From Sample Realizations to Random Processes

Life as a (somewhat frustrated) measure non-theorist

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Random processes
What is a random process?
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Family of random variables

\{ f(t) : t \in T \}

T: parameter space (or index set)
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\[ \{ f(t) : t \in T \} \]

\( T \): parameter space (or index set)

But how are they characterized?
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**Assumption:** All finite joint probability measures are given:

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for all finite $T_i \subset T$
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Hope: These will somehow identify a unique object, in some sense.
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\[(f(t))_{t \in T_i} \iff M_{T_i} \text{ on } (IR^{T_i}, G_{T_i})\]

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Also: I want to compute global probabilities for the entire process.
\( \pi_{T_1} : R^T \to R^{T_1} \) [Natural projection]

Cylinder set: \( \pi_{T_1}^{-1} A \)

where \( T_1 \) finite & \( A \in \mathcal{E}_T \)
\[ \pi_{T_1} : \mathbb{R}^T \rightarrow \mathbb{R}^{T_1} : f \mapsto (f(t_0), \ldots, f(t_{n-1})) \]

**Natural projection**

**Cylinder set**: \( \pi_{T_1}^{-1} A \)

where \( T_1 \) finite & \( A \in \mathcal{B}_{T_1} \)

**Example**: \( T_1 = \{0, 1, 2\} \)

\[ A = [-1, 1] \times [0, 1] \times [-\frac{1}{2}, \frac{1}{2}] \]

\( \pi_{T_1}^{-1} A \) = set of all functions that satisfy

\[ f(0) \in [-1, 1], \quad f(1) \in [0, 1], \quad f(2) \in [-\frac{1}{2}, \frac{1}{2}] \]
Precise formulation of the problem:

We have a finitely additive set function on the algebra $A = \bigcup_{T \in \mathcal{T}} \pi_i^{-1}(G_T)$ defined as

$$m(A) = m_{\pi_i}^{-1}(\pi_i^{-1}(G_T)) \text{ for } A \in \pi_i^{-1}(G_T).$$
Precise formulation of the problem:

We have a finitely additive set function on the algebra \( A = \bigcup_{T_i \in T} \pi_{T_i}^{-1}(G_{T_i}) \) defined as

\[ \mu(A) = \mu_{T_i}(\pi_{T_i}^{-1}(A)) \text{ for } A \in \pi_{T_i}^{-1}(G_{T_i}). \]

Q:\ Can \( \mu \) be extended to a \( \sigma \)-additive set function (= measure) on some \( G = \sigma(A) \)?
Precise formulation of the problem:

\[ Q: \text{Can } \mu \text{ be extended to a } \sigma\text{-additive set function (}= \text{measure}) \text{ on some } G = \sigma(A)? \]

\[ \sigma\text{-algebra of cylinder sets} \]

Kolmogorov's Extension Theorem (Kolmogorov, 1933):

for \( G = \sigma(A) \):

Yes (if the finite measures are consistent).

\[ \text{small print} \]

It's must possess an approximately compact class, which all Borel measures on \( \mathbb{R}^+ \), finite, as possible.
Consistency: Given $T_2 \subset T_1 \subset T$ with $\#T_1 < \alpha$, 

$$M_{T_2}(A) = M_{T_1}(T_{T_1 \rightarrow T_2}^{-1} A) \quad \forall A \in G_{T_2}$$

where $T_{T_1 \rightarrow T_2} : R_{T_1} \rightarrow R_{T_2}$ is the natural projection $R_{T_1} \rightarrow R_{T_2}$. 
Why I think unsatisfactory?
Why is this unsatisfactory?

Sets in $\mathcal{S}(\mathcal{A})$ are countably determined (σ-combinations of finitely determined cylinder sets)...

Many interesting sets are left out (cannot compute their probability):

e.g. $\{f : |f| < a\}$, $C = \{f \text{ continuous on } \mathbb{R}^3\}$
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( remove the beard )
Measures on linear spaces
Random processes on linear spaces

$E$ is linear space;

$f$ is a random element of $E^*$, the algebraic dual of $E$

($E$ is a random linear functional on $E$);

$\mu$ is a measure on $\sigma$-algebra of Borel cylinder sets in $E^*$ associated with $f$.

notation: $f(e^i) = \langle e^i, f \rangle \rightarrow \mathbb{R}$

How to characterize $\mu$?
RECALL:

Given any linear independent vector \( \mathbf{e}_i = (e_1, \ldots, e_n) \in \mathbb{E}^n \)

Define

Projection \( \pi_i : \mathbb{E}^n \rightarrow \mathbb{R}^n : f \mapsto (f(e_1), \ldots, f(e_n)) \)

For \( A \in \text{Borel}(\mathbb{R}^n) \),

\( \pi_i A \) is a Borel cylinder set.
Consistency for linear processes: Given

\[ E_i = (e_i, \ldots, e_n) \in \mathbb{E}^n \quad \text{and} \quad \mathbf{E} \in \mathbb{R}^{n \times m} \]

\[ \mathbf{E}_i = (\tilde{e}_i, \ldots, \tilde{e}_m) = \mathbf{M} E_i, \]

\[ \exists \mathbf{M} \in \mathbb{R}^{n \times m} \]

Equivalently:

\[ f(\sum \alpha_i e_i) = \sum \alpha_i f(e_i) \]

in distribution
Zorn's lemma

\( \iff E \) has a

(= Axiom of Choice) \implies \ (Hamel) basis \( T \)

\[ f \text{ is uniquely determined by } f(e), e \in T \]

Problem is reduced to Kolmogorov's thm.
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enter Generalized Random Processes

of Gel'fand & Co.
Generalized Random Process:

Family of R.V.S \( \{f(e) \mid e \in E\} \)

where \( E \) is a topological vector space.

- Consistency defined as before;
- New notion: Continuity:

\[
\begin{align*}
\varepsilon, k &\to e_1 \\
\vdots &
\end{align*}
\]

\[
\begin{align*}
e_{n,k} &\to e_n \\
\varepsilon &\in E
\end{align*}
\]

\[
\Rightarrow (f(e_{n,k}), \ldots, f(e_{n,k})) \rightarrow (f(e_1), \ldots, f(e_n))
\]

in distribution

\[
M_{E_{1,k}} \rightarrow M_{E_n}
\]
Naturally, if \( \mu \) is a measure on \( E' \) (continuous dual), consistency and continuity are satisfied.

\[ Q: \text{ When are 1,2 sufficient for extending a cylinder measure to a } \text{ additive one on } E'? \]

(RECALL: consistency alone was sufficient to have a measure on \( E' \).)
Minko (1958) – proof of Sel’fand’s conjecture:

For finite-dim. measures w/ \( 1, 2 \), to uniquely extend to a measure on \( E' \), it is sufficient that \( E \) be nuclear.
Examples of nuclear spaces:

$S$: Schwartz space

$D$: space of compactly supp. test functions

$\rightarrow$ random processes in $S'$

$\rightarrow$ random distributions
characteristic functionals:

one way to define/construct finite dim. measures fulfilling $\{1,2\}$.

NOT TODAY...
QUESTIONS?
Note on notation:

\[ A^B = \text{set of all functions } B \to A \]

Examples:

\[ \mathbb{R}^\mathbb{Z} = \text{set of all real sequences} \ldots \ldots \]

\[ \mathbb{R}^{\mathbb{R}} = \text{set of all functions } \mathbb{R} \to \mathbb{R} \ldots \ldots \]

\[ \mathbb{R}^{[0,1]} = \text{set of all functions } [0,1] \to \mathbb{R} \]

\[ 0 \quad 1 \]
Probability measure:
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\[ G : \text{set system} \subseteq \mathcal{P}(X) \]

closed under \( \cup \) \( \cap \) \( \setminus \) \( \Rightarrow \) algebra

\( \sigma \)-algebra
Probability measure:

\( G : \) set system \( \subset \mathcal{P}(\mathbb{X}) \)

closed under \( U \cap \setminus \) \( \rightarrow \) algebra

\( \sigma \)-algebra

\( m : G \rightarrow [0,1] \)

\( \text{increasing, } m(\emptyset) = 0 \)

pairwise disjoint \( A \rightarrow m(\bigcup A) = \sum_{a \in A} m(a) \)

\( A \) finite \( \rightarrow \) additivity

\( A \) countable \( \rightarrow \) \( \sigma \)-additivity