Wavelets, Filterbanks, and the Karhunen-Loève transform

Michael Unser

Biomedical Imaging Group, Swiss Federal Institute of Technology,
CH-1015, Lausanne, Switzerland.
Email : Michael.Unser@epfl.ch

SUMMARY

Most orthogonal signal decompositions, including block transforms, wavelet transforms, wavelet packets, and perfect reconstruction filterbanks in general, can be represented by a paraunitary system matrix. Here, we consider the general problem of finding the optimal P x P paraunitary transform that minimizes the approximation error when a signal is reconstructed from a reduced number of components $Q<P$. This constitutes a direct extension of the Karhunen-Loeve transform which provides the optimal solution for block transforms (unitary system matrix). We discuss some of the general properties of this type of solution. We review different approaches for finding optimal and sub-optimal decompositions for stationary processes. In particular, we show that the solution can be determined analytically in the unconstrained case. If one includes order or length constraints, then the optimization problem turns out to be much more difficult.

1. INTRODUCTION

Transform domain processing is a powerful concept that is used in many signal processing algorithms. There are three major application areas where transforms are commonly used in image processing [13]. The first and most obvious one in data compression which capitalizes on the energy compaction properties of some classes of linear transforms; in particular, the DCT [17]. The second is data processing; for example, generalized filtering for noise reduction. The main property that is exploited there is that the underlying signal tends to get concentrated into few coefficients while the noise is spread out more evenly. Hence, working in the transform domain has the advantage of increasing the signal-to-noise ratio. The third application is data analysis; for example, feature extraction.

Here we are interested in transform methods that are local so that the processing can be adaptive [27]. The simplest approach is to subdivide the signal into adjacent non-overlapping blocks of size P, and to apply a unitary transform to each block. A possible refinement is to allow for overlapping basis functions which can be achieved using lapped orthogonal transforms [10]. These approaches can also be viewed as subband decompositions. Another very natural way to achieve locality is to use a wavelet transform which provides a hierarchical multiscale decomposition of a signal [9, 15, 23]. The wavelet transform provides a constant Q subband decomposition where each channel is approximately one octave wide. By choosing to chain the wavelet filters differently, one can also generate a whole variety of wavelet packets corresponding to different tillings of the time-frequency plane [24].

What these approaches have in common is that they can all be viewed as special cases of a critically sampled perfect reconstruction filter bank (PRFB)[22, 19]. A schematic representation of such a system is shown in Fig. 1. If the synthesis filters are such that $\tilde{H}_i(z)=H_i(z^{-1})$, then the system is lossless and defines an orthogonal transform of the input signal. Such a transform is entirely characterized by a $P \times P$ *paraunitary* system matrix $\mathbf{H}(z)$. It is therefore of interest to characterize paraunitary transforms that are optimal in the sense that they minimize the approximation error when the signal is reconstructed from a reduced number of components $Q<P$. In the simplest case in which the filters are non-overlapping (standard block transform), the solution is the Karhunen-Loève transform (KLT). The purpose of this paper is to extend this analysis to the more general paraunitary case and review signal-adapted solutions for stationary processes.

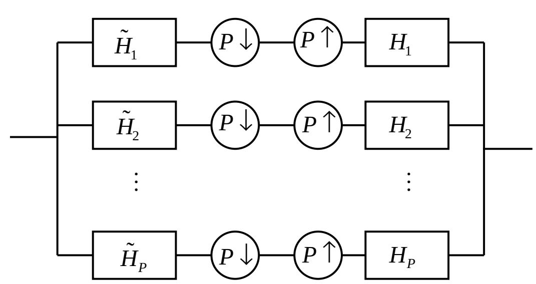

Fig. 1 : Block diagram of a P-band paraunitary PRFB.

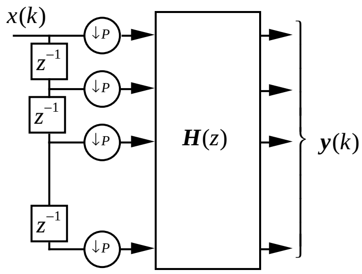

Fig. 2: Polyphase representation of a *P*-band wavelet analysis filterbank.

2. POLYPHASE REPRESENTATION

Most wavelet transforms and filterbank decomposition algorithms can be conveniently described as a multivariate filtering operation using the so-called *polyphase* representation [23]. The corresponding filterbank system is shown in Fig. 2. In this diagram, $x(k)$ represents the input signal and the y's are the various wavelet channels or subband components. In the standard dyadic case, there are only two channels (P=2), but the concept is also valid for larger values of P (P-band perfect reconstruction filterbank) [19, 22]. It turns out that the transformation is orthogonal if and only if the $P\times P$ transfer function matrix $\mathbf{H}(z)$ satisfies the paraunitary condition:

$$\mathbf{H}(z)\cdot\mathbf{H}^T(z^{-1})=\mathbf{I}_P, \tag{1}$$

where $\mathbf{I}_P$ is the $P\times P$ identity matrix. Note that for traditional block transforms, the matrix $\mathbf{H}(z)$ does not depend on z (i.e., the various blocks are processed independently of each other).

3. OPTIMAL FILTER BANKS

Let us now consider the specification of the optimal filterbank for the decomposition of a signal $x(k)$, $k\in Z$, which is a realization of a wide sense stationary process with covariance $c_{xx}(l)$, $l\in Z$. The covariance entries for the equivalent P-variate blocked representation $\mathbf{x}(k)=\left(x(kP),\cdots,x(kP-P+1)\right)$ are given by

$$\left[\mathbf{C}_{xx}(l)\right]_{i,j}=c_{xx}(Pl+i-j) \tag{2}$$

For a lossless (or paraunitary) system, we have

$$\mathrm{trace}[\mathbf{C}_{yy}(0)]=\sum_{i=1}^{P}\sigma_i^2=\mathrm{trace}[\mathbf{C}_{xx}(0)]=\mathrm{const} \tag{3}$$

where $\mathbf{C}_{xx}(l)$ and $\mathbf{C}_{yy}(l)$ are the covariance matrix sequences of the input and output vector signals $\boldsymbol{x}(k)$ and $\boldsymbol{y}(k)$, respectively, and where $\sigma_i^2:=[\mathbf{C}_{yy}(0)]_{i,i}$ represents the variance of the ith component.

The relation between the input and output covariance matrices is most conveniently described in the z-transform domain:

$$\hat{\mathbf{C}}_y(z)=\mathbf{H}(z)\cdot\hat{\mathbf{C}}_x(z)\cdot\mathbf{H}^T(z^{-1}). \tag{4}$$

where $\hat{\mathbf{C}}_x(z)=\sum_{k\in Z}\mathbf{C}_{xx}(k)z^{-k}$ and $\hat{\mathbf{C}}_y(z)=\sum_{k\in Z}\mathbf{C}_{yy}(k)z^{-k}$ are the $P\times P$ matrix z-transforms of $\mathbf{C}_{xx}(l)$ and $\mathbf{C}_{yy}(l)$, respectively.

The crucial parameters for an analysis using a paraunitary P-band filterbank are the channel variances σ_i^2, i=1,…,P, which correspond to the diagonal terms of the output covariance matrix $\mathbf{C}_{yy}(0)$. Because the system is lossless (cf (3)), the approximation error that results from the transmission of $Q<P$ bands instead of P is simply

$$\varepsilon_Q^2=\sum_{i=Q+1}^{P}\sigma_i^2\ . \tag{5}$$

We can therefore define the optimal filterbank as the one that minimizes the error for any number of bands $Q<P$. This definition is essentially the same as that of the KLT, except that the present analysis system is more general than a simple unitary matrix. Such a transformation is also optimal under a variety of other criteria; for instance, any general performance index of the form

$$\xi_f=\sum_{i=1}^{P}f(\sigma_i^2)\ , \tag{6}$$

where $f(\cdot)$ is a continuous function that is monotonously increasing convex or decreasing concave [17]. A performance measure widely used for coding applications is the transform coding gain [6]

$$G_{TC}=10\left(\log_{10}\left(\frac{1}{P}\sum_{i=1}^{P}\sigma_i^2\right)-\frac{1}{P}\sum_{i=1}^{P}\log_{10}(\sigma_i^2)\right), \tag{7}$$

which is an indicator of the SNR improvement that is obtained by applying an optimal separate quantizer for each component, as compared to coding the initial signal values directly. Note that (7) is a particular case of (6) with $f(\cdot)=a_0-a_1\log(\cdot)$.

In our determination of the optimal filterbank, we will have to distinguish between two cases depending on whether or not we impose length or order constraints on our filters. It turns out that the unconstrained solution is tractable mathematically but leads to IIR solutions that are difficult to implement in practice (ideal filters). For practical applications, it is usually more appropriate to constrain the solution to be FIR (filterbank of degree N). Unfortunately, there is no corresponding analytical solution and the optimization has to be performed numerically. Before considering these solutions, we will briefly identify some of their properties.

Property 1 : *An optimal P-band filterbank (constrained or unconstrained) will generally outperform the KLT with a block size P.*

This property follows directly from the fact that the simplest P-band filterbank (degree zero) is a unitary transform with block size P : $\boldsymbol{H}_0(z)=\boldsymbol{U}_0$, which includes the KLT as a special case. In general, there are more degrees of freedom than in the case of an orthogonal transform, and the different components of the system can be optimized for improved performance.

Property 2 : *An optimal P-band filterbank (constrained or unconstrained) will result in uncorrelated components.*

Proof : Let $\mathbf{C}_{yy}(0)$ be the PxP covariance matrix (at lag l=0) of the optimally filtered vector **y**. If the non-diagonal components of this covariance matrix are non-zero, we can apply an additional orthogonal transformation to diagonalize this matrix, which will result in some performance improvement (without increasing the order of the system). This is in contradiction with the initial assumption of optimality, which proves the desired result.

It is important to note that unlike the case of orthogonal P-block transforms (KLT), decorrelatedness is no longer a sufficient condition for optimality.

4. UNCONSTRAINED SOLUTIONS

For simplicity, we will only describe the solution for the two channel case in detail following the treatment given in [18]. For the more general P-band case, we refer to the independent work of Tsatsanis and Giannakis who arrived at the solution using an elegant principal component formulation in the frequency domain [16].

For P=2, the determination of the optimal filterbank is simplified because it is sufficient to consider one branch only. In other words, the optimal solution can be obtained by maximizing (resp. minimizing) the energy in the lowpass (resp. highpass) branch. For the system to be lossless, the corresponding lowpass filter in Fig. 1 must satisfy the standard power complementary condition [14]

$$\left|H_1(e^{j2\pi f})\right|^2+\left|H_1(-e^{j2\pi f})\right|^2=2. \tag{19}$$

The highpass filter is simply obtained by modulation : $H_2(z)=zH_1(-z^{-1})$. The optimal filterbank is then specified as follows:

Theorem 1 : *For a stationary process with (univariate) spectral power density (SPD) $C_{xx}(e^{j2\pi f})$, the optimal decomposition is obtained with the ideal filter*

$$\left|H_1(e^{j2\pi f})\right|=\begin{cases}\sqrt{2}, & \text{if } \ f\in\Omega_1\\ 0, & \text{otherwise.}\end{cases}$$

with bandpass region

$$\Omega_1=\left\{f\in\left[0,\tfrac{1}{4}\right):\ C_{xx}(e^{j2\pi f})\geq C_{xx}(e^{j2\pi f+\pi})\right\}\ \cup \left\{f\in\left(\tfrac{1}{4},\tfrac{1}{2}\right]:\ C_{xx}(e^{j2\pi f})>C_{xx}(e^{j2\pi f+\pi})\right\}$$

Thus the optimal solution corresponds to an ideal half-band decomposition whenever the SPD is such that $C_{xx}(e^{j2\pi f})\geq C_{xx}(e^{j(2\pi f+\pi)}),\ \forall f\in\left[0,\tfrac{1}{4}\right]$. In particular, this condition is satisfied when $C_{xx}(e^{j2\pi f})$ is a non-increasing function of f. The half-band decomposition is therefore optimal for the whole class of Markov-1 processes.

Interestingly, there are a number of wavelet transform constructions that converge asymptotically to this limit. The better known example is the family of Battle-Lemarié spline wavelets which converge to an ideal bandpass filter as the order of the spline goes to infinity [8, 1]. Daubechies wavelets also exhibit similar convergence properties [3]. This partially explains why higher order wavelets usually result in smaller approximation errors.

Since most signals encountered in practice tend to be predominantly lowpass, these results provide a good justification for the standard QMF design techniques which aim at obtaining a filter H_1 with good lowpass characteristics [21, 25, 20, 5].

For P>2, the solution is similar and depends on the spectral characteristics of the input signal [16]; it has also the form of an ideal filter with pure "on" and "off" frequency bands. If the power spectral density is monotone, then the optimal solution is the ideal filterbank with P uniformly-spaced subbands.

These theoretical results would suggest that the use of an adaptive design may not offer any substantial advantages, at least for the class of stationary processes satisfying the condition identified above. In practice, this is only partially true because the total complexity of the system is usually limited. For filters with a fixed number of taps, the use of a signal-dependent approach will in general provide better performance because it has the capacity to adapt its limited resources to the spectral characteristics of the input signal. Another potential disadvantage of using ideal filters (or some close approximations) is that they may induce ringing artifacts. They also require more computations.

Perhaps, the most interesting application of this theorem is to provide an asymptotic performance limit that is easy to calculate and that can be used to access the performance of suboptimal (or constrained) decompositions of the type considered next.

A final remarkable property of ideal filterbank solutions is that the output components are uncorrelated over all time lags.

This is much stronger than the component-wise decorrelation of the KLT which only holds at a given instant k.

5. CONSTRAINED DESIGN

Despite their optimality in terms of decorrelation and energy compaction, the ideal filter solutions that have just been described are not particularly useful for implementation purposes (slowly decaying impulse responses, Gibbs oscillations). This provides a good motivation for investigating more constrained solutions. Unfortunately, there appears to be no general closed form solution as soon as one forces the filters to be FIR (order constraint). This type of constrained design gives rise to a rather difficult numerical optimization problem. This is still a very active area of research; it started with the work of Delsarte et al. [4], and Caglar et al. [2]. Recently, there seems to have been some progress in designing optimum FIR compaction filters thanks in part to the use of more sophisticated optimization techniques. Moulin et al. [11, 12] have obtained optimized filters from the spectral factors of the solution of a semi-infinite programming problem (SIP). Xuan and Bamberger [26] have investigated the design of 2D principal component filters and proposed an optimization technique that uses sequential quadratic programming (SQL). Kirac and Vaidyanathan [7] were able to give an analytical FIR solution for a restricted class of random processes in the two channel case. They also proposed suboptimal design techniques for the P-band case. While there are now several design methods available, the problem is not closed yet. There is still room for finding a simple and universal design method that is more directly applicable in practice. There are also many underlying issues that are not fully resolved.

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