ON THE APPROXIMATION POWER OF SPLINES:
ORTHOGRAPHY VERSUS HEXAGONAL LATTICES

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Extended abstract

Recently, we have proposed a novel family of bivariate, non-separable splines [1]. These splines, called “hex-splines” have been designed to deal with hexagonally sampled data. Incorporating the shape of the Voronoi cell of a hexagonal lattice, they preserve the twelve-fold symmetry of the hexagon tiling cell. Similar to B-splines, we can use them to provide a link between the discrete and the continuous domain, which is required for many fundamental operations such as interpolation and resampling [2]. The question we answer in this paper is “How well do the hex-splines approximate a given function in the continuous domain?” and more specifically “How do they compare to separable B-splines deployed on a lattice with the same sampling density?”

A general signal space, spanned by shifted versions of a function \( \varphi(x) \) (such as a spline) on a lattice described by a matrix \( R = [r_1, r_2] \), contains all signals

\[
s(x) = \sum_k c(k) \varphi(x - Rk); \quad c(k) \in l_2(\mathbb{Z}^2). \tag{1}
\]

In general, the coefficients \( c(k) \) are determined as

\[
c(k) = \int g(x) \tilde{\varphi}(x - Rk) dx,
\]

where \( g \) is the original function and \( \tilde{\varphi} \) is the prefilter. The optimal choice, i.e., corresponding to an orthogonal projection into the function space, is the dual filter \( \tilde{\varphi}_d = \varphi/\hat{\varphi} \).

Here \( \hat{\varphi} \) is the Fourier transform of the sampled autocorrelation function of \( \varphi \). Another common choice is the interpolation prefilter, which selects \( c(k) \) such that \( s(Rk) = g(Rk) \).

Separable B-splines are a perfect fit to be used as basis functions on conventional rectangular lattices. For hexagonal lattices, one can use a “slanted” version of the B-splines. Their support correspond to a rhomboid. Recently, we proposed the use of hex-splines, which are inspired on the Voronoi cell indicator function and exhibit a hexagonal support. Higher order hex-splines are constructed by successive two-dimensional convolutions. First, we want to compare hex-splines versus slanted B-splines on the same hexagonal lattice. Second, we also compare hex-splines on a hexagonal lattice against separable B-splines on a square lattice with the same sampling density. These comparisons are done from an approximation theory point of view.

Approximation theory provides us with a convenient way to quantify the approximation error by integration with an error kernel \( E(\omega) \) in the Fourier domain: [3]

\[
||s(x) - g(x)||^2 = \frac{1}{4\pi^2} \int |\tilde{g}(\omega)|^2 E(\omega) d\omega. \tag{3}
\]

This error kernel is composed out of two parts:

\[
E(\omega) = E_{\text{min}} + E_{\text{res}} = 1 - \left| \frac{\phi(\omega)}{\hat{\varphi}(\omega)} \right|^2 + E_{\text{res}}.
\]

Most important, in the case of using the optimal prefilter (orthogonal projection), this kernel reduces to \( E_{\text{min}} \).

The asymptotic behavior tells us how well the approximation converges to the original \( g \) when the sampling lattice is made finer by a scaling factor \( h \). In this case, the argument of the error kernel under the integral of Eq. (3) is scaled accordingly as \( \omega \rightarrow \omega h \). So by analyzing \( E(\omega) \) around \( 0 \) we obtain

\[
||s(x) - g(x)||^2 \propto E(h\omega) \propto h^{2L} O(||\omega||^{2L}), \tag{4}
\]

where \( L \) is the order of approximation. The constants in front of \( \omega \) allow us to compare the behavior of \( E(\omega) \) between different signal spaces when the order of approximation is the same.

After a brief introduction on hex-splines and approximation theory, this paper concentrates on orthogonal projection, i.e., using the optimal prefilter \( \tilde{\varphi}_d \). As mentioned before, this is the best possible way to approach a function in the spline space. First, we compute the asymptotic behavior of
when the sampling grid gets denser. Second, we compare the asymptotic constants against the ones we obtain for slanted B-splines (with the same order of approximation $L$ as the hex-splines) on the same lattice and B-splines (again the same order) on a square lattice with the same sampling density. As result, we find that the asymptotic constants for the hex-splines are smaller than those for the B-splines in both cases. Therefore, hex-splines are asymptotically a better representation than splines on hexagonal lattices. Second, hex-splines on a hexagonal lattice allow to obtain a better approximation for a given function than B-splines on an orthogonal lattice with the same sampling density.

References

