Dear Dr. Liebling,

I am pleased to inform you that you were selected to receive the 2004 Research Award of the Swiss Society of Biomedical Engineering for your thesis work "On Fresnelets, interference fringes, and digital holography". The award will be presented during the general assembly of the SSBE, September 3, Zurich, Switzerland.

Please let us know if
1) you will be present to receive the award,
2) you would be willing to give a 10 minutes presentation of the work during the general assembly.

The award comes with a cash prize of 1000.- CHF.

Would you please send your banking information to the treasurer of the SSBE, Uli Diermann (Email: uli.diermann@bfh.ch), so that he can transfer the cash prize to your account?

I congratulate you on your achievement.

With best regards,

Michael Unser, Professor
Chairman of the SSBE Award Committee

cc: Ralph Mueller, president of the SSBE; Uli Diermann, treasurer

Dr. Michael Liebling
Biological Imaging Center
California Inst. of Technology
Mail Code 139-74
Pasadena, CA 91125, USA

Invited seminar, Korea Advanced Inst. of Science & Technology (KAIST), July 17-18, Seoul, Korea

20th century statistical signal processing

Hypothesis: Signal = stationary Gaussian process

Karhunen-Loève transform (KLT) is optimal for compression

DCT asymptotically equivalent to KLT

(Ahmed-Rao, 1975; U., 1984)

(Pearl et al., IEEE Trans. Com 1972)
20th century statistical signal processing

Hypothesis: Signal = Gaussian process

\[ y = Hs + n \]

Noise: i.i.d. Gaussian with variance \( \sigma^2 \)

Signal covariance: \( C_s = \mathbb{E}\{s \cdot s^T\} \)

Wiener filter is **optimal** for restoration/denoising

\[
s_{\text{LMMSE}} = C_s H^T (H C_s H^T + \sigma^2 I)^{-1} y = F_{\text{Wiener}} y
\]

\( \uparrow \quad L = C_s^{-1/2} \): Whitening filter

Wiener (LMMSE) solution = Gauss MMSE = Gauss MAP

\[
s_{\text{MAP}} = \arg\min_s \frac{1}{\sigma^2} \| y - Hs \|^2_2 + \left( \| C_s^{-1/2} s \|^2_2 \right)
\]

**Data Log likelihood**

\[ \] **Gaussian prior likelihood**

\( \Leftrightarrow \) quadratic regularization (Tikhonov) \( \lambda \| Ls \|^2_2 \)

Then came wavelets ... and sparsity

![Image](image.png)

**Applications**

**Sparsity**

**Compressed sensing**

Alfred Haar

1910

1982

Ingrid Daubechies

1994

1987-88

Stéphane Mallat

Yves Meyer

Martin Vetterli

David Donoho

Emmanuel Candès

2006
### Fact 1: Wavelets can outperform Wiener filter

**Communications**

Filtering Noise from Images with Wavelet Transforms

J. B. Weaver,* Yansun Xu,* D. M. Healy, Jr.,† and L. D. Cromwell*

*Department of Radiology, Dartmouth-Hitchcock Medical Center, and †Department of Mathematics, Dartmouth College, Hanover, New Hampshire 03755

Received April 12, 1991

A new method of filtering MR images is presented that uses wavelet transforms instead of Fourier transforms. The new filtering method does not reduce the sharpness of edges. However, the new method does eliminate any small structures that are similar in size to the noise eliminated. There are many possible extensions of the filter. © 1991 Academic Press, Inc.

\[ \hat{w} = T(x, w) \]

### Fact 2: Wavelet coding can outperform jpeg

\[ f(x) = \sum_{i,k} \psi_{i,k}(x) w_{i,k} \]

(Shapiro, IEEE-IP 1993)

- Wavelet transform
- Inverse wavelet transform
- 66.4 dB
- 0.00%
- Discarding “small coefficients”
Fact 3: $l_1$ schemes can outperform $l_2$ optimization

$$s^* = \arg\min \underbrace{\|y - Hs\|_2^2}_{\text{data consistency}} + \underbrace{\lambda R(s)}_{\text{regularization}}$$

- Wavelet-domain regularization
  
  Wavelet expansion: $s = Wv$ (typically, sparse)
  
  Wavelet-domain sparsity-constraint: $R(s) = \|v\|_1$ with $v = W^{-1}s$
  
  \text{(Nowak et al., Daubechies et al. 2004)}

- $\ell_1$ regularization (Total variation) \text{(Rudin-Osher, 1992)}
  
  $R(s) = \|Ls\|_1$ with $L$: gradient

- Compressed sensing/sampling \text{(Candes-Romberg-Tao; Donoho, 2006)}

SPARSE STOCHASTIC MODELS: The step beyond Gaussianity

Requirements for a comprehensive statistical framework

- Backward compatibility

- Continuous-domain formulation
  
  piecewise-smooth signals, translation and scale-invariance, sampling . . .

- Predictive power
  
  Can wavelets really outperform sinusoidal transforms (KLT) ?

- Statistical justification and refinement of current algorithms
  
  Sparsity-promoting regularization, $\ell_1$ norm minimization
1.3 FROM SPLINES TO STOCHASTIC PROCESSES

- Splines and Legos revisited
- Higher-order polynomial splines
- Random splines, innovations, Lévy processes
- Wavelet analysis of Lévy processes
- Lévy’s synthesis of Brownian motion

Splines and Legos revisited

- Cardinal spline of degree 0: piecewise-constant

\[ f_1(t) = \sum_{k \in \mathbb{Z}} f_1[k] \beta^0_+(t - k) \]

\[ \beta^0_+(t) = \begin{cases} 
1, & \text{for } 0 \leq t < 1 \\
0, & \text{otherwise.} 
\end{cases} \]

Notion of D-spline:

\[ Df_1(t) = \sum_{k \in \mathbb{Z}} a_1[k] \delta(t - k) \]
### B-spline and derivative operator

**Derivative**
\[ Df(t) = \frac{df(t)}{dt} \]

**Finite difference operator**
\[ D_d f(t) = f(t) - f(t - 1) \]
\[ = (\beta_0^1 \ast D)f(t) \]

**B-spline of degree 0**
\[ \beta_0^1(t) = D_d D^{-1} \delta(t) = D_d \mathbb{1}_+(t) \]
\[ \downarrow \]
\[ \beta_0^1(\omega) = \frac{1 - e^{-j\omega}}{j\omega} \]

### Random splines and innovations

**cardinal**

**non-uniform**

\[ Ds(t) = \sum_n a_n \delta(t - t_n) = w(t) \]

Random weights \( \{a_n\} \) i.i.d. and random knots \( \{t_n\} \) (Poisson with rate \( \lambda \))

- **Anti-derivative operators**
  
  **Shift-invariant solution:**
  \[ D^{-1}_- \varphi(t) = (\mathbb{1}_+ \ast \varphi)(t) = \int_{-\infty}^{t} \varphi(\tau) d\tau \]
  
  **Scale-invariant solution:**
  \[ D_0^{-1} \varphi(t) = \int_{0}^{t} \varphi(\tau) d\tau \]
**Innovation-based synthesis**

\[ L = \frac{d}{dt} = D \Rightarrow L^{-1}: \text{integrator} \]

\[ \delta(t) \rightarrow L^{-1}\{\cdot\} \rightarrow \rho = L^{-1}\delta \]

Translation invariance

\[ \delta(t - t_0) \rightarrow L^{-1}\{\cdot\} \rightarrow \rho(t - t_0) \]

Linearity

\[ \sum_n a_n \delta(t - t_n) \rightarrow L^{-1}\{\cdot\} \rightarrow s(t) = \sum_n a_n \rho(t - t_n) \]

---

**Compound Poisson process**

- **Stochastic differential equation**

  \[ Ds(t) = w(t) \]

  with boundary condition \( s(0) = 0 \)

  Innovation: \( w(t) = \sum_n a_n \delta(t - t_n) \)

- **Formal solution**

  \[
  s(t) = D_0^{-1}w(t) = \sum_n a_n D_0^{-1}\{\delta(\cdot - t_n)\}(t) \\
  = \sum_n a_n (1_+ (t - t_n) - 1_+ (-t_n))
  \]

  (impose boundary condition)
Generalization: Lévy processes

Generalized innovations: white Lévy noise with \( \mathbb{E}\{w(t)w(t')\} = \sigma^2_w \delta(t-t') \)

\[
Ds = w \quad \text{(unstable SDE !)}
\]

\[
s = D_0^{-1}w \iff \forall \varphi \in \mathcal{S}, \quad \langle \varphi, s \rangle = \langle D_0^{-1} \varphi, w \rangle
\]

### White noise (innovation)

<table>
<thead>
<tr>
<th>Gaussian</th>
</tr>
</thead>
<tbody>
<tr>
<td>Integrator</td>
</tr>
<tr>
<td>Impulsive</td>
</tr>
<tr>
<td>( s(t) = \int_0^t d\tau )</td>
</tr>
</tbody>
</table>

### Lévy process

- Brownian motion
- Compound Poisson
- Lévy flight

(Paul Lévy circa 1930)

Decoupling Lévy processes: increments

Increment process:

\[
u(t) = D_d s(t) = D_d D_0^{-1} w(t) = (\beta_0^0 * w)(t).
\]

Increment process is stationary with autocorrelation function

\[
R_u(\tau) = \mathbb{E}\{u(t)u(t+\tau)\} = (\beta_0^0 * (\beta_0^0)^\vee * R_w)(y) = \sigma^2_w \beta_1^1(\tau - 1)
\]

Discrete increments

\[
u[k] = s(k) - s(k-1) = \langle w, \beta_0^0(\cdot - k) \rangle.
\]

\( u[k] \) are i.i.d. because

- \( \{\beta_0^0(\cdot - k)\} \) are non-overlapping
- \( w \) is independent at every point (white noise)
Wavelet analysis of Lévy processes

- **Haar wavelets**

\[
\psi_{\text{Haar}}(t) = \begin{cases} 
1, & \text{for } 0 \leq t < \frac{1}{2} \\
-1, & \text{for } \frac{1}{2} \leq t < 1 \\
0, & \text{otherwise.}
\end{cases}
\]

\[
\psi_{i,k}(t) = 2^{-i/2}\psi_{\text{Haar}}\left(\frac{t - 2^ik}{2^i}\right)
\]

- **Wavelets as multi-scale derivatives**

\[
\psi_{i,k} = 2^{i/2-1}D\phi_{i,k}
\]

\[
D_0^{-1}\psi_{i,k} = 2^{i/2-1}\phi_{i,k}.
\]

- **Wavelet coefficients of Lévy process**

\[
Y_{i,k} = \langle s, \psi_{i,k} \rangle \propto \langle s, D\phi_{i,k} \rangle
\]

\[
\propto \langle D^* s, \phi_{i,k} \rangle = -\langle w, \phi_{i,k} \rangle
\]
M-term approximation: wavelets vs. KLT

Wavelet-based synthesis of Brownian motion

- White Gaussian noise
  \[ w = \sum_{i \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} Z_{i,k} \psi_{i,k} \text{ with } Z_{i,k} = \langle w, \psi_{i,k} \rangle \]

- Brownian motion
  \[ s(t) = D^{-1}_0 w \]
  \[ = \sum_{i \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} Z_{i,k} D^{-1}_0 \psi_{i,k}(t) \]
  \[ = \sum_{i \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} 2^{i/2-1} Z_{i,k} \phi_{i,k}(t) \]
Providing a novel approach to sparse stochastic processes, this comprehensive book presents the theory of stochastic processes that are ruled by stochastic differential equations, and that admit a parsimonious representation in a matched wavelet-like basis.

Two key themes are the statistical property of infinite divisibility, which leads to two distinct types of behavior – Gaussian and sparse – and the structural link between linear stochastic processes and spline functions, which is exploited to simplify the mathematical analysis. The core of the book is devoted to investigating sparse processes, including a complete description of their transform-domain statistics. The final part develops practical signal-processing algorithms that are based on these models, with special emphasis on biomedical image reconstruction.

This is an ideal reference for graduate students and researchers with an interest in signal/image processing, compressed sensing, approximation theory, machine learning, or statistics.

MICHAEL UNSER is Professor and Director of EPFL’s Biomedical Imaging Group, Switzerland. He is a member of the Swiss Academy of Engineering Sciences, a fellow of EURASIP, and a fellow of the IEEE.

POUYA D. TAFTI is a researcher at Qlaym BmbH, Düsseldort, and a former member of the Biomedical Imaging Group at EPFL, where he conducted research on the theory and applications of probabilistic models for data.

Extension to fractional orders: B-splines

\[ L = D^\gamma \] (fractional derivative)

\[ L_d = \Delta_+^\gamma \] (fractional differences)

\[
\begin{align*}
\beta_+^0(r) &= \Delta_+ r_+^0 \\ &\mapsto \frac{1 - e^{-j\omega}}{j\omega}
\end{align*}
\]

\[
\begin{align*}
\vdots &\vdots \\
\beta_+^{\alpha}(r) &= \frac{\Delta_+^{\alpha+1} r_+^{\alpha}}{\Gamma(\alpha + 1)} \\ &\mapsto \left(1 - e^{-j\omega}\right)^{\alpha+1}
\end{align*}
\]

One-sided power function:

\[ r_+^\alpha = \begin{cases} 
 r^\alpha, & r \geq 0 \\
 0, & r < 0
\end{cases} \]

Degree \( \alpha = \gamma - 1 \)
Mandelbrot’s fractional Brownian motion

Solution of fractional SDE (Blu-U., 2007)

\[ D^\gamma s = w \text{ where } w \text{ is Gaussian white noise} \quad \Rightarrow \quad s = \mathcal{I}_2^\gamma w \]

\[ \mathcal{I}_2^\gamma : \text{scale-invariant, right-inverse of } D^\gamma \]

Hurst exponent: \( H = \gamma - \frac{1}{2} \)

---

Continuous-domain innovation model

Main outcome: non-Gaussian solutions are \textbf{necessarily sparse (infinitely divisible)}

\[ w(x), x \in \mathbb{R}^d \]

\[ s(x), x \in \mathbb{R}^d \]

Why? ... as will explained in next chapters ...

(invoking powerful theorems in functional analysis: Bochner-Minlos, Gelfand, Schoenberg & Lévy-Khinchine)
2.1 DECOUPLING OF SPARSE PROCESSES

\[ s = L^{-1}w \quad \Leftrightarrow \quad w = Ls \]

- Discrete approximation of operator
- Operator-like wavelet analysis

Decoupling: Linear combination of samples

Input: \( s(k), k \in \mathbb{Z}^d \) (sampled values) \( s = L^{-1}w \)

Discrete approximation of whitening operator: \( L_d \)

\[ L_d \delta(x) = \sum_{k \in \mathbb{Z}^d} d_L[k] \delta(x - k) \]

Discrete increment process:

\[ u[k] = L_d s(x)|_{x=k} = (\beta_L * w)(x)|_{x=k} = \langle \beta_L^\vee (\cdot - k), w \rangle \]

**Generalized B-spline:**

\[ \beta_L(x) = L_d L^{-1} \delta(x) \]

A-to-D translator
Decoupling: Wavelet analysis

\[ Ls = w \]

Generalized operator-like wavelets:

\[ \psi_i(x) = L^* \phi_i(x) \]

(Khalidov-U. 2006, Ward-U. JFAA 2013)

Operator-like wavelet analysis of sparse process:

\[ \langle \psi_i(\cdot - x_0), s \rangle = \langle L^* \phi_i(\cdot - x_0), s \rangle \]
\[ = \langle \phi_i(\cdot - x_0), Ls \rangle \]
\[ = \langle \phi_i(\cdot - x_0), w \rangle = (\phi_i^\vee * w)(x_0) \]