Fast Gabor-Like Windowed Fourier and Continuous Wavelet Transforms

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***Abstract*—Fast algorithms for the evaluation of running windowed Fourier and continuous wavelet transforms are presented. The analysis functions approximate complex-modulated Gaussians as closely as desired and may be optimally localized in time and frequency. The Gabor filtering is performed indirectly by convolving a premodulated signal with a Gaussian-like window and demodulating the output. The window functions are either B-splines dilated by an integer factor m or quasi-Gaussians of arbitrary size generated from the n-fold convolution of a symmetrical exponential. Both approaches result in a recursive implementation with a complexity independent of the window size ($O(N)$).**

I. Introduction

TWO fundamental tools in signal analysis are the windowed (or short-time) Fourier transform (WFT) and the continuous wavelet transform (CWT) [1], [3]. Both methods decompose a signal by performing inner products with a collection of running analysis functions. Each of these templates is predominantly localized in a certain region of the time-frequency plane. The optimum joint localization, as specified by the uncertainty principle, is achieved when the analysis functions are complex modulated Gaussians, which are also referred to as Gabor functions [4]. In the case of the WFT, the time and frequency resolutions are both fixed, which makes this approach particularly suitable for the analysis of signals with slowly varying periodic or stationary characteristics. In the case of the CWT, the analysis functions are obtained by dilation of a single (bandpass) wavelet [5]. This property enables the CWT to "zoom in" on singularities and makes it very attractive for the analysis of transient signals.

A major practical issue is the computational efficiency of these techniques. Fast recursive algorithms for the WFT have been developed for certain special windows, including rectangular [6], [7], Hamming and Hanning [8], and certain all-pole structures [9]. In each of these cases, the complexity per frequency component is $O(N)$, where N is the length of the input signal. Most CWTs can be computed as efficiently using the so-called "á trous" algorithm, provided that one restricts the analysis to scales that are powers of two [10], [11].

The purpose of this paper is to present a new recursive procedure for performing both running Fourier and *complex* wavelet signal analyses. This method is as efficient as previous recursive implementation of the WFT. However, it is unique in the sense that the time-frequency localization of the analysis functions can be chosen arbitrarily close to the optimum. When applied to the CWT, it is at least as performant as previous multiresolution-based algorithms [10], [11] with the advantage that it is also applicable to nondyadic scales. Finally, it is substantially faster than standard FFT-based techniques that require $O(N\log(N))$ operations.

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II. A Common Algorithmic Formulation

This section briefly describes the modulation technique that is the basis for the present approach.

A. Windowed Fourier Transform

Let $\{g(k)\}_{k\in Z}$ denote a discrete-time signal, and let $w_K(k)$ represent a window sequence. The WFT of g is then defined as

$$F_{w_K}g(l,\omega_n)=\sum_{k\in Z}g(k)w_K(k-l)e^{-j\omega_n(k-l)} \qquad (1)$$

where $\omega_n=\frac{2\pi n}{K}$, $n=0,\ldots,K-1$. Other definitions have been used, but they only differ by a phase factor. If $w_K(k)$ is of length K, then $\{F_{w_K}g(l,\omega_n)\}_{n=0,\ldots,K-1}$ represents the discrete Fourier transform of the portion of the signal that is viewed through an observation window positioned at the index l.

The WFT may be decomposed as

$$\begin{aligned}F_{w_K}g(l,\omega_n)&=e^{j\omega_n l}\sum_{k\in Z}w_K(k-l)g_{\omega_n}(k)\\&=e^{j\omega_n l}(w_K^T*g_{\omega_n})(l)\end{aligned} \qquad (2)$$

where $g_{w_n}(k)$ is the modulated signal

$$g_{w_n}(k)=g(k)e^{-j\omega_n k} \qquad (3)$$

and where $w_K^T(k)=w_K(-k)$ is the time-transpose of the window sequence. These last two equations suggest an implementation based on a modulator followed by a low-pass windowing filter [12].

B. Continuous Wavelet Transform

The CWT in the discrete case is defined as

$$W_\psi g(l,a)=\frac{1}{\sqrt{a}}\sum_{k\in Z}g(k)\psi\left(\frac{k-l}{a}\right) \qquad (4)$$

where a is the *scale* parameter, and ψ is a Gabor-wavelet obtained by modulation of a window function $w(x)$

$$\psi(x)=w(x)e^{-j\Omega x}. \qquad (5)$$

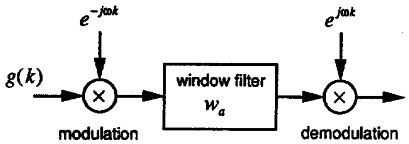

Fig. 1. Modulation approach to Gabor filtering.

Defining $w_a(x) = w(x/a)$, and the auxiliary modulated signal

$$g_a(k) = \frac{1}{\sqrt{a}} g(k) e^{-j\Omega k/a} \tag{6}$$

we rewrite (4) as

$$W_\psi g(l, a) = e^{j\Omega l/a} \sum_{k \in Z} g_a(k) w_a(k-l) = e^{j\Omega l/a}(w_a^T * g_a)(l). \tag{7}$$

This equation is almost identical to (2) except for the coupling that now exists between the size of the window and the modulation frequency. Accordingly, the components of both transforms can be evaluated using the general block diagram in Fig. 1.

III. Fast Recursive WFT and CWT Algorithms

Apart from a pre- and post-multiplication, the computational cost of the modulation approach in Fig. 1 depends entirely on the efficiency of the convolution with the windowing kernel. We now consider two approaches for implementing a symmetrical window filter ($w_a^T = w_a$) that is as close as we wish to a Gaussian and yet has a complexity $O(N)$ that is independent of its size.

A. B-Spline Window

The central B-spline of degree n ($\beta^n(x)$) is generated from the $(n+1)$-fold convolution of a unit rectangular pulse [13]. It is a symmetrical, compactly supported function that converges to a Gaussian uniformly and in all L_q-norms as n tends to infinity (cf. Theorem 1 of [14]). The corresponding discrete B-spline window is obtained by enlarging $\beta^n(x)$ by an integral factor m and sampling at the integers

$$w_m(k) = \beta^n(x)|_{x=k/m}. \tag{8}$$

If n is odd, then the transfer function of this filter can be factorized as (cf. [13])

$$W_m(z) = \left(\sum_{k=-[n/2]}^{+[n/2]} \beta^n(k) z^{-k} \right) \cdot \left(\frac{z^{k_0}}{m^n} \left(\sum_{k=0}^{m-1} z^{-k} \right)^{n+1} \right) \tag{9}$$

where $k_0 = (n+1)(m-1)/2$ is a proper offset. The first of these factors represents the transfer function of a symmetrical FIR filter (discrete B-spline filter), whereas the second corresponds to the cascade of $(n+1)$ moving sums. Each moving sum filter is implemented recursively with as few as two additions per sample. Hence, the total complexity of B-spline filtering is independent of the window size m [13].

A choice of modulation that is especially suitable for constructing wavelets is $\Omega = \mp 2\pi$; this ensures that ψ is an admissible wavelet in the sense that its Fourier transform has at least one zero at the origin [15]. The resulting complex B-spline wavelet[1] is

$$\psi(x) = \beta^n(x) e^{j2\pi x} \longleftrightarrow \hat{\psi}(f) = \text{sinc}^{n+1}(f-1). \tag{10}$$

The time-frequency localization of this function improves rapidly with n since $\beta^n(x)$ converges to a Gaussian. For $n = 3$, the variance product is already within 0.5% of the limit specified by the uncertainty principle (cf. Table II of [14]). This cubic spline wavelet transform can be computed with as few as eight (real) multiplications and 22 additions per sample. The corresponding impulse response for $m = 30$ is shown in Fig. 2(a). Because of its simplicity, this technique constitutes the method of choice for computing complex CWT's with integer dilation factors. Recently, we have also proposed a similar algorithm (which does not use modulation) for the evaluation of real-valued CWT's [16].

B. Approximation of a Gaussian Using Exponentials

The approach that is presented next can approximate an arbitrary Gaussian window

$$w_a(k) \cong \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-k^2}{2a^2}\right). \tag{11}$$

This is achieved by cascading n symmetrical exponential filters

$$W_a(z) = a \cdot H_\alpha(z)^n \tag{12}$$

with

$$H_\alpha(z) = \frac{(1-\alpha)^2}{(1-\alpha z)(1-\alpha z^{-1})} \overset{z}{\longleftrightarrow} h_\alpha(k) = \left(\frac{1-\alpha}{1+\alpha}\right) \alpha^{|k|}. \tag{13}$$

These kernels are chosen because they can be implemented recursively from the cascade of simple first-order causal and anticausal filters with as few as two additions and two multiplies per sample (cf. II.B of [17]). $h_\alpha(k)$, $\alpha \geq 0$, is a sequence that is symmetrical, positive, and has a sum that is one. It may therefore be interpreted as a discrete probability density function (PDF). The variance of this distribution is

$$\mu_2 = \sum_{k \in Z} k^2 h_\alpha(k) = \frac{\alpha^2}{(1-\alpha)^2}. \tag{14}$$

The normalized window function $w_a(k)/a$ is the n-fold convolution of this density; it also represents the PDF of the sum of n independent exponentially distributed random variables. As n increases, this sequence converges to a Gaussian as a consequence of the central limit theorem. In addition, its variance is simply the sum of the variances of the individual terms, i.e., $\sigma^2 = n \cdot \mu_2$. This property is used to determine the appropriate value of a by solving (14) as a function of μ_2, which yields

$$\alpha = 1 + \frac{1}{\mu_2} - \frac{\sqrt{1+2\mu_2}}{\mu_2} \tag{15}$$

[1] This complex wavelet should not be confused with the real-valued B-spline wavelets that have been constructed by us previously [14]. These latter wavelets provide bases of L_2 and are therefore much more constrained mathematically. They are also well localized in the sense that they converge to a cosine-Gabor function.

TABLE I
RELATIVE TIME-FREQUENCY BANDWIDTH PRODUCT AS A FUNCTION OF THE MULTIPLICITY FOR THE SECOND ALGORITHM ($a = 2^{7/2}$)

$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$	$n = 10$
1.5653	1.0882	1.0395	1.0225	1.0146	1.0102	1.0075	1.0058	1.0046	1.0037

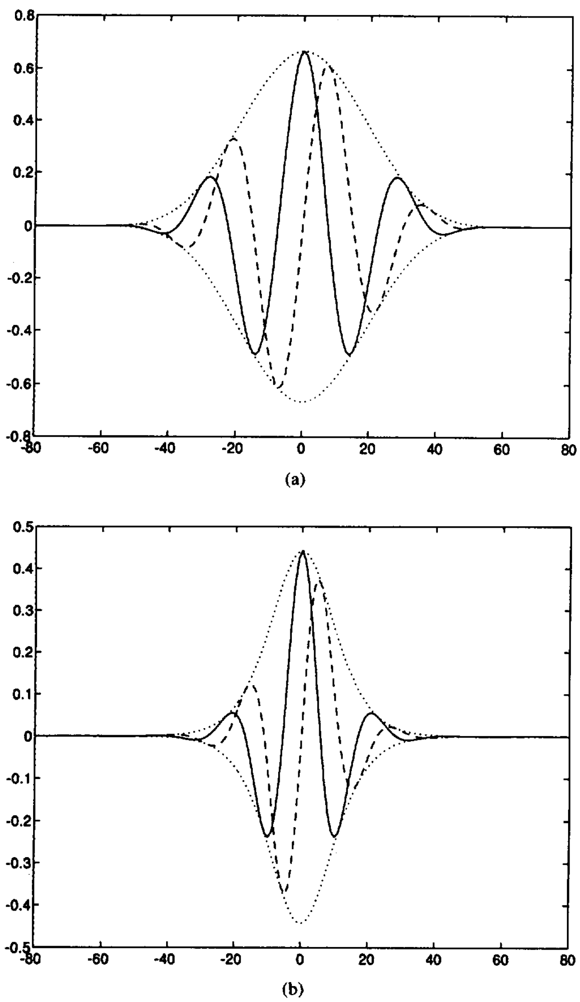

Fig. 2. Examples of impulse responses (real, imaginary, ± magnitude) of wavelet filters implemented using the modulation approach and recursive window filtering: (a) Complex cubic B-spline ($n = 3$ and $\Omega = 2\pi$ with a scale factor $m = 30$; (b) quasi-Gabor wavelet (four cascaded exponentials and $\Omega = \pi$) with a nonrational scale factor $a = 2^{2/7}$.

with $\mu_2 = a^2/n$. The response of a wavelet filter with a nonrational scaling factor ($a = 2^{7/2}$) that was implemented using this technique is shown in Fig. 2(b). The results in Table I illustrate that the localization of this wavelet filter improves with n; a similar convergence behavior has also been observed for other scales. The convergence is not as fast as in the B-spline case. The total complexity of this algorithm is $(4n + 6)$ multiplies + $(4n + 2)$ adds per wavelet coefficient.

Unlike the B-spline approach, this algorithm puts no restriction on the selection of the scale parameter. However, there is a small price to be paid for this increased flexibility. First, the corresponding wavelet does not satisfy the admissibility condition $\int \psi(x)dx = 0$, which may require a small dc correction. Second, there is no exact analytic wavelet formula, although the Gabor approximation $\psi(x) \cong (2\pi)^{-1/2}e^{-x^2/2}e^{-j\Omega x}$ should be sufficient for most applications.

IV. CONCLUSION

Two fast recursive algorithms have been proposed for computing Gabor-like WFT's as well as complex CWT's. Their most attractive features are summarized as follows:

- The complexity per channel (scale or frequency component) is $O(N)$; it is the same for all Gabor functions.
- The resulting analysis functions approximate complex-modulated Gaussians as closely as desired. Hence, the decomposition is optimally localized in both time and frequency.
- Unlike previous approaches, the analysis is not restricted to scales that are powers of two. The B-spline algorithm can handle any integer scale, whereas the exponential one has no restrictions at all.
- Both algorithms are noniterative and have a very regular structure. This makes them ideally suited for parallel implementation with one processor per channel.

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