Fractional Brownian Vector Fields

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Scalar fractional Brownian motion (fBm)
- Invariances
- Fractional PDE formulation (innovation model)

Fractional Brownian vector fields
- Vector invariances
- Generalized fractional Laplacians
- Characterization of vector fBm
- Some properties
- Parameter estimation with wavelets
Scalar Fractional Brownian Motion
Non-stationary random field on $\mathbb{R}^d$ with

- Gaussian statistics;
- zero mean;
- zero boundary conditions ($B_H(0) = 0$);
- stationary increments with variance

$$\mathbb{E}\{|B_H(x) - B_H(y)|^2\} \propto |x - y|^{2H}$$

($H \in (0, 1)$: Hurst exponent).
Invariance properties

Statistical invariances:

• Scaling:

\[ S_{\sigma} B_H = \sigma^H B_H \quad \text{in law}, \]

\( (S_{\sigma} : f \mapsto f(\sigma^{-1} \cdot), \ \sigma \in \mathbb{R}_+) \);

• Scalar rotation (and reflection):

\[ R^{\text{scalar}}_{\Omega} B_H = B_H \quad \text{in law}, \]

\( (R^{\text{scalar}}_{\Omega} : f \mapsto f(\Omega^T \cdot), \ \Omega \text{ orthogonal}) \).
Whitening/innovation modelling

- Characterization/generalization by means of a \textit{whitening equation}:

\[ U^* B_H = W \]

where:

- \( W \) is \textit{white Gaussian noise};
- \( U^* \) is the whitening operator.

\Rightarrow \text{Non-stationary generalization of spectral shaping.}
Whitening/innovation modelling: Steps

1. Identify $U$ (using invariances);

2. Find a \textit{continuous linear left inverse} $L : S \to \mathcal{L}^2$:

   \[
   LU = \text{identity};
   \]

3. Define $B_H$ as a particular solution (generalized random field):

   \[
   \langle B_H, \phi \rangle := \langle W, L\phi \rangle \quad (\ast)
   \]

   \textbf{Justification:}

   \[
   (\ast) \implies \langle B_H, U\psi \rangle = \langle W, LU\psi \rangle = \langle W, \psi \rangle
   \]

   \[
   \implies U^* B_H = W.
   \]
1. The fractional Laplacian $U^\gamma \stackrel{\mathcal{F}}{\leftrightarrow} \kappa_\gamma |\omega|^{2\gamma}$ satisfies

\[
\begin{align*}
U^\gamma S_\sigma &= \sigma^{2\gamma} S_\sigma U^\gamma; \\
U^\gamma R^\text{scalar}_\Omega &= R^\text{scalar}_\Omega U^\gamma.
\end{align*}
\] (homogeneity) (rotation invariance)

2. Continuous linear left inverse ($S \rightarrow \mathcal{L}_2$):

\[
L^\gamma : f \mapsto \frac{1}{\kappa_\gamma (2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle x, \omega \rangle} \frac{1}{|\omega|^{2\gamma}} \left( \hat{f}(\omega) - \sum_{|k| \leq 2\gamma - \frac{d}{2}} \frac{\hat{f}(k)(0) \omega^k}{k!} \right) d\omega.
\]

Invariances: Like $U$, $L$ is homogeneous and rotation-invariant.
3. Innovation/whitening model:

- Captures the inverse power-law spectrum of $B_H$;
- Generalizes to $H > 1$;
- Non-Gaussian $W \Rightarrow$ non-Gaussian models à la Lévy motion (may need to redefine $L$).
Fractional Brownian Vector Fields
Fractional Brownian vector fields

How to define fractional Brownian vector fields?

- **Trivial definition**: Vector of independent scalar fBms.
  
  No constraints on the interdependency of the components;

  ⇒ Hence no control over directional behaviour.

- **Solution**: More general definition based on *invariances*. 
Vector invariances

- **Vector rotation**: Rotate the domain, but keep directions fixed.

Rotation by $\Omega \in O(n)$:

$$R_{\Omega}^{\text{vector}} : f \mapsto \Omega f(\Omega^T).$$

- Desired invariances for vector fBm:

  $$S_\sigma B_H = \sigma^H B_H \quad \text{in law,}$$

  $$R_{\Omega}^{\text{vector}} B_H = B_H \quad \text{in law}.$$
**Imposing invariances**

**Idea:** Whitening/innovation model as before:

\[ \mathbf{U}^* \mathbf{B}_H = \mathbf{W}, \]

\( \mathbf{W} \): vector of white noises; \( \mathbf{U} \) is:

- **Homogeneous:**
  \[ \mathbf{U} \begin{bmatrix} \sigma^{-2} \\ \sigma \end{bmatrix} = \sigma^2 \begin{bmatrix} \sigma \\ \sigma \end{bmatrix} \mathbf{U}; \]

- **Vector rotation invariant:**
  \[ \mathbf{U} \mathbf{R}^{\text{vector}}_{\Omega} = \mathbf{R}^{\text{vector}}_{\Omega} \mathbf{U}. \]
Theorem (Arigovindan & Unser ’05, PDT & Unser ’10): A vector convolution operator with the said invariances has a Fourier multiplier of the form

\[
U_\gamma^{(\xi_1, \xi_2)} \overset{\mathcal{F}}{\longleftrightarrow} \kappa_\gamma \Phi_\xi^{\gamma}(\omega) := \kappa_\gamma |\omega|^{2\gamma} \left[ e^{\xi_1} \frac{\omega\omega^T}{|\omega|^2} + e^{\xi_2} \left( I - \frac{\omega\omega^T}{|\omega|^2} \right) \right].
\]

Interpretation:

- \(|\omega|^{2\gamma} \): fractional Laplacian
- \(\frac{\omega\omega^T}{|\omega|^2} \): projection onto the curl-free component
- \(I - \frac{\omega\omega^T}{|\omega|^2} \): projection onto the div-free component
Fractional vector Laplacians (2)

Properties of $\Phi_{\xi}^\gamma$:

- Homogeneity: $S_\sigma \Phi_{\xi}^\gamma = \sigma^{2\gamma} \Phi_{\xi}^\gamma$;

- Rotation contra-variance: $R^\text{vector}_\Omega \Phi_{\xi}^\gamma = \Phi_{\xi}(\cdot) \Omega$;

- Inversion: $\Phi_{\xi}^\gamma(\omega) \Phi_{-\xi}^{-\gamma}(\omega) = 1$, $\omega \neq 0$;

- Fourier transform: $\mathcal{F}\{\Phi_{\xi}^\gamma\} = \Phi_{\hat{\xi}}^{-\gamma-d}$;

- Products: $\Phi_{\xi_1}^\gamma \Phi_{\xi_1}^\gamma = \Phi_{\xi_1+\xi_2}^{\gamma_1+\gamma_2}$.
Fractional vector Laplacians (3)

- Continuous linear left inverse defined same as before:

\[ L_\gamma^\xi : f \mapsto \frac{1}{\kappa_\gamma (2\pi)^d} \int_{\mathbb{R}^d} e^{\langle x,\omega \rangle} \Phi_{-\xi}^{-\gamma}(\omega) \left( \hat{f}(\omega) - \sum_{|k| \leq \lfloor 2\gamma - \frac{d}{2} \rfloor} \frac{\hat{f}^{(k)}(0) \omega^k}{k!} \right) d\omega. \]

Key properties:

- Homogeneous;
- Vector rotation invariant;
- Continuous \( S^d \to \mathcal{L}^d_2. \)
Self-similar and rotation invariant solution of

\[
(U^{\frac{H+d}{2}} + d^2) \ast B_{H, \xi} = \mathbf{W};
\]

(\(\mathbf{W}\) is vector of white noise).

- Coordinates are no longer independent (unless \(\xi_1 = \xi_2\)).

- \(\xi_1 - \xi_2\) controls vectorial behaviour:
  - \(\xi_1 - \xi_2 \to +\infty\): solenoidal (div-free);
  - \(\xi_1 - \xi_2 \to -\infty\): irrotational (curl-free).

- Interpreted as a generalized random field (Gel’fand \& al.).
Generalized random fields (1)

- $\langle B_{H,\xi}, \phi \rangle$, $\phi \in S^d$, are R.V.s with consistent finite-dimensional prob. measures.

- The stochastic law (prob. measure) of $B_{H,\xi}$ is derived from its characteristic functional:

**Theorem (Bochner-Minlos):** There is a one-to-one correspondence between positive-definite and continuous characteristic functionals $Z_B(\phi)$, $\phi \in \mathcal{E}$ (a nuclear space), and probability measures $P_B$ on $\mathcal{E}'$, via the relation

$$Z_B(\phi) = \mathbb{E}\{e^{i\langle B, \phi \rangle}\} = \int_{\mathcal{E}'} e^{i\langle \chi, \phi \rangle} P_B(d\chi).$$
Generalized random fields (2)

Example (white Gaussian noise):

\[ Z_W(\phi) = e^{-\frac{1}{2}||\phi||^2} \]

Properties:

- Independent values at every point (whiteness):
  \[ \langle W, \phi \rangle, \langle W, \psi \rangle \text{ independent if } \text{Supp } \phi \cap \text{Supp } \psi = \emptyset; \]

- Jointly Gaussian finite-dim. distributions for all
  \[ \langle W, \phi_i \rangle, \quad 1 \leq i \leq N. \]
Characterization of vector fBm

Reminder: Solution in the sense of distributions

\[ \langle B_{H, \xi}, \phi \rangle := \langle W, L_{\xi}^{H+d^4} \phi \rangle \implies \left( U_{\xi}^{H+d^4} \right)^* B_{H, \xi} = W. \]

Characteristic functional:

\[ Z_{B_{H, \xi}}(\phi) = \mathbb{E}\{ e^{i\langle B_{H, \xi}, \phi \rangle} \} = \mathbb{E}\{ e^{i\langle W, L\phi \rangle} \} = Z_W(L_{-\xi}^{\frac{H}{2}-\frac{d}{4}} \phi) \]

(requires continuity \( S^d \to \mathcal{L}_2^d \)).
Some properties of vector fBm (1)

Scale and rotation invariance of $L_{\xi}^{H+\frac{d}{4}}$\

- Self-similarity:
  \[ S_{\sigma} B_H = \sigma^H B_H \text{ in law}; \]

- Rotation invariance:
  \[ R_{\Omega}^{\text{vector}} B_H = B_H \text{ in law}. \]
Some properties of vector fBm (2)

- Generalization to $H > 1$

$$B_{H,\xi} = (L_{\xi}^{H+d/4})^* W$$

also valid for $H > 1$ (non-integer).

- Stationary $n$th-order increments for $n \geq \lfloor H \rfloor + 1$;

- Covariance structure of increments for $0 < H < 1$:

$$\mathbb{E}\{[B_{H,\xi}(x) - B_{H,\xi}(y)] [B_{H,\xi}(x) - B_{H,\xi}(y)]^T\} \propto \Phi_{(\eta_1,\eta_2)}^H(x - y)$$

- Vectorial behaviour:
  - $\xi_1 - \xi_2 \to +\infty \Rightarrow$ div-free;
  - $\xi_1 - \xi_2 \to -\infty \Rightarrow$ curl-free;
  - $\xi_1 = \xi_2 \Rightarrow$ independent coordinates.
Examples

(a) $H = 0.60, \xi_1 = \xi_2 = 0$
   (indep. coordinates)

(b) $H = 0.60, \xi_1 = 0, \xi_2 = 100$
   (curl-free)

(c) $H = 0.60, \xi_1 = 100, \xi_2 = 0$
   (div-free)
Wavelet analysis of vector fBm (1)

Vector Wavelets

Let $E \xrightarrow{\mathcal{F}} \omega \omega^T/|\omega|^2$ (curl-free projection).

Define vector wavelets (matrix-valued):

- Smoothing kernel $\Phi$ (matrix-valued, usu. diagonal);

- Wavelets:

$$\Psi = U^\gamma \Phi = U^\gamma [E + (\text{Id} - E)] \Phi$$

$$\Psi_1: \text{captures curl-free comp.} \quad \Psi_2: \text{captures div-free comp.}$$
Wavelet analysis of vector fBm (2)

**Parameter Estimation**

- \( \log(\text{wavelet energy}) \) varies linearly across scales; slope depends on \( H \).

  \( \Rightarrow \) Estimates of \( H \).

- Ratio between \( \Psi_1 \) and \( \Psi_2 \) energy depends on \( \xi_1 - \xi_2 \).

  \( \Rightarrow \) Estimates of vectorial character \((\xi_1 - \xi_2)\).
Thank you.