Lausanne, August 19, 2004

Dear Dr. Liebling,

I am pleased to inform you that you were selected to receive the 2004 Research Award of the Swiss Society of Biomedical Engineering for your thesis work “On Fresnelets, interference fringes, and digital holography”. The award will be presented during the general assembly of the SSBE, September 3, Zurich, Switzerland.

Please, let us know if
1) you will be present to receive the award,
2) you would be willing to give a 10 minutes presentation of the work during the general assembly.

The award comes with a cash prize of 1000.- CHF. Would you please send your banking information to the treasurer of the SSBE, Uli Diermann (Email: uli.diermann@bfh.ch), so that he can transfer the cash prize to your account?

I congratulate you on your achievement.

With best regards,

Michael Unser, Professor
Chairman of the SSBE Award Committee

cc: Ralph Mueller, president of the SSBE; Uli Diermann, treasurer

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Sparsity and optimality of splines: Deterministic vs. statistical justification

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Mathematics and Image Analysis (MIA), 18-20 January 2016, Paris, France.

OUTLINE

- **Sparsity and signal reconstruction**
  - Inverse problems in bio-imaging
  - Compressed sensing: towards a continuous-domain formulation

- **Deterministic formulation**
  - Splines and operators
  - New optimality results for generalized TV

- **Statistical formulation**
  - Sparse stochastic processes
  - Derivation of MAP estimators

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erc
Inverse problems in bio-imaging

- Linear forward model
  \[ y = Hs + n \]

- Reconstruction as an optimization problem
  \[ s_{\text{rec}} = \arg \min \|y - Hs\|_2^2 + \lambda \|Ls\|_p^p, \quad p = 1, 2 \]
  
  \[ -\log \text{Prob}(s) : \text{prior likelihood} \]

Inverse problems in imaging: Current status

- **Higher reconstruction quality**: Sparsity-promoting schemes almost systematically outperform the classical linear reconstruction methods in MRI, x-ray tomography, deconvolution microscopy, etc...  
  \[(\text{Lustig et al. 2007})\]

- **Increased complexity**: Resolution of linear inverse problems using \( \ell_1 \) regularization requires more sophisticated algorithms (iterative and non-linear); efficient solutions (FISTA, ADMM) have emerged during the past decade.  
  \[(\text{Chambolle 2004}; \text{Figueiredo 2004}; \text{Beck-Teboulle 2009}; \text{Boyd 2011})\]

- The paradigm is supported by the theory of **compressed sensing**  
  \[(\text{Candès-Romberg-Tao; Donoho, 2006})\]

- **Outstanding research issues**
  - Beyond \( \ell_1 \) and TV: Connection with **statistical modeling & learning**
  - Beyond matrix algebra: **Continuous-domain** formulation
  - Guarantees of **optimality** (either deterministic or statistical)
Sparsity and continuous-domain modeling

- Compressed sensing (CS)
  - Generalized sampling and infinite-dimensional CS
    (Adcock-Hansen, 2011)
  - Xampling: CS of analog signals
    (Eldar, 2011)

- Splines and approximation theory
  - $L_1$ splines
    (Fisher-Jerome, 1975)
  - Locally-adaptive regression splines
    (Mammen-van de Geer, 1997)
  - Generalized TV
    (Steidl et al. 2005; Bredies et al. 2010)

- Statistical modeling
  - Sparse stochastic processes
    (Unser et al. 2011-2014)

The spline connection

Photo courtesy of Carl De Boor
Splines are intrinsically sparse

\( \mathcal{L}\{\cdot\} : \) admissible differential operator
\( \delta(\cdot - x_0) : \) Dirac impulse shifted by \( x_0 \in \mathbb{R}^d \)

**Definition**

The function \( s : \mathbb{R}^d \to \mathbb{R} \) is a (non-uniform) \( \mathcal{L} \)-spline with knots \( \{x_k\}_{k=1}^K \) if

\[
\mathcal{L}\{s\} = \sum_{k=1}^{K} a_k \delta(\cdot - x_k) = w_\delta : \text{spline's innovation}
\]

Splines and operators

**Definition**

A linear operator \( \mathcal{L} : \mathcal{X} \to \mathcal{Y} \), where \( \mathcal{X} \supset \mathcal{S}(\mathbb{R}^d) \) and \( \mathcal{Y} \) are appropriate subspaces of \( \mathcal{S}'(\mathbb{R}^d) \), is called spline-admissible if

1. it is linear shift-invariant (LSI);
2. its null space \( \mathcal{N}_\mathcal{L} = \{ p \in \mathcal{X} : \mathcal{L}\{p\} = 0 \} \) is finite-dimensional of size \( N_0 \);
3. there exists a function \( \rho_\mathcal{L} : \mathbb{R}^d \to \mathbb{R} \) of slow growth (the Green’s function of \( \mathcal{L} \)) such that \( \mathcal{L}\{\rho_\mathcal{L}\} = \delta \).

- Structure of null space: \( \mathcal{N}_\mathcal{L} = \text{span}\{p_n\}_{n=1}^{N_0} \)
- Admits some basis \( p = (p_1, \cdots, p_{N_0}) \)
- Only includes elements of the form \( x^m e^{i(\omega_0, x)} \) with \( |m| = \sum_{i=1}^{d} m_i \leq n_0 \)
**Spline synthesis: example**

\[ L = D = \frac{d}{dx}, \quad N_D = \text{span}\{p_1\}, \quad p_1(x) = 1 \]

\[ \rho_D(x) = 1_+(x): \text{Heaviside function} \]

\[ w_\delta(x) = \sum_k a_k \delta(x - x_k) \]

\[ s(x) = b_1 p_1(x) + \sum_k a_k 1_+(x - x_k) \]

**Spline synthesis: generalization**

\[ L: \text{spline admissible operator (LSI)} \]

\[ \rho_L(x): \text{Green's function of } L \]

\[ N_L = \text{span}\{p_n\}_{n=1}^{N_0} \]

Spline's innovation: \[ w_\delta(x) = \sum_k a_k \delta(x - x_k) \]

\[ \Rightarrow s(x) = \sum_k a_k \rho_L(x - x_k) + \sum_{n=1}^{N_0} b_n p_n(x) \]

Requires specification of boundary conditions
Principled operator-based approach

- Biorthogonal basis of $\mathcal{N}_L = \text{span}\{p_n\}_{n=1}^{N_0}$
  - $\phi = (\phi_1, \cdots, \phi_{N_0})$ such that $\langle \phi_m, p_n \rangle = \delta_{m,n}$
  - $p = \sum_{n=1}^{N_0} \langle \phi_n, p \rangle p_n$ for all $p \in \mathcal{N}_L$

- Operator-based spline synthesis
  - Boundary conditions: $\langle s, \phi_n \rangle = b_n, \; n = 1, \cdots, N_0$
  - Spline’s innovation: $L\{s\} = w_\delta = \sum_{k} a_k \delta(\cdot - x_k)$
    $$s(x) = L^{-1}_\phi \{w_\delta\}(x) + \sum_{n=1}^{N_0} b_n p_n(x)$$

- Existence of $L^{-1}_\phi$ as a stable right-inverse of $L$? (see Theorem 1)
  - $LL^{-1}_\phi w = w$
  - $\langle \phi, L^{-1}_\phi w \rangle = 0$

Beyond splines ...
From Dirac impulses to Borel measures

\( S(\mathbb{R}^d) \): Schwartz’s space of smooth and rapidly decaying test functions on \( \mathbb{R}^d \)

\( S'(\mathbb{R}^d) \): Schwartz’s space of tempered distributions

**Space of real-valued, countably additive Borel measures on \( \mathbb{R}^d \)**

\[ \mathcal{M}(\mathbb{R}^d) = (C_0(\mathbb{R}^d))^\prime = \left\{ w \in S'(\mathbb{R}^d) : \| w \|_{TV} = \sup_{\varphi \in S(\mathbb{R}^d) : \| \varphi \|_\infty = 1} \langle w, \varphi \rangle < \infty \right\}, \]

where \( w : \varphi \mapsto \langle w, \varphi \rangle = \int_{\mathbb{R}^d} \varphi(r) dw(r) \)

**Equivalent definition of “total variation” norm**

\[ \| w \|_{TV} = \sup_{\varphi \in C_0(\mathbb{R}^d) : \| \varphi \|_\infty = 1} \langle w, \varphi \rangle \]

**Basic inclusions**

- \( \delta(\cdot - x_0) \in \mathcal{M}(\mathbb{R}^d) \) with \( \| \delta(\cdot - x_0) \|_{TV} = 1 \) for any \( x_0 \in \mathbb{R}^d \)
- \( \| f \|_{TV} = \| f \|_{L_1(\mathbb{R}^d)} \) for all \( f \in L_1(\mathbb{R}^d) \) \( \Rightarrow \) \( L_1(\mathbb{R}^d) \subseteq \mathcal{M}(\mathbb{R}^d) \)

Generalized Beppo-Levi spaces

\( L \): spline-admissible operator

**Generalized “total variation” semi-norm (gTV)**

\[ gTV(f) = \| L\{f\} \|_{TV} \]

**Generalized Beppo-Levi spaces**

\[ \mathcal{M}_L(\mathbb{R}^d) = \left\{ f : \mathbb{R}^d \to \mathbb{R} : \| Lf \|_{TV} < \infty \right\} \]

\( f \in \mathcal{M}_L(\mathbb{R}^d) \Leftrightarrow L\{f\} \in \mathcal{M}(\mathbb{R}^d) \)

- **Classical Beppo-Levi spaces:** \( (\mathcal{M}(\mathbb{R}^d), L) \to (L_p(\mathbb{R}), D^n) \) \( \text{(Deny-Lions, 1954)} \)

- **Inclusion of non-uniform \( L \)-splines**

\[ s = \sum_{k} a_k \rho_L(\cdot - x_k) + \sum_{n=1}^{N_0} b_n p_n \Rightarrow L\{s\} = \sum_{k} a_k \delta(\cdot - x_k) \]

\[ gTV(s) = \| L\{s\} \|_{TV} = \sum_{k} |a_k| = \| a \|_{\ell_1} \]
New optimality result: universality of splines

\( L \): spline-admissible operator

\[ \mathcal{M}_L(\mathbb{R}) = \{ f : gTV(f) = \| L\{f\} \|_{TV} = \sup_{\| \varphi \|_\infty \leq 1} \langle L\{f\}, \varphi \rangle < \infty \} \]

**Generalized total variation**: \( gTV(f) = \| L\{f\} \|_{L_1} \) when \( L\{f\} \in L_1(\mathbb{R}^d) \)

**Linear measurement operator** \( \mathcal{M}_L : \mathbb{R}^M : f \mapsto z = \nu(f) \)

\[ \Leftrightarrow z_m = \langle f, \nu_m \rangle \]

**Theorem** [U.-Fageot-Ward, 2015]: The generic linear-inverse problem

\[ \min_{f \in \mathcal{M}_L(\mathbb{R})} \left( \| y - \nu(f) \|_2^2 + \lambda \| L\{f\} \|_{TV} \right) \]

admits a global solution of the form

\[ f(x) = \sum_{k=1}^{K} a_k \rho_L(x - x_k) + \sum_{n=1}^{N_0} b_n p_n(x) \]

with \( K < M \), which is a non-uniform \( L \)-spline with knots \( (x_k)_{k=1}^{K} \).

---

Optimality result for Dirac measures

- \( F \): linear continuous map \( \mathcal{M}(\mathbb{R}^d) \rightarrow \mathbb{R}^M \)
- \( C \): convex compact subset of \( \mathbb{R}^M \)
- Generic constrained TV minimization problem

\[ \mathcal{V} = \arg \min_{w \in \mathcal{M}(\mathbb{R}^d) : F(w) \in C} \| w \|_{TV} \]

**Generalized Fisher-Jerome theorem**

The solution set \( \mathcal{V} \) is a convex, weak*-compact subset of \( \mathcal{M}(\mathbb{R}^d) \) with extremal points of the form

\[ w_\delta = \sum_{k=1}^{K} a_k \delta(\cdot - x_k) \]

with \( K \leq M \) and \( x_k \in \mathbb{R}^d \).

**Jerome-Fisher, 1975**: Compact domain & scalar intervals
Existence of stable right-inverse operator

$$L_{\infty,n_0}(\mathbb{R}^d) = \{ f : \mathbb{R}^d \to \mathbb{R} : \sup_{x \in \mathbb{R}^d} (|f(x)|(1 + |x|)^{n_0}) < +\infty \}$$

**Theorem 1** [U.-Fageot-Ward, preprint]

Let $L$ be a spline-admissible operator with a $N_0$-dimensional null space $N_L \subseteq L_{\infty,-n_0}(\mathbb{R}^d)$ such that $p = \sum_{n=1}^{N_0} \langle p, \phi_n \rangle p_n$ for all $p \in N_L$. Then, there exists a unique and stable operator $L^{-1}_\phi : \mathcal{M}(\mathbb{R}^d) \to L_{\infty,-n_0}(\mathbb{R}^d)$ such that, for all $w \in \mathcal{M}(\mathbb{R}^d)$,

- Right-inverse property: $LL^{-1}_\phi w = w,$
- Boundary conditions: $\langle \phi, L^{-1}_\phi w \rangle = 0$ with $\phi = (\phi_1, \cdots, \phi_{N_0})$.

Its generalized impulse response $g_\phi(x,y) = L^{-1}_\phi \delta(x - y)(x)$ is given by

$$g_\phi(x,y) = \rho_L(x - y) - \sum_{n=1}^{N_0} p_n(x)q_n(y)$$

with $\rho_L$ such that $L\rho_L = \delta$ and $q_n(y) = \langle \phi_n, \rho_L \cdot - y \rangle$.

Characterization of generalized Beppo-Levi spaces

- Regularization operator $L : \mathcal{M}_L(\mathbb{R}^d) \to \mathcal{M}(\mathbb{R}^d)$

\[ f \in \mathcal{M}_L(\mathbb{R}^d) \iff g_{TV}(f) = \|L\{f\}\|_{TV} < \infty \]

**Theorem 2** [U.-Fageot-Ward, preprint]

Let $L$ be a spline-admissible operator that admits a stable right-inverse $L^{-1}_\phi$ of the form specified by Theorem 1. Then, any $f \in \mathcal{M}_L(\mathbb{R}^d)$ has a unique representation as

$$f = L^{-1}_\phi w + p,$$

where $w = L\{f\} \in \mathcal{M}(\mathbb{R}^d)$ and $p = \sum_{n=1}^{N_0} \langle \phi_n, f \rangle p_n \in N_L$ with $\phi_n \in (\mathcal{M}_L(\mathbb{R}^d))'$. Moreover, $\mathcal{M}_L(\mathbb{R}^d) \subseteq L_{\infty,-n_0}(\mathbb{R}^d)$ and is a Banach space equipped with the norm

$$\|f\|_{\mathcal{M}_L,\phi} = \|L\{f\}\|_{TV} + \|\langle f, \phi \rangle\|_2.$$

- Generalized Beppo-Levi space:

\[ \mathcal{M}_{L,\phi}(\mathbb{R}^d) = \mathcal{M}_{L,\phi}(\mathbb{R}^d) \oplus N_L \]

\[ \mathcal{M}_{L,\phi}(\mathbb{R}^d) = \{ f \in \mathcal{M}_L(\mathbb{R}^d) : \langle \phi, f \rangle = 0 \} \]

\[ N_L = \{ p \in \mathcal{M}_L(\mathbb{R}^d) : L\{p\} = 0 \} \]
**Link with sparse stochastic processes**

**Random spline: archetype of sparse signal**

![Non-uniform spline of degree 0](image)

\[ D_s(t) = \sum_n a_n \delta(t - t_n) = w(t) \]

Random weights \( \{a_n\} \) i.i.d. and random knots \( \{t_n\} \) (Poisson with rate \( \lambda \))

- Anti-derivative operators

- Shift-invariant solution: \( D^{-1} \varphi(t) = (1_+ * \varphi)(t) = \int_{-\infty}^{t} \varphi(\tau) d\tau \)

- Scale-invariant solution: \( D_{\phi_1}^{-1} \varphi(t) = \int_0^t \varphi(\tau) d\tau \quad (\text{see Theorem 1 with } \phi_1 = \delta) \)
**Compound Poisson process**

- Stochastic differential equation
  \[ Ds(t) = w(t) \]
  with boundary condition \( s(0) = \langle \phi_1, s \rangle = 0 \) with \( \phi_1 = \delta \)

- Innovation:
  \[ w(t) = \sum_n a_n \delta(t - t_n) \]

- Formal solution
  \[ s(t) = D^{-1}_{\phi_1} w(t) = \sum_n a_n D^{-1}_{\phi_1} \{\delta(\cdot - t_n)\}(t) \]
  \[ = b_1 + \sum_n a_n \mathbb{1}_+(t - t_n) \]

---

**Lévy processes: all admissible brands of innovations**

Generalized innovations: white Lévy noise with
\[ \mathbb{E}\{w(t)w(t')\} = \sigma_w^2 \delta(t - t') \]

\[ Ds = w \quad \text{(perfect decoupling!)} \]

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**White noise (innovation)**
- Gaussian
- Impulsive
- \( \mathcal{S}_\alpha \mathcal{S} \) (Cauchy)

**Lévy process**
- Brownian motion (Wiener 1923)
- Compound Poisson
- Lévy flight (Paul Lévy circa 1930)
Generalized innovation model

Theoretical framework: Gelfand’s theory of generalized stochastic processes

Generic test function \( \varphi \in \mathcal{S} \) plays the role of index variable

Solution of SDE (general operator)

1. \( X = \langle \varphi, w \rangle \)
2. \( L^{-1} \)
3. \( s = L^{-1}w \)
4. \( Y = \langle \varphi, s \rangle = \langle \varphi, L^{-1}w \rangle = \langle L^{-1}\varphi, w \rangle \)

Proper definition of \textit{continuous-domain} white noise

(\textit{Unser et al., IEEE-IT 2014})

Main feature: inherent sparsity (few significant coefficients)

Regularization operator vs. wavelet analysis

From Dirac impulses to innovation processes

\( w \) is a generalized innovation process (or continuous-domain white noise) in \( \mathcal{S}'(\mathbb{R}^d) \) if

1. \textit{Observability}: \( X = \langle \varphi, w \rangle \) is an ordinary random variable for any \( \varphi \in \mathcal{S}(\mathbb{R}^d) \).
2. \textit{Stationarity}: \( X_{x_0} = \langle \varphi(\cdot - x_0), w \rangle \) is identically distributed for all \( x_0 \in \mathbb{R}^d \).
3. \textit{Independent atoms}: \( X_1 = \langle \varphi_1, w \rangle \) and \( X_2 = \langle \varphi_2, w \rangle \) are independent whenever \( \varphi_1 \) and \( \varphi_2 \) have non-intersecting support.

\textbf{Theorem} (under mild technical conditions) (\textit{Amini-U., IEEE-IT 2014})

\( w \) is an innovation process in \( \mathcal{S}'(\mathbb{R}^d) \)

\[ \Rightarrow \quad X = \langle \varphi, w \rangle \text{ is well defined and infinitely divisible for any } \varphi \in L_p(\mathbb{R}^d) \]

\textbf{Definition}: A random variable \( X \) with generic pdf \( p_{id}(x) \) is \textit{infinitely divisible} (id) iff., for any \( N \in \mathbb{Z}^+ \), there exist i.i.d. random variables \( X_1, \ldots, X_N \) such that \( X \overset{d}{=} X_1 + \cdots + X_N \).

\[
X = \langle w, \text{rect} \rangle = \langle \varphi, \text{rect} \rangle = \sum_{n=1}^{\infty} \frac{1}{n} \]

\( \overset{\text{i.i.d.}}{\rightarrow} \)

\[
\text{rect} \quad + \cdots + \text{rect} \]

1. \text{i.i.d.}
Probability laws of innovations are infinite divisible

- Canonical observation through a rectangular test function

\[ X_{id} = \langle w, \text{rect} \rangle = \langle \underbrace{\ldots}, \underbrace{1} \rangle \]

- \( w \) innovation process \( \Leftrightarrow X_{id} = \langle w, \text{rect} \rangle \) infinitely divisible
  - with canonical Lévy exponent \( f(\omega) = \log \hat{p}_{id}(\omega) \)

- Statistical description of white Lévy noise \( w \) (innovation)

  - Generic observation: \( X = \langle \varphi, w \rangle \) with \( \varphi \in L_p(\mathbb{R}^d) \)

\[ X = \langle w, \varphi \rangle = \langle \underbrace{\ldots}, \underbrace{\varphi} \rangle \triangleq \lim_{n \to \infty} \langle \underbrace{\ldots}, \underbrace{\sum_{n}^{\infty}} \rangle \]

\[ = \lim_{n \to \infty} \langle \underbrace{\ldots}, \underbrace{\sum_{n}^{\infty}} \rangle + \ldots + \langle \underbrace{\ldots}, \underbrace{\varphi} \rangle \]

- \( X \) is infinitely divisible with (modified) Lévy exponent

\[ f_{\varphi}(\omega) = \log \hat{p}_{X}(\omega) = \int_{\mathbb{R}^d} f(\omega \varphi(x)) \, dx \]

Probability laws of sparse processes are id

- Analysis: go back to innovation process: \( w = Ls \)

  - Generic random observation: \( X = \langle \varphi, w \rangle \) with \( \varphi \in \mathcal{S}(\mathbb{R}^d) \) or \( \varphi \in L_p(\mathbb{R}^d) \) (by extension)

  - Linear functional: \( Y = \langle \psi, s \rangle = \langle \psi, L^{-1} w \rangle = \langle L^{-1*} \psi, w \rangle \)

    If \( \phi = L^{-1*} \psi \in L_p(\mathbb{R}^d) \) then \( Y = \langle \psi, s \rangle = \langle \phi, w \rangle \) is infinitely divisible
    - with (modified) Lévy exponent \( f_{\phi}(\omega) = \int_{\mathbb{R}^d} f(\omega \phi(x)) \, dx \)

\[ \Rightarrow \quad p_Y(y) = \mathcal{F}^{-1} \{ e^{f_{\phi}(\omega)} \}(y) = \int_{\mathbb{R}} e^{f_{\phi}(\omega)-i\omega y} \frac{d\omega}{2\pi} \]

= explicit form of pdf
Examples of infinitely divisible laws

\[ p_{\text{id}}(x) \]

(a) Gaussian

\[ p_{\text{Gauss}}(x) = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \]

(b) Laplace

\[ p_{\text{Laplace}}(x) = \frac{\lambda}{2} e^{-\lambda|x|} \]

(c) Compound Poisson

\[ p_{\text{Poisson}}(x) = \mathcal{F}^{-1}\{e^{\lambda(p_A(\omega)-1)}\} \]

(d) Cauchy (stable)

\[ p_{\text{Cauchy}}(x) = \frac{1}{\pi (x^2 + 1)} \]

Characteristic function:

\[ \hat{p}_{\text{id}}(\omega) = \int_{\mathbb{R}} p_{\text{id}}(x) e^{j\omega x} dx = e^{f(\omega)} \]

Examples of id noise distributions

\[ p_{\text{id}}(x) \]

Observations: \( X_n = \langle w, \text{rect}(\cdot - n) \rangle \)

(a) Gaussian

\[ f(\omega) = -\frac{\sigma^2}{2} |\omega|^2 \]

(b) Laplace

\[ f(\omega) = \log \left( \frac{1}{1+\omega^2} \right) \]

(c) Compound Poisson

\[ f(\omega) = \lambda \int_{\mathbb{R}} (e^{j\omega x} - 1) p(x) dx \]

(d) Cauchy (stable)

\[ f(\omega) = -s_0 |\omega| \]

Complete mathematical characterization:

\[ \mathcal{F}_w(\varphi) = \exp \left( \int_{\mathbb{R}^d} f(\varphi(x)) dx \right) \]
Aesthetic sparse signal: the Mondrian process

\[ L = D_x D_y \xrightarrow{\mathcal{F}} (j\omega_x)(j\omega_y) \]

Scale- and rotation-invariant processes

Stochastic partial differential equation:

\[ (-\Delta)^{\frac{H+1}{2}} s(x) = w(x) \]

Gaussian

Sparse (generalized Poisson)

(U.-Tafti, IEEE-SP 2010)
Powers of ten: from astronomy to biology

High-level properties of SSP

- **Infinite divisible probability laws**: broadest class of distributions preserved through linear transformation.

- **Explicit calculations**: Analytical determination of transform-domain statistics (including, joint pdfs).

- **Unifying framework**: includes all traditional families of stochastic processes (ARMA, fBm), as well as their non-Gaussian generalizations.

- **Sparsifying transforms / ICA**: SSP are (approximately) decoupled in a matched operator-like wavelet basis. (Pad-U., *IEEE-SP 2015*)

- **N-term approximation properties**: SSP are truly “sparse” as described by their inclusion in (weighted) Besov spaces. (Fageot et al., *ACHA 2015*)
STATISTICAL SIGNAL RECONSTRUCTION

- Discretization of reconstruction problem
- Signal reconstruction algorithm (MAP)

Discretization of reconstruction problem

Spline-like reconstruction model: \( s(r) = \sum_{k \in \Omega} s[k] \beta_k(r) \quad \leftrightarrow \quad s = (s[k])_{k \in \Omega} \)

- Innovation model

\[
\begin{align*}
Ls &= w \\
s &= L^{-1}w
\end{align*}
\]

Discretization \( u = Ls \) (matrix notation)

\( p_U \) is part of infinitely divisible family

- Physical model: image formation and acquisition

\[
y_m = \int_{R^d} s_1(x) \eta_m(x) dx + n[m] = \langle s_1, \eta_m \rangle + n[m], \quad (m = 1, \ldots, M)
\]

\[
y = y_0 + n = Hs + n
\]

\( n: \) i.i.d. noise with pdf \( p_N \)

\[
[H]_{m,k} = \langle \eta_m, \beta_k \rangle = \int_{R^d} \eta_m(r) \beta_k(r) dr: \quad (M \times K) \text{ system matrix}
\]
Posterior probability distribution

\[ p_{S|Y}(s|y) = \frac{p_{Y|S}(y|s)p_S(s)}{p_Y(y)} = \frac{p_N(y - Hs)p_S(s)}{p_Y(y)} = \frac{1}{Z} p_N(y - Hs)p_S(s) \]  

\( u = Ls \Rightarrow p_S(s) \propto p_U(Ls) \approx \prod_{k \in \Omega} p_U([Ls]_k) \)

- Additive white Gaussian noise scenario (AWGN)

\[ p_{S|Y}(s|y) \propto \exp \left( -\frac{||y - Hs||^2}{2\sigma^2} \right) \prod_{k \in \Omega} p_U([Ls]_k) \]

\[ \ldots \text{and then take the log and maximize} \ldots \]

General form of MAP estimator

\[ s_{MAP} = \arg \min \left( \frac{1}{2} ||y - Hs||^2 + \sigma^2 \sum_n \Phi_U([Ls]_n) \right) \]

- Gaussian: \( p_U(x) = \frac{1}{\sqrt{2\pi\sigma_0}} e^{-x^2/(2\sigma_0^2)} \Rightarrow \Phi_U(x) = \frac{1}{2\sigma_0} x^2 + C_1 \)

- Laplace: \( p_U(x) = \frac{1}{2} e^{-|x|} \Rightarrow \Phi_U(x) = \lambda|x| + C_2 \)

- Student: \( p_U(x) = \frac{1}{B(r, \frac{1}{2})} \left( \frac{1}{x^2 + 1} \right)^{r+\frac{1}{2}} \Rightarrow \Phi_U(x) = (r + \frac{1}{2}) \log(1 + x^2) + C_3 \)

Potential: \( \Phi_U(x) = -\log p_U(x) \)
3D deconvolution with sparsity constraints

Maximum intensity projections of $384 \times 448 \times 260$ image stacks;
Leica DM 5500 widefield epifluorescence microscope with a $63 \times$ oil-immersion objective;
C. Elegans embryo labeled with Hoechst, Alexa488, Alexa568;


Cryo-electron tomography (real data)

Standard Fourier-based reconstruction

High-resolution Fourier-based reconstruction

High-resolution reconstruction with sparsity

6.185 Å
SUMMARY: Sparsity in infinite dimensions

- Continuous-domain formulation
  - Linear measurement model
  - Linear signal model: PDE
  - L-splines = signals with “sparsest” innovation

- Deterministic optimality result
  - gTV regularization: favors “sparse” innovations

- Statistical model that supports sparsity
  - Statistical decoupling:
    Gaussian vs. sparse innovations (Poisson, student, $S\alpha S$)
  - Unifying framework: “sparse stochastic processes”
  - MAP enforces sparsity through non-quadratic regularization

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- Dr. Cédric Vonesch
- ....

Preprints and demos: 

http://bigwww.epfl.ch/
Gaussian vs. Sparse

Fourier analysis

Spines

Wavelet analysis

Norbert Wiener

Isaac Schoenberg

Paul Lévy

References

Theory of sparse stochastic processes


Algorithms and imaging applications