Shift-Orthogonal Wavelet Bases Using Splines

Michael Unser, Philippe Thévenaz, and Akram Aldroubi

***Abstract*— We present examples of a new type of wavelet basis functions that are orthogonal across shifts but not across scales. The analysis functions are piecewise linear while the synthesis functions are polynomial splines of degree n (odd). The approximation power of these representations is essentially as good as that of the corresponding Battle–Lemarié orthogonal wavelet transform, with the difference that the present wavelet synthesis filters have a much faster decay. This last property, together with the fact that these transformations are almost orthogonal, may be useful for image coding applications.**

I. Introduction

THE theory of the wavelet transform has resulted in the construction of many multiresolution bases of L_2 (the space of square integrable functions) [1]–[4]. The earlier examples of wavelet bases were orthogonal (e.g., Daubechies, and Battle–Lemarié wavelets). It was soon realized that one could gain in flexibility by relaxing the intrascale orthogonality constraint; this led to the construction of semi-orthogonal wavelets, which are still orthogonal across scale [5]–[7]. The versatility of semi-orthogonal wavelet basis allows one to introduce many interesting properties [7] and almost any desirable shape [8]. A noteworthy example in this category are the B-spline wavelets that exhibit near-optimal time-frequency localization [9]. The next evolutionary step was to drop the orthogonality requirement altogether, which led to the construction of biorthogonal wavelets [10], [11]. The advantage of this last category is that the wavelet filters can be shorter; in particular, they can be both FIR and linear phase, which is typically not possible otherwise.

In this letter, we consider another possibility that has been neglected so far, namely, wavelets that are orthogonal to their translates within the same scale but not across scales; this property will be called shift-orthogonality. Subband coding is an area where this feature could be appealing. Orthogonality is required for the quantization error in the wavelet domain to be a valid indicator of the final distortion [4]. In our case, we insist on preserving orthogonality within the wavelet channels, which is consistent with independent channel processing. In particular, this justifies discarding small coefficients, which is the typical procedure for achieving the greatest compression savings.

Shift-orthogonality is less constraining than full orthogonality. This gives us more freedom for designing wavelets. For example, we can obtain shorter wavelet synthesis filters, which is a useful feature for reducing reconstruction artifacts (e.g., spreading of coding errors and ringing around sharp transitions).

The next consideration is the choice of an approximation space, the order of which determines the rate of decay of the error as a function of scale. The approximation power of the representation depends primarily on the synthesis scaling function [12], [13]. Recent error bound comparisons for several types of wavelets suggest that splines have the best approximation properties [12], [13]; therefore, we will consider these particular functions in our design. To get the best quality approximation, we use higher order splines on the synthesis side. Conversely, we use lower order splines on the analysis side to reduce the size of wavelet synthesis filters.

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The authors are with the Biomedical Engineering and Instrumentation Program, National Center for Research Resources, National Institutes of Health, Bethesda, MD 20892-5766 USA.
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II. Spline Scaling Functions and Wavelets

A. Spline Spaces and Notations

We consider the two spline multiresolution analyses of L_2 $\{V_i^1\}_{i\in Z}$ and $\{V_i^n\}_{i\in Z}$ (n odd), where

$$s_j(x) \in V_j^n \Leftrightarrow s_j(2^j x) \in V_0^n = \left\{ s_0(x) = \sum_{k\in Z} c(k)\beta^n(x-k) | c \in l_2 \right\}. \tag{1}$$

V_0^n is the basic space of splines of degree n, that is, the subspace of functions that are $(n-1)$ continuously differentiable and are polynomial of degree n in each interval $(k, k+1)$ $k \in Z$. The generating function $\beta^n(x)$ is Schoenberg's central B-spline of degree n, which is obtained from the $(n+1)$-fold convolution of a unit rectangular pulse. For n odd, the B-spline of degree n satisfies the two-scale relation (cf. [5])

$$\beta^n \frac{x}{2} = \sum_{k\in Z} u_2^n(k)\beta^n(x-k) \tag{2}$$

where u_2^n is the binomial filter of order $n+1$, whose transfer function is

$$u_2^n(k) \overset{z}{\longleftrightarrow} U_2^n(z) = z^{(n+1)/2} \cdot \frac{(1+z^{-1})^{n+1}}{2^n}. \tag{3}$$

We also use the following notation to represent the discrete B-spline of degree n

$$b_1^n(k) = \beta^n(x)|_{x=k} \overset{z}{\longleftrightarrow} B_1^n(z). \tag{4}$$

B. Construction of Dual Spline Scaling Functions

The basic approximation (or synthesis) space is V_0^n, where n is odd. The parameter important for performance is the order of the representation $p = n+1$, which is one more than the degree [13]. The corresponding orthogonal generating function is the Battle–Lemarié scaling function, which can be expressed as (cf. [5])

$$\phi(x) + \sum_{k\in Z} (b_1^{2n+1})^{-1/2}(k)\beta^n(x-k), \quad (5)$$

where $(b_1^{2n+1})^{-1/2}(k) \overset{z}{\longleftrightarrow} 1/\sqrt{B_1^{2n+1}(z)}$ denotes the square-root convolution inverse of the symmetric sequence $b_1^{2n+1}(k)$.

Instead of performing a decomposition in V_0^n by using an orthogonal projection as in the Battle–Lemarié case [1], we use an oblique projection perpendicular to V_0^1 (the space of piecewise-linear splines). We could also have used a higher order analysis space, but our motivation here is to have the simplest possible analysis functions. The corresponding (unique) dual function $\tilde{\phi}(x) \in V_0^1$ must satisfy the biorthogonality constraint $\langle \phi(x), \tilde{\phi}(x-k)\rangle = \delta[k]$, where $\delta[k]$ is the discrete unit impulse [14]. This leads to the following characterization:

$$\tilde{\phi}(x) = \sum_{k\in Z} (b_1^{2n+1})^{1/2} * (b_1^{n+2})^{-1}(k)\ \beta^1(x-k) \quad (6)$$

where $(b_1^{n+2})(k) \overset{z}{\longleftrightarrow} 1/B_1^{n+2}(z)$ is the convolution inverse of the cross-correlation sequence $(b_1^{n+2})(k) = \langle \beta^n(x), \beta^1(x-k)\rangle$. Thus, the projection P_i of a function $f \in L_2$ into V_1^n perpendicular to V_i^1 can be written as

$$P_i f(x) = \sum_{k\in Z} \langle f, \tilde{\phi}_{i,k}\rangle \phi_{i,k} \quad (7)$$

where we use the standard short-form notation $\phi_{i,k} = 2^{-i/2}\phi(x/2^i - k)$.

C. Construction of Shift-Orthogonal Spline Wavelets

We now consider the characterization of the corresponding wavelets at resolution level $i = 1$. The synthesis wavelet $\psi(x/2)$ must satisfy the following conditions:

i) $\psi(x/2) \in V_0^n$; i.e., $\psi(x/2)$ is a spline of degree n.

ii) $\langle \psi(x/2), \beta^1(x/2-k)\rangle = 0$ because $\psi(x/2)$ is perpendicular to V_1^1.

iii) $\langle 2^{-1/2}\psi(x/2), 2^{-1/2}\psi(x/2-k)\rangle = \delta[k]$ (intrascale orthogonality).

It turns out that there is a unique function ψ that verifies all those conditions; it is given by

$$\psi\frac{x}{2} = \sum_{k\in Z} [p]_{\uparrow 2} * q(k)\beta^n(x-k), \quad (8)$$

where the sequences q and p are defined as follows:

$$q(k+1) = (-1)^k \cdot (u_2^1 * b_1^{n+1})(k) \quad (9)$$

$$p(k) = \sqrt{2}([q * q^T * b_1^{2n+1}]_{\downarrow 2})^{-1/2}(k). \quad (10)$$

The operators $[\bullet]_{\uparrow 2}$ and $[\bullet]_{\downarrow 2}$ denote upsampling and downsampling by a factor of two, respectively, and $q^T(k) = q(-k)$. The dual analysis wavelet $\tilde{\psi}(x/2)$ must satisfy a similar set of conditions:

iv) $\tilde{\psi}(x/2) \in V_0^1$; i.e., $\tilde{\psi}(x/2)$ is a piecewise-linear spline.

v) $\langle \tilde{\psi}(x/2), \beta^n(x/2-k)\rangle = 0$ because $\tilde{\psi}(x/2)$ is perpendicular to V_1^n.

vi) $\langle 2^{-1/2}\tilde{\psi}(x/2), 2^{-1/2}\psi(x/2-k)\rangle = \delta[k]$ (biorthogonality).

After some algebraic manipulations, we obtain the similar expression

$$\tilde{\psi}\frac{x}{2} = \sum_{k\in Z} [\tilde{p}]_{\uparrow 2} * \tilde{q}(k)\beta^1(x-k), \quad (11)$$

where

$$\tilde{q}(k+1) = (-1)^k \cdot (u_2^n * b_1^{n+2})(k) \quad (12)$$

$$\tilde{p}(k) = 2 \cdot (p * [q * \tilde{q}^T * b_1^{n+2}]_{\downarrow 2})^{-1}(k) \quad (13)$$

with p and q defined in (10) and (9). Since the maximum angle between the analysis and synthesis spaces V_0^1 and V_0^n is less than 90° [14], the corresponding wavelet subspaces and oblique projection operators are well defined [15]. Hence, the convolution and square-root inverses in (10) and (13) are well posed, and the resulting digital filters p and $\tilde{p}$ are stable and invertible. The corresponding dual pairs of scaling functions and wavelets for $n = 3$ are shown in Fig. 1.

III. Wavelet Transform and Filterbank Algorithm

Let W_i^n be the complementary wavelet space of V_i^n in V_{i-1}^n perpendicular to V_i^1, i.e., $V_{i-1}^n = W_i^n \oplus V_i^n$ with $W_i^n \perp V_i^1$. It is not difficult to show that $\{\psi_{i,k} = 2^{-i/2}\psi(x/2^{-i} - k)\}_{k\in Z}$ is an orthogonal basis of W_i^n. Since it is well known that $\cup_{i\in Z} V_i^n$ is dense in L_2, it follows that the set $\{\psi_{i,k}\}_{(i,k)\in Z^2}$ is a Riesz basis of L_2 and that every function $f \in L_2$ can be represented by its shift-orthogonal wavelet expansion

$$f(x) = \sum_{i\in Z}\sum_{k\in Z} \langle f, \tilde{\psi}_{i,k}\rangle \psi_{i,k}. \quad (14)$$

The special feature of this decomposition is that the basis functions are orthogonal with respect to shifts (index k) but not across scales or dilations (index i). We should note, however, that the residual correlations across scales should be not be too significant because the angles between the various spline spaces are relatively small (cf. (55) and table III in [14]). For these reasons, we can expect the shift-orthogonal decomposition (14) to provide essentially the same type of energy compaction as the corresponding orthogonal Battle–Lemarié wavelet transform.

The wavelet transform (14) can be implemented iteratively using a standard tree-structured perfect reconstruction filterbank [4]. The corresponding symmetric analysis and synthesis filters $(\tilde{h}, \tilde{g})$ and (h, g), respectively, are defined as follows:

$$\begin{cases} \tilde{h}(k) = \frac{1}{2}\left\langle \tilde{\phi}\frac{x}{2}, \phi(x+k)\right\rangle \\ \tilde{g}(k) = \frac{1}{2}\left\langle \tilde{\psi}\frac{x}{2}, \phi(x+k)\right\rangle \\ h(k) = \left\langle \phi\frac{x}{2}, \tilde{\phi}(x-k)\right\rangle \\ g(k) = \left\langle \psi\frac{x}{2}, \tilde{\phi}(x-k)\right\rangle \end{cases} \quad (15)$$

We have derived explicit filter formulas in both time and frequency domains. All filters are infinite but decay exponentially fast. The filter coefficients for $n = 3$ (cubic splines)

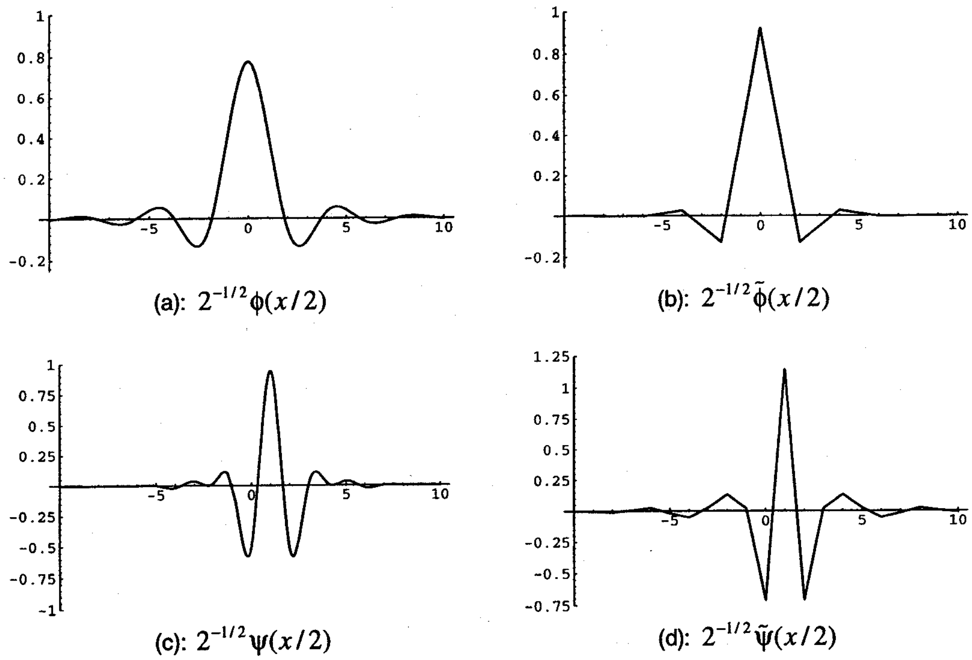

Fig. 1. Dual sets of cubic spline and piecewise-linear scaling functions and wavelets at the first resolution level: (a) Orthogonal cubic spline Battle–Lemarié scaling function; (b) dual linear spline scaling function; (c) shift-orthogonal cubic spline wavelet; (d) dual linear spline wavelet. The basis functions in (c) and (d) (resp., (a) and (b)) are polynomial splines with knots at the integers (resp., at the even integers).

TABLE I
FILTER COEFFICIENTS FOR THE CUBIC SPLINE SHIFT-ORTHOGONAL WAVELET TRANSFORM. THE FILTERS ARE ALL SYMMETRIC. THE LAST COLUMN DISPLAYS THE BATTLE–LEMARIÉ WAVELET FILTER FOR $n = 1$.

k	$\tilde{h}(k)$	$\tilde{g}(k-1)$	$h(k)$	$g(k+1)$	Battle–Lemarié $n = 1$
0	0.582529	0.54633	1.08347	1.15485	1.15633
±1	0.282665	−0.308251	0.613659	−0.563151	−0.561863
±2	−0.0519285	−0.0384306	−0.070996	−0.0973574	−0.0977235
±3	−0.0395953	0.0804651	−0.155616	0.0753237	0.0734618
±4	0.0123611	0.0227514	0.0453692	0.024683	0.0240007
±5	0.00839683	−0.0313868	0.0594936	−0.0147667	−0.0141288
±6	−0.00132849	−0.0108077	−0.024291	−0.00582268	−0.00549176
±7	−0.00166831	0.013063	−0.0254308	0.00323499	0.00311403
±8	−0.000919721	0.00476525	0.0122829	0.00135813	0.00130584
±9	0.000117659	−0.00555301	0.0115986	−0.0007871	−0.000723563
±10	0.000956435	−0.00206644	−0.00615726	−0.00034943	−0.000317203
±11	0.000182244	0.00237672	−0.00549058	0.00018464	0.000173505
±12	−0.000648131	0.000891964	0.00309248	0.0000834622	0.0000782857
±13	−0.000167137	−0.00102066	0.00266174	−0.0000478199	−0.0000424422
±14	0.000377941	−0.000384402	−0.00156092	−0.0000222962	−0.0000195427
±15	0.000109648	0.00043885	−0.00131126	0.0000115052	0.0000105279
±16	−0.000206646	0.000165587	0.00079187	$5.38729\ 10^{-6}$	$4.92118\ 10^{-6}$
±17	−0.0000631112	−0.000188834	0.00065353	$-3.07609\ 10^{-6}$	$-2.63837\ 10^{-6}$
±18	0.000108551	−0.0000713083	−0.000403594	$-1.47358\ 10^{-6}$	$-1.2477\ 10^{-6}$
±19	0.0000342583	0.0000812759	−0.000328589	$7.48299\ 10^{-7}$	$6.6641\ 10^{-7}$
±20	−0.0000557311	0.0000307068	0.000206534	$3.57617\ 10^{-7}$	$3.18076\ 10^{-7}$

are given in Table I. The lowpass filter h is the same as the Battle–Lemarié filter described in [1]. Interestingly, the wavelet synthesis filter g decays significantly faster and turns out to be very similar to a Battle–Lemarié filter of degree 1 ($n = 1$), which is also given for comparison.

IV. CONCLUSION

We have presented a new class of hybrid spline wavelet transforms. The motivation behind this proposal was to use lower order analysis functions while essentially preserving the approximation and orthogonality properties of higher order Battle–Lemarié wavelets. In contrast with previous semi- and biorthogonal constructions, we have only relaxed the orthogonality constraint in between resolution levels. The basis functions are still orthogonal within a given wavelet channel (or scale), a property that is quite desirable for quantization purposes. The advantage over the Battle–Lemarié family is that the wavelet synthesis filter decays substantially faster, which is a property that could be useful for reducing ringing artifacts in coding applications. All the underlying basis functions (including duals) have been characterized explicitly in the time domain (polynomial spline representation). Such direct formulas are, in general, not available for other wavelet bases, except for the class of semi-orthogonal splines considered in [5]. The wavelets presented here are examples of a new type of shift-orthogonal wavelet that will be completely characterized in a forthcoming paper. We will also describe shift-orthogonal spline wavelets for n even.

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