Proc. SPIE Vol. 3169, *Wavelets Applications in Signal and Image Processing V*, 1997, pp. 422-431.

TEN GOOD REASONS FOR USING SPLINE WAVELETS

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ABSTRACT

The purpose of this note is to highlight some of the unique properties of spline wavelets. These wavelets can be classified in four categories: othogonal (Battle-Lemarié), semi-orthogonal (e.g., B-spline), shift-orthogonal, and biorthogonal (Cohen-Daubechies-Feauveau). Unlike most other wavelet bases, splines have explicit formulae in both the time and frequency domain, which greatly facilitates their manipulation. They allow for a progressive transition between the two extreme cases of a multiresolution: Haar's piecewise constant representation (spline of degree zero) versus Shannon's bandlimited model (which corresponds to a spline of infinite order). Spline wavelets are extremely regular and usually symmetric or anti-symmetric. They can be designed to have compact support and to achieve optimal time-frequency localization (B-spline wavelets). The underlying scaling functions are the B-splines, which are the shortest and most regular scaling functions of order L. Finally, splines have the best approximation properties among all known wavelets of a given order L. In other words, they are the best for approximating smooth functions.

Keywords: splines, wavelet basis, biorthogonal wavelets, regularity, smoothness, time-frequency localization, approximation properties.

1. INTRODUCTION

Researchers are now faced with an ever increasing variety of wavelet bases to choose from. While the choice of the "best" wavelet is obviously application-depend, it can be useful to isolate a number of properties and features that are of general interest to the user. The purpose of this paper is to present a list of arguments in favor of splines, which are unique in a number of ways.

Splines have had a significant impact on the theory of the wavelet transform. The earliest example is the Haar[11] wavelet which is a spline of degree 0. This construction was extended to higher order splines by Strömberg[24], eventhough his work remained largely unoticed until wavelets became what they are today. Perhaps the best known examples of spline wavelets are the orthogonal Battle-Lemarié functions [5, 12], which can be seen as percursors of Mallat's multiresolution theory of the wavelet transform [13]. Splines have also been used to illustrate many of the later constructions of non-orthogonal wavelet bases (semi-orthogonal, bi-orthogonal, and more recently, shift-orthogonal). Dropping the orthogonality requirement was found to be advantageous for recovering many desirable wavelet properties that could not be achieved otherwise. Noteworthy examples are the B-spline wavelets [6, 29] which are compactly supported and achieve a near optimal time-frequency localization. Finally, the most popular representatives of the Cohen-Daubechies-Feauveau class of biorthogonal wavelets [7] are splines as well. This is because the iteration of the binomial refinement filter — which is a crucial component in any wavelet construction — converges to the B-spline which is the generating function for polynomial splines.

2. SPLINES AND WAVELETS

Spline wavelets stand apart in the general theory of the wavelet transform. Their construction starts with the specification of the underlying multiresolution function spaces (polynomial splines). Thus, spline wavelets can be characterized explicitly; this is in contrast with most other constructions where the scaling function is specified indirectly via a two-scale relation. The main advantage of an explicit construction is that one does not have to worry about the delicate issues of the convergence of the iterated filterbank. It also makes the study of regularity much more transparent.

2.1 Polynomial splines

A polynomial spline of degree n is made up of polynomial segments of degree n that are connected in a way that garantees the continuity of the function and of its derivative up to order n-1. The joining points between the polynomial segments are called *knots*. In the context of the wavelet transform, the knots are equally-spaced and typically positioned at the integers. One can thus define a hierarchy of spline subspaces of degree n, $\{V_i^n\}_{i\in Z}$, where V_i^n is the subspace of L_2-functions that are (n-1) times continuously differentiable and are polynomials of degree n in each interval $\left[2^i k, 2^i(k+1)\right)$, $k \in Z$. The spacing between the knot points 2^i is controlled by the scale index i. Clearly, a function $f(x) \in V_{i_0}^n$ that is piecewise polynomial on each segment $\left[2^{i_0} k, 2^{i_0}(k+1)\right)$ is also included in any of the finer subspaces V_i^n with $i \le i_0$. Thus, we have the following inclusion property

$$L_2 \supset \cdots \supset V_{-1}^n \supset V_0^n \cdots \supset V_i^n \cdots \supset \{0\}. \tag{1}$$

Furthermore, it is well known that one can approximate any L_2-function by a spline as closely as one wishes by letting the knot spacing (or scale) go to zero ($i \to -\infty$). This means that the above sequence of nested subspaces is dense in L_2 and therefore meets all the requirements for a multiresolution analysis of L_2 in the sense defined by Mallat [13, 14]. This implies that it is indeed possible to construct wavelet bases that are polynomial splines.

The best way to proceed is to use Schoenberg's representation of splines in terms of the B-spline basis functions [17, 18]. In order to satisfy the multiresolution inclusion property for any degree n, we will use the so-called *causal B-splines* which can be constructed from the (n+1)-fold convolution of the indicator function in the unit interval (causal B-spline of degree 0)

$$\varphi^n(x) = \underbrace{\varphi^0 * \cdots * \varphi^0}_{(n+1)\ \text{times}}(x), \tag{2}$$

where

$$\varphi^0(x) = \begin{cases} 1, & 0 \le x < 1 \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The B-spline of degree n satisfies the two-scale relation [31]

$$\varphi^n(x/2) = \sqrt{2} \sum_{k\in Z} h^n(k)\varphi^n(x-k), \tag{4}$$

where $h^n(k)$ is the binomial filter of order n+1 whose transfer function is

$$h^n(k) \quad \stackrel{z}{\longleftrightarrow} \quad H^n(z) = \sqrt{2} \cdot \left(\frac{1+z^{-1}}{2}\right)^{n+1}. \tag{5}$$

In 1946, Schoenberg proved that any polynomial spline of degree n with knots at the integers could be represented as a linear combination of shifted B-splines [17]. Thus, our basic spline space V_0^n can also be specified as

$$V_0^n = \left\{ s_0(x) = \sum_{k\in Z} c(k)\varphi^n(x-k) \mid c \in l_2 \right\}, \tag{6}$$

where the weights $c(k)$ are the so-called B-spline coefficients of the spline function $s_0(x)$. In addition, it can be shown that the B-splines $\left\{\varphi^n(x-k)\right\}_{k\in Z}$ constitute a Riesz basis of V_0^n in the sense that there exist two constants $A_n > 0$ and $B_n < +\infty$ such that

$$\forall c \in l_2,\ A_n \cdot \|c\|_{l_2}^2 \le \left\| \sum_{k\in Z} c(k)\varphi^n(x-k) \right\|_{L_2}^2 \le B_n \cdot \|c\|_{l_2}^2. \tag{7}$$

The lower inequality implies that the B-splines are linearly independent (i.e., $s_0(x) = 0 \ \Rightarrow c(k) = 0$). The upper inequality guarantees that $V_0^n \subset L_2$. Hence, any polynomial spline has a unique representation in terms of its B-spline coefficients $c(k)$. Schoenberg also proved that the B-splines are the shortest possible spline functions [19]. This, together with the fact that these functions have a simple analytical form, makes the B-spline representation one of the preferred tools for the study and characterization of splines [10].

2.2 Biorthogonal wavelets

In the most general case, the construction of biorthogonal wavelet bases involves two multiresolution analyses of L_2: one for the analysis, and one for the synthesis [7]. These are usually denoted by $\{V_i(\tilde{\varphi})\}_{i\in Z}$ and $\{V_i(\varphi)\}_{i\in Z}$, where $\tilde{\varphi}(x)$ and $\varphi(x)$ are the analysis and synthesis scaling functions, respectively. Note that $\tilde{\varphi}$ and φ can be arbitrary solutions of a two-scale relation and not necessarily the causal B-splines φ^n defined previously. The corresponding analysis and synthesis wavelets $\tilde{\psi}(x)$ and $\psi(x)$ are then contructed by taking linear combinations of these scaling functions

$$\tilde{\psi}(x/2) = \sqrt{2}\sum_k \tilde{g}(k)\tilde{\varphi}(x-k) \qquad (8)$$

$$\psi(x/2) = \sqrt{2}\sum_k g(k)\varphi(x-k). \qquad (9)$$

They form a biorthogonal set in the sense that

$$\langle \psi_{i,k}, \tilde{\psi}_{j,l}\rangle = \delta_{i-j,k-l}, \qquad (10)$$

with the short form convention $\psi_{i,k} = 2^{-i/2}\psi(2^{-i}x-k)$. This allows us to obtain the wavelet expansion of any L_2-function as

$$\forall f \in L_2, \quad f = \sum_{i\in Z}\sum_{k\in Z}\langle f, \tilde{\psi}_{i,k}\rangle \psi_{i,k}. \qquad (11)$$

Note that the underlying basis functions are usually specified indirectly in terms of the four sequences $h(k)$, $\tilde{h}(k)$, $g(k)$ and $\tilde{g}(k)$, which are the filters for the fast wavelet transform algorithm.

2.3 Spline wavelets

We have a spline wavelet transform whenever the synthesis functions ($\psi(x)$ and $\varphi(x)$) are polynomial splines of degree n. This means that the synthesis wavelet can also be represented by its B-spline expansion

$$\psi(x/2) = \sum_{k\in Z} w(k)\varphi^n(x-k). \qquad (12)$$

It is important to observe that the underlying scaling function $\varphi(x) \in V_0^n$ is not necessarily the B-spline of degree n — unless $h(k)$ is precisely the binomial filter (5). This function is usually specified indirectly as the solution of the two-scale relation

$$\varphi(x/2) = \sqrt{2}\sum_{k\in Z} h(k)\varphi(x-k), \qquad (13)$$

where $h(k)$ is the corresponding (lowpass) reconstruction filter. However, in the spline case, there will always exist a sequence $p(k)$ such that

$$\varphi(x) = \sum_{k\in Z} p(k)\varphi^n(x-k). \qquad (14)$$

Such specific B-spline characterizations for various kinds of spline scaling functions (orthogonal, dual, or interpolating) can be found elsewhere [32]. Note that the sequence $p(k)$ defines an invertible convolution operator from l_2 into l_2 which performs the change from one coordinate system to the other (i.e., φ to φ^n). The basic requirement for $\{\varphi(x-k)\}_{k\in Z}$ to form a Riesz basis of V_0^n is that there exist two constants $A_p > 0$ and $B_p < +\infty$ such that $A_p \le |P(e^{j\omega})|^2 \le B_p$ *almost everywhere*, where $P(e^{j\omega})$ denotes the Fourier transform of p.

If we combine (9) with (14), we obtain the B-spline coefficients of the wavelet $\psi(x)$:

$$w(k) = (p * g)(k) \quad \overset{z}{\longleftrightarrow} \quad W(z) = P(z)G(z). \qquad (15)$$

These spline wavelets are quite attractive because they are extremely regular. In fact, any spline wavelets of degree n is (n+1) times differentiable almost everywhere. It has a Sobolev regularity index $s_{max}=n+1/2$ meaning that all its fractional derivatives up to s_{max} are well-defined in the L_2-sense [23].

A very important wavelet parameter is the order of the representation determined from the zero-properties of the refinement filter h. By definition, the order L is the largest integer such that the transfer function of h can be factorized as , where $Q(z)$ is the *z*-transform of a stable filter. Thus, splines have an order of approximation $L=n+1$ which is one more than the degree. This order property has some remarkable consequences such as the vanishing moments of the analysis wavelet, the ability of the scaling function to reproduce polynomials of degree $n=L-1$, and the special eigen-structure of the two-scale transition operator (cf. Strang-Nguyen [23], Chapter 7).

Many kinds of spline wavelets have been described in the literature. The four primary types can be differentiated on the basis of their orthogonality properties; they are summarized in Table 1. Four corresponding examples of cubic spline wavelets and their duals are salso shown in Fig. 1.

TABLE I: CLASSIFICATION OF SPLINE WAVELETS WITH THEIR MAIN PROPERTIES.

Wavelet type	**Orthogonality**	**Compact support**	**Key properties**	**Implementation**
Orthogonal splines (Battle-Lemarié, Mallat)	Yes	No	• Symmetry & regularity + Orthogonality	IIR/IIR
Semi-orthogonal splines (B-splines) (Chui-Wang, Unser-Aldroubi)	Inter-scale	Analysis or Synthesis	• Symmetry & regularity + Optimal time-frequency localization	Recursive IIR/FIR
Shift-orthogonal splines (Unser-Thévenaz-Aldroubi)	Intra-scale	No	• Symmetry & regularity + Quasi-orthogonality + Fast decaying wavelet	IIR/IIR
Biorthogonal splines (Cohen-Daubechies-Feauveau)	No	Yes	• Symmetry & regularity + Compact support	FIR/FIR

• *Orthogonal spline wavelets:* The wavelets in this first category were constructed independently by Battle [5] and Lemarié [12]. They were also investigated by Mallat [13, 14] to illustrate his general multiresolution theory of the wavelet transform. Like most spline wavelets, the Battle-Lemarié functions are very regular, smooth and symmetric. Unfortunately, they are not compactly supported, eventhough they decay exponentially fast.

• *Semi-orthogonal spline wavelets*: The wavelets in this category retain the inter-scale orthogonality, but there is no requirement for the basis functions to be orthogonal to their translates within the same resolution level. Chui-Wang [6] and Unser-Aldroubi-Eden [29] independently constructed the first such example: the compactly supported B-spline wavelets (cf. Fig. 1b), which are the wavelet counterparts of the classical B-splines. It was then realized that one could still generate many other semi-orthogonal spline wavelets by taking suitable linear combinations [31]. These wavelets are versatile because is possible to choose the sequence $p(k)$ in (14) and $w(k)$ in (12) such that the underlying functions have some specific property [3] (e.g. interpolation, orthogonality or optimal time-frequency localization). In fact, it is possible to construct semi-orthogonal spline wavelets with virtually any prescribed shape [1]. Some of the wavelets and scaling functions in the semi-orthogonal case can be compactly supported; however, the dual analysis functions, which are splines as well, are generally not. Thus, the implementation usually requires some infinite impulse response filters. We note, however, that a recursive implementation is usually possible [31], except in the orthogonal case where the filters need to be truncated to finite length.

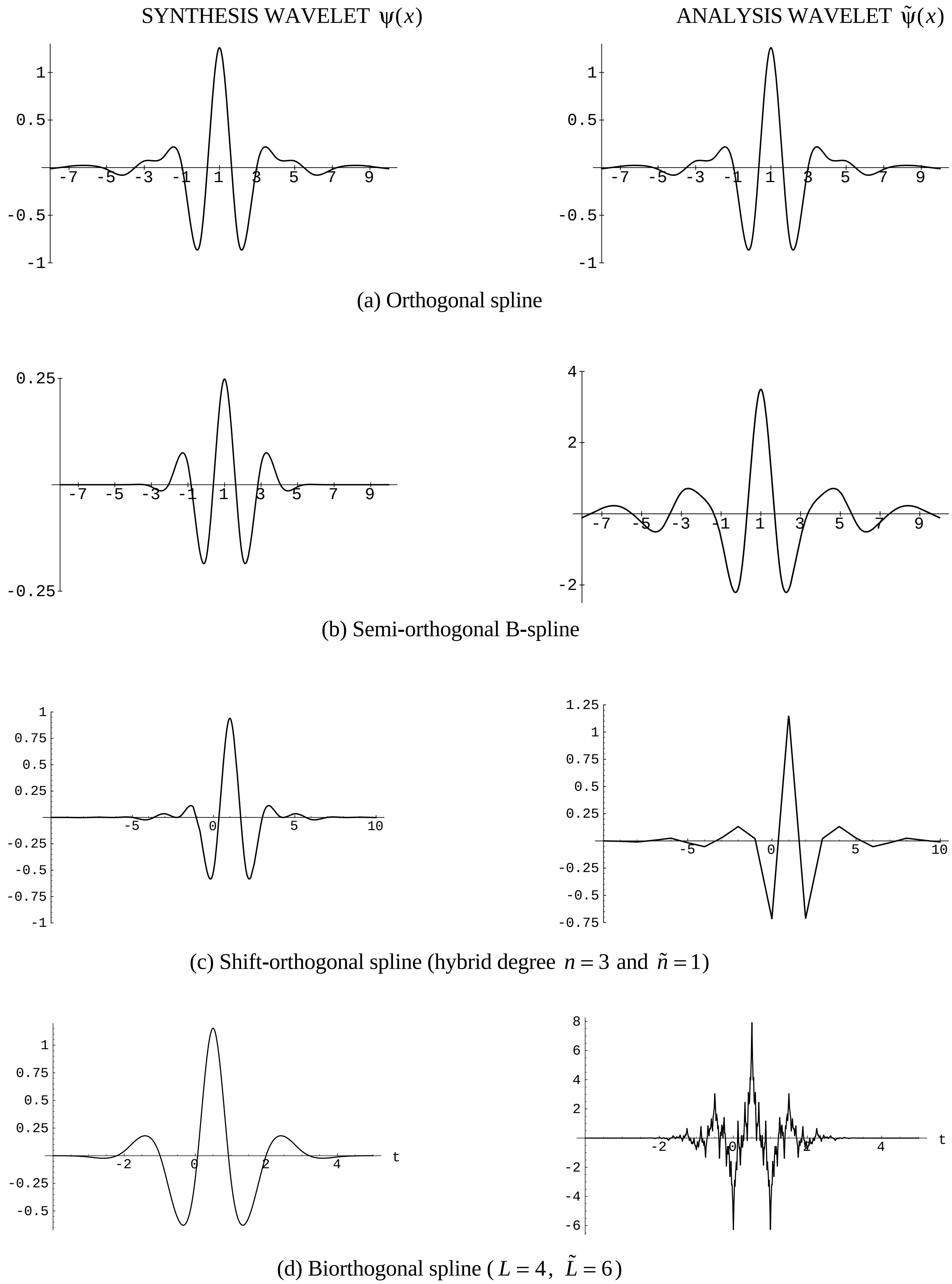

Fig. 1: Examples of four different types of cubic splines wavelets and their corresponding duals.

• *Biorthogonal spline wavelets*: Biorthogonal wavelet basis were introduced by Cohen-Daubechies-Feauveau [7] in order to obtain wavelet pairs that are symmetric, regular and compactly supported. Unfortunately, this is incompatible with the orthogonality requirement that has to be dropped altogether. Biorthogonal wavelets build with splines are especially attractive because of their short support and regularity. These wavelets turn out to be quite popular for coding applications [36]. In particular, the symmetry and short support properties are very valuable for reducing truncation artifacts in the reconstructed images. It is obviously also possible to construct other non-compactly supported spline wavelets since this type of construction is the least constrained of all.

• *Shift-orthogonal spline wavelets*: This category explores a last possibility which is to retain the intra-scale orthogonality requirement alone [35]. This idea was first investigated with the construction of spline wavelets of hybrid degree allowing for a very direct visualization of the two underlying multiresolution; for example, piecewise linear for the analysis and cubic for the synthesis [34, 35] (cf. Fig. 1c). Their main advantage is that it is possible to reduce the decay of the wavelet while essentially retaining the orthogonality and approximation properties of the Battle-Lemarié spline wavelets. These features can potentially be of interest for subband coding. The orthogonality property is required for the quantization error in the wavelet domain to be an exact indicator of the reconstructed image's final distortion. Orthogonality with respect to shifts, in particular, is consistent with the idea of independent channel processing (scalar quantization). Likewise, having shorter wavelet synthesis filters is advantageous for reducing reconstruction artifacts (e.g., spreading of coding errors, ringing around sharp transitions). These possibilities remain to be tested experimentally.

3. SPLINE AND WAVELET PROPERTIES

We have seen that all spline wavelets are linear combination of B-splines. Thus, they will inherit most of the properties of these basis functions. The list below start with the most basic properties that are common to all splines and then proceeds with others that are more specific. Because of the initial title of this paper, we have limited ourselves to ten such properties. We gladly invite the reader to find more.

3.1 Closed form solution

The B-splines, which have been defined as the $(n+1)$-fold convolution of a unit box function, have simple explicit forms in both the time and frequency domain. To derive such formulae, we start by expressing the convolution property (2) by a product in the Fourier domain, which yields

$$\hat{\varphi}^n(\omega) = \left(\hat{\varphi}^0(\omega)\right)^{n+1} = \left(\frac{1-e^{-j\omega}}{j\omega}\right)^{n+1}, \tag{16}$$

where $\hat{\varphi}^n(\omega)$ denotes the Fourier transform of $\varphi^n(x)$. This may also be rewritten as

$$\hat{\varphi}^n(\omega) = e^{-j\omega\frac{(n+1)}{2}} \operatorname{sinc}^{n+1}(\omega/2\pi), \tag{17}$$

which involves the $(n+1)$th power of the sinc function. Next, we expand (16) using the binomial expansion

$$\hat{\varphi}^n(\omega) = \sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^k \frac{e^{-j\omega k}}{(j\omega)^{n+1}}. \tag{18}$$

The crucial step is then to identify $(j\omega)^{-(n+1)}$ as the Fourier transform of the $(n+1)$-fold integral of the Dirac delta; i.e., the function $(x)_+^n/n!$ where $(x)_+^n = \max(0,x)^n$. By interpreting the complex exponentials as shift factors, we can get back to the time domain

$$\varphi^n(x) = \frac{1}{n!}\sum_{k=0}^{n+1}(-1)^k \binom{n+1}{k} (x-k)_+^n. \tag{19}$$

This formula shows that $\varphi^n(x)$ is piecewise polynomial of degree n. It is also clear from (18) that $\varphi^n(x)$ can be differentiated n times before one starts uncovering discontinuities at the integer knots. This is because the $(n+1)$th derivative of $\varphi^n(x)$

(multiplication by $(j\omega)^{n+1}$ in the frequency domain) is a sequence of Dirac impulses with the alternating binomial weights $(-1)^k \binom{n+1}{k}$.

Another interesting observation is that (19) can also be interpreted as the (n+1)th forward difference of the one-sided power function $(x)_+^n$. In other words, we have

$$\varphi^n(x) = \Delta_{n+1}(x)_+^n, \tag{20}$$

where Δ_{n+1} denotes the (n+1) iteration of the forward difference operator $\Delta f(x) = f(x) - f(x-1)$. The important point here is that these formulae together with (12) provide an explicit characterization of all spline wavelets. By contrast, most other scaling functions are only defined through an infinite product in the frequency domain [9, 23, 36]

$$\hat{\varphi}(\omega) = \prod_{i=1}^{+\infty} \left(\frac{1}{\sqrt{2}} H(e^{j\omega/2^i}) \right). \tag{21}$$

Applying this latter result to the B-spline case yields the identity

$$\left(\frac{1-e^{-j\omega}}{j\omega} \right)^{n+1} = \prod_{i=1}^{+\infty} \left(\frac{1+e^{-j\omega/2^i}}{2} \right)^{n+1}. \tag{22}$$

3.2 Simple manipulation

Splines are piecewise polynomial, which greatly simplifies their manipulation [10, 16]. In particular, it is straightforward to obtain spline derivatives and integrals. For instance, the first derivative of a causal B-spline of degree n is given by

$$\frac{d\varphi^n}{dx}(x) = \varphi^{n-1}(x) - \varphi^{n-1}(x-1). \tag{23}$$

This result may be derived by multiplying (18) by $j\omega$ (differentiation in the Fourier domain). Thus, differentiation corresponds to a reduction of the spline degree by one. Similarly, integration will result in a corresponding increase in the degree. This type of relation may be useful if one uses spline wavelets for solving differential equations.

3.3 Symmetry

The B-splines are symmetric. It is therefore easy to construct spline wavelets that are either symmetric or anti-symmetric by selecting the sequence with the appropriate symmetry in (12). The advantage is that the corresponding wavelet transform can be implemented using mirror boundary conditions which reduces boundary artifacts [27]. This is usually not possible for non-linear phase wavelets such as the celebrated Daubechies wavelets [8].

3.4 Shortest scaling function of order *L*

We recall that the refinement filter for an Lth order wavelet transform can be factorized as [9]

$$H(z) = (1+z^{-1})^L \cdot Q(z), \tag{24}$$

where $Q(z)$ is the transfer function of a stable filter. The B-spline of degree n=L-1 corresponds to the shortest Lth order refinement filter with $Q(z)$=1. One can therefore conclude that the B-spline of degree n=L-1 is the shortest possible scaling function of order L.

3.5 Maximum regularity for a given order *L*

In general, the regularity (or Sobolev smoothness) of a scaling function cannot be greater than the order L. We also know that the Sobolev smoothness [23] of a B-spline of degree n=L-1 is $s_{\max} = L - \frac{1}{2}$. However, the B-splines are are not the smoothest scaling functions of order L — they are only optimal if we take into account the filter length. For instance, we can consider the refinement filter (24) with $Q(z) = (1 + \varepsilon + z^{-1})$, which, for ε>0 sufficiently small (but non-zero), can achieve the maximum smoothness

$s_{\max} = L$. But this also shows that the B-splines are the smoothest scaling for a refinement filter of a given length: the example above with ε=0 is a B-spline of order L+1 which has a smoothness $s_{\max} = L + \frac{1}{2}$ — this is 1/2 better than anything else of the same length !

3.6 *m*-scale relation

The B-splines satisfy a two-scale relation for any integer m. Unlike most other scaling functions, m is not restricted to a power of two. We will derive this property by first considering a B-spline of degree 0 expanded by a factor of m, which can obviously be represented as

$$\varphi^0(x/m) = \sum_{k=0}^{m-1} \varphi^0(x-k). \qquad (25)$$

We rewritte this equation in the Fourier domain as

$$\hat{\varphi}^0(m\omega) = H_m^0(z) \cdot \hat{\varphi}^0(\omega)$$

where $H_m^0(z) = \sum_{k=0}^{m-1} z^{-k} / m$. If we now apply the convolution properties of B-splines, we get

$$\hat{\varphi}^n(m\omega) = \left(H_m^0(z)\right)^{n+1} \cdot \hat{\varphi}^n(\omega),$$

or, equivalently,

$$\varphi^m(x/m) = m \sum_{k \in Z} h_m^n(k) \varphi^n(x-k), \qquad (26)$$

where

$$H_m^n(z) = \frac{1}{m^n} \left(\sum_{k=0}^{m-1} z^{-k} \right)^{n+1}. \qquad (27)$$

Thus, (26) provides us with a two-scale relation that is valid for any integer m. In addition, the refinement filter $h_m^n(k)$ can be interpreted as a cascade of (n+1) moving sum filters. Each of these filters can be implemented very efficiently using a recursive updating strategy that requires no more than two additions per sample points. This property can be advantageous for implementation purposes. For instance, it is the basis for an efficient filtering procedure for zooming up signal or images by an integer factor m [28]. Another application is the fast implementation of the continuous wavelet transform with integer scales [33].

3.7 Variational properties

Splines provide a "natural" signal interpolant that is optimal in the sense that it has the least oscillating energy. This property is a consequence of the *first integral relation* [2], which states that for any function $f(x)$ whose mth derivative is square integrable, we have

$$\int_{-\infty}^{+\infty} \left(f^{(m)}\right)^2 dx = \int_{-\infty}^{+\infty} \left(s^{(m)}\right)^2 dx + \int_{-\infty}^{+\infty} \left(f^{(m)} - s^{(m)}\right)^2 dx \qquad (28)$$

where $s(x)$ is the spline interpolant of degree n=2m-1 such that $s(k) = f(k)$. In particular, if we apply this decomposition to the problem of the interpolation of a given data sequence *f(k)*, we see that the spline interpolant $s(x)$ minimizes the norm of the mth derivative among all possible interpolants $f(x)$, which is a rather remarkable result [19]. In this sense, the spline is the interpolating function that oscillates the least. For $m=2$, the energy function in (28) is a good approximation to the integral of the curvature for a curve $y=f(x)$. Thus, cubic spline interpolants exhibit a *minimum curvature property*, which justifies the analogy with the draftman's spline, or French curve. This latter device is a thin elastic beam that is constrained to pass through a given set of points.

3.8 Best approximation properties

For an Lth order wavelet, the approximation error decreases with the Lth power of the scale $a = 2^i$ (cf. Strang[21, 22]). Specifically, we can derive the following asymptotic relation [26]

$$\lim_{a \to 0} \| f - P_a f \| = C_\varphi \cdot a^L \cdot \| f^{(L)} \|, \tag{29}$$

where $P_a f$ denotes the projection (orthogonal or oblique) of f onto the multiresolution space at scale a. This error formula becomes valid as soon as the sampling step a is sufficiently small with respect to the smoothness scale of $f(x)$. The constant C_φ is the same for all spline wavelet transforms of a given order L, and is given by [26]

$$C_\varphi = \sqrt{\frac{|B_{2L}|}{(2L)!}}, \tag{30}$$

where B_{2L} is Bernouilli's number of order $2L$. This turns out, by far, to be the smallest constant among all known wavelet transforms of the same order L (cf. Table I in the above mentioned reference). This is partly due to the fact that the B-splines basis functions are very regular. For comparison, the Daubechies constants are so much worse that one would need to sample the signal at more than twice the rate to reach the same asymptotic L_2-error. Thus, it appears that splines are the best for approximating smooth functions. Another interesting consequence is that the asymptotic approximation error will be the same irrespective of the orthogonality properties of the transform.

3.9 Optimal time-frequency localization

The direct wavelet counterparts of the B-splines are the B-spline wavelets which are compactly supported as well. We have shown in our earlier work [29] that these wavelet converge to a cosine-modulated Gaussian (or Gabor) function as the degree of the spline goes to infinity. Specifically, if $\psi^n(x)$ denotes the B-spline wavelet of degree n, then we have the following approximate formula (cosine-modulated Gaussian)

$$\psi^n(x) \cong \frac{4b^{n+1}}{\sqrt{2\pi(n+1)}\sigma_w} \cos\left(2\pi f_0 (2x-1)\right) \exp\left(-\frac{(2x-1)^2}{2\sigma_w^2 (n+1)} \right), \tag{31}$$

with b=0.697066, f_0=0.409177 and σ_w^2=0.561145. The quality of this Gabor approximation improves rapidly with increasing n; for n=3, the approximation error is less than 3%. The implication is that there are spline wavelet bases that can be optimally localized in time and frequency. In other words, we can get as close as we wish to the time-frequency localization limit specified by the uncertainty principle. For the cubic spline example in Fig. 1b, the product of the time and frequency uncertainties is already within 2% of the limit specified by Heisenberg's uncertainty principle.

3.10 Convergence to the ideal lowpass filter

Splines provide a convenient framework that allows for a progressive transition between the two extreme cases of a multiresolution: the piecewise constant model (with n=0) and the bandlimited model corresponding to a spline of degree infinite [4, 15, 20]. In particular, it can be shown [30] that the Battle-Lemarié scaling functions converge to sinc(x) — the impulse response of the ideal lowpass filters — as the order of the spline goes to infinity. Likewise, their corresponding orthogonal wavelets converge to the ideal bandpass filter [3] (modulated sinc). In practice, their approximation of an ideal filter starts to be good for $n \geq 3$ (relative error < 5 %). This type of convergence properties may be relevant for coding applications. For instance, it has been shown that the ideal bandpass decomposition is optimal in the sense that it maximizes the coding gain for all stationary processes with non-increasing spectral power density [25].

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