideal lowpass filter as n tends to infinity:

$$\lim_{n\to\infty} \mathring{H}^n(f) = \text{Rect}(f) = \begin{cases} 1, & |f| < 1/2, \\ 1/2, & |f| = 1/2, \\ 0, & |f| > 1/2. \end{cases} \qquad (4.8)$$

Moreover, $\mathring{H}^n(f)$ converge to Rect (f) in $L_p(-\infty, +\infty)$ as n goes to infinity $\forall p \in [1, \infty)$.

To prove this theorem, we also use the following result that was established by us previously [5].

Theorem 2: The Fourier transforms of the polynomial spline interpolators $H^n(f)$ converge pointwise to an ideal lowpass filter as n tends to infinity:

$$\lim_{n\to\infty} H^n(f) = \text{Rect}(f). \qquad (4.9)$$

Moreover, $H^n(f)$ converge to Rect (f) in $L_p(-\infty, +\infty)$ as n goes to infinity $\forall p \in [1, \infty)$.

Not too surprisingly, we get an equivalent result for a representation of splines in term of orthogonal basis functions as follows.

Theorem 3: The Fourier transforms of the orthogonal spline basis functions $F^n(f)$ converge pointwise to an ideal lowpass filter as n tends to infinity:

$$\lim_{n\to\infty} F^n(f) = \sqrt{\text{Rect}(f)}. \tag{4.10}$$

Moreover, $F^n(f)$ converge to Rect (f) in $L_p(-\infty, +\infty)$ as n goes to infinity $\forall p \in [1, \infty)$.

Using a well-known theorem in transform theory [12] that states that for $1 \le p \le 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, the Fourier transform is a bounded linear operator from $L_p(-\infty, +\infty)$ into $L_q(-\infty, +\infty)$, we derive the following theorem.

Theorem 4: The spline filter functions $\eta^n(t)$, $\mathring{\eta}^n(t)$, and $\phi^n(t)$ converge in $L_q(-\infty, +\infty)$ to sinc (t) (the impulse response of the ideal lowpass filter) as n goes to infinity, $\forall q \in [2, \infty]$.

Proof of Lemma 1: Let us define the functions of the continuous variables x and f,

$$u_1^n(x, f) = \frac{1}{(x/f + 1)^{n+1}} \tag{4.11}$$

and

$$u_2^n(x, f) = \frac{1}{|x/f - 1|^{n+1}}. \tag{4.12}$$

Clearly, we have that

$$|U^n(f)| \le U_1^n(f) + U_2^n(f), \tag{4.13}$$

where

$$U_i^n(f) = \sum_{k=1}^{+\infty} u_i^n(k, f), \qquad i = 1, 2. \tag{4.14}$$

For any given f, the function $u_1^n(x, f)$ is strictly decreasing for $x \in (0, +\infty)$ and so is $u_2^n(x, f)$ for $x \in (f, +\infty)$. This implies that, for any k within those intervals,

$$u_i^n(k, f) < \int_{k-1}^{k} u_i^n(x, f)\, dx.$$

It follows that

$$U_1^n(f) < u_1^n(1, f) + \int_1^{+\infty} u_1^n(x, f)\, dx$$
$$= \frac{1}{(1/f + 1)^{n+1}} + \frac{f}{n(1/f + 1)^n} \tag{4.15}$$

and

$$\sum_{k=k_0+1}^{+\infty} u_2^n(k, f) < u_2^n(k_0 + 1, f) + \int_{k_0+1}^{+\infty} u_2^n(x, f)\, dx$$
$$= \frac{1}{|1/f - 1|^{n+1}} + \frac{f}{n|(k_0 + 1)/f - 1|^n}, \tag{4.16}$$

where $k_0 = \lfloor f \rfloor$. For $f \in (0, 1)$, k_0 is zero and the left term of this latter expression is $U_2^n(f)$. By combining (4.15) and (4.16), we get the inequality

$$|U^n(f)| < \frac{1}{(1/f + 1)^{n+1}} + \frac{f}{n(1/f + 1)^n} + \frac{1}{(1/f - 1)^{n+1}} + \frac{f}{n(1/f - 1)^n}, \qquad \forall f \in (0, 1), \tag{4.17}$$

which immediately leads to the first part of Lemma 1. Similarly, $u_2^n(x, f)$ is strictly increasing for $x \in (0, f)$, which implies that

$$u_2^n(k, f) < \int_k^{k+1} u_2^n(x, f)\, dx, \qquad \text{for } k = 0, \cdots, k_0 - 1,$$

where $k_0 = \lfloor f \rfloor$. It then follows that

$$\sum_{k=1}^{k_0} u_2^n(k, f) < \int_1^{k_0} u_2^n(x, f)\, dx + u_2^n(k_0, f)$$
$$= \frac{f}{n|k_0/f - 1|^n} - \frac{f}{n|1/f - 1|^n} + \frac{1}{|k_0/f - 1|^{n+1}}. \tag{4.18}$$

By defining the quantity $0 \le \epsilon = f - k_0 \le 1$ and combining (4.15), (4.16), and (4.18), we get the inequality

$$|U^n(f)| < \frac{1}{(1/f + 1)^{n+1}} + \frac{f}{n(1/f + 1)^n} + \frac{1}{|1/f - 1|^{n+1}} + \frac{f}{n(\epsilon/f)^n} + \frac{f}{n((1 - \epsilon)/f)^n} + \frac{1}{(\epsilon/f)^{n+1}}, \qquad \forall f \in (1, +\infty). \tag{4.19}$$

The second part of Lemma 1 simply follows from the fact that the largest term of the right-hand side is $(f/\gamma)^{n+1}$, where $\gamma = \min(\epsilon, 1 - \epsilon)$. □

The proof of Lemma 2 is trivial and follows from the fact that the individual terms in $U^{2n+1}(f)$ are all positive and that the largest one is precisely $(f/\gamma)^{2n+2}$.

Proof of Theorem 1: The transfer function $\mathring{H}^n(f)$ can be decomposed as

$$\mathring{H}^n(f) = \frac{1}{1+U^{2n+1}(f)} + \frac{U^n(f)}{1+U^{2n+1}(f)} = H^{2n+1}(f) + R^n(f). \quad (4.20)$$

The essential part of the proof will be to show that the second term of this decomposition converges to zero since we already know from Theorem 2 [5] that $H^{2n+1}(f)$ converges pointwise as well as in all L_p-norms to Rect(f). Three distinct cases have to be considered.

a) $f \in (0, 1/2)$.
From Lemma 1 and 2, we have that

$$\left|\frac{U^n(f)}{1+U^{2n+1}(f)}\right| < |U^n(f)| < \frac{4}{(1/f-1)^{n+1}}. \quad (4.21)$$

For $f \in (0, 1/2)$, this expression converges to zero as n goes to infinity, which proves the pointwise convergence. To get an estimate for the L_p-norm, we raise the right term to the power p and integrate to get

$$\|R^n(f)\|^p_{L_p(0,1/2)} < \int_0^{1/2} \frac{4}{(1/f-1)^{p(n+1)}}\,df \leq \frac{2}{np+p+1}, \quad (4.22)$$

which also tends to zero as n goes to infinity.

b) $f \in (1/2, 1)$.
From Lemma 1 and 2, $R^n(f)$ is bounded from above as

$$\left|\frac{U^n(f)}{1+U^{2n+1}(f)}\right| < \left|\frac{U^n(f)}{U^{2n+1}(f)}\right| < 4(\gamma/f)^{n+1},$$

where $0 < \gamma = (1-f) < 1/2$. Since $f > 1/2$, we also have

$$R^n(f) < 4(2\gamma)^{n+1}, \quad (4.23)$$

which convergences to zero for $\gamma < 1/2$. An estimate of the L_p-norm is provided by

$$\|R^n(f)\|^p_{L_p(1/2,1)} < \int_0^{1/2} 4(2\gamma)^{p(n+1)}\,d\gamma = \frac{2}{np+p+1}, \quad (4.24)$$

which tends to zero as n goes to infinity.

c) $f \in (1, +\infty)$.
From Lemma 1 and 2, it follows that

$$\left|\frac{U^n(f)}{1+U^{2n+1}(f)}\right| < \frac{|U^n(f)|}{U^{2n+1}(f)} < 6(\gamma/f)^{n+1} < 6/f^{n+1} \quad (4.25)$$

since $0 \leq \gamma \leq 1$. For $f > 1$, the right-most term converge to zero, which proves the pointwise convergence. An estimate of the L_p-norm is provided by

$$\|R^n(f)\|^p_{L_p(1,+\infty)} < 6\int_1^{+\infty} \frac{1}{f^{p(n+1)\,df}} = \frac{6}{np+p-1}, \quad (4.26)$$

which tends to zero as n goes to infinity. □

Proof of Theorem 3: The procedure is very similar to that just used. The essential steps are to show that the absolute value of the difference with the ideal response converges to zero pointwise and that the corresponding L_p-norm estimates tend to zero as n goes to infinity. The use of Lemmas 1 and 2 within the three frequency intervals leads to the following inequalities:

a) $f \in (0, 1/2)$:

$$|\text{Rect}(f) - F^n(f)| = \left|1 - \frac{1}{\sqrt{1+U^{2n+1}(f)}}\right| < \sqrt{U^{2n+1}(f)} < \frac{2}{(1/f-1)^{n+1}} \quad (4.27)$$

$$\|F^n(f) - \text{Rect}(f)\|^p_{L_p(0,1/2)} < \frac{1}{np+p+1}. \quad (4.28)$$

b) $f \in (1/2, 1)$:

$$|F^n(f) - \text{Rect}(f)| = \frac{1}{\sqrt{1+U^{2n+1}(f)}} < \frac{1}{\sqrt{U^{2n+1}(f)}} < (\gamma/f)^{n+1} < (2\gamma)^{n+1} \quad (4.29)$$

$$\|F^n(f) - \text{Rect}(f)\|^p_{L_p(1/2,1)} < \frac{1/2}{np+p+1}, \quad (4.30)$$

where $\gamma < 1/2$.

c) $f \in (1, +\infty)$:

$$|F^n(f) - \text{Rect}(f)| = \frac{1}{\sqrt{1+U^{2n+1}(f)}} < \frac{1}{\sqrt{U^{2n+1}(f)}} < (\gamma/f)^{n+1} < 1/f^{n+1} \quad (4.31)$$

$$\|F^n(f) - \text{Rect}(f)\|^p_{L_p(1,+\infty)} < \frac{1}{np+p-1}. \quad (4.32)$$

V. Conclusion

We have considered the problem of the approximation and interpolation of a signal using polynomial splines from a signal processing perspective. In particular, we have shown that the least-squares spline approximation of an arbitrary signal can be obtained in three steps: 1) prefiltering, 2) sampling, and 3) postfiltering. This process is, therefore, very similar to the classical filtering/sampling/interpolation paradigm for the representation of bandlimited signals, which is dictated by Shannon's sampling theorem. An aspect that has been emphasized is that both prefiltering and interpolation algorithms should be selected consistently in order to minimize the loss of information occurring during discretization. For instance, it turns out that the use of a conventional anti-aliasing lowpass filter is only optimal when it is used in conjunction with a sinc interpolator and that other prefilters are better suited for other forms of interpolation.

We have determined the frequency responses of optimal prefilters and their corresponding postfilters for polynomial splines of any order using three equivalent continuous signal representations. One of these representations uses sampled signal values as expansion coefficients and is best suited for conventional discrete signal processing. We have shown that the corresponding prefilters and spline interpolators tend to an ideal lowpass filter (pointwise and in all L_p-norms) as the order of the splines goes to infinity. These results establish the asymptotic equivalence with Shannon's well known sampling theorem and provide some useful extensions for polynomial splines of any order.

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