ÉCOLE POLYTECHNIQUE
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Sampling and interpolation for biomedical imaging: Part II

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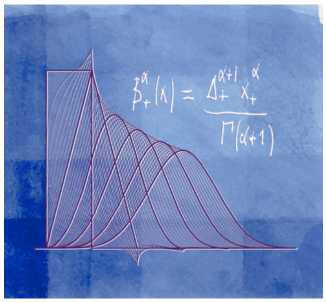
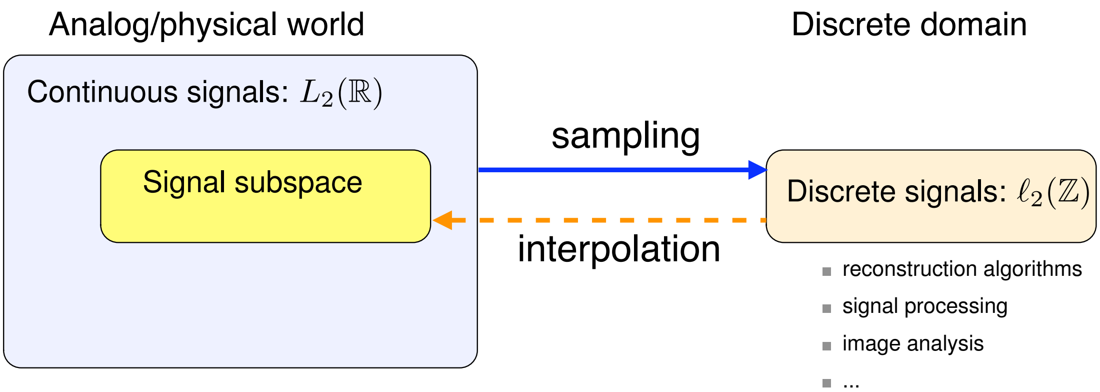


ISBI 2006, Tutorial, Washington DC, April 2006

SAMPLING: 50+ years after Shannon



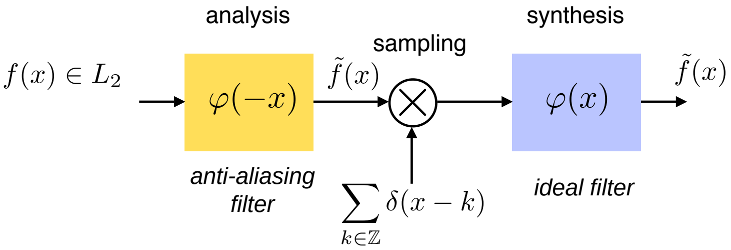
- Introduction: Shannon revisited
- Sampling preliminaries
- Sampling revisited
- Quantitative approximation theory

$\left.\right\}$ Review paper on sampling

- Interpolation/approximation in the presence of noise

Shannon's sampling reinterpreted

- Generating function: $\varphi(x) = \mathrm{sinc}(x)$
- Subspace of bandlimited functions: $V(\varphi) = \mathrm{span}\{\varphi(x-k)\}_{k\in\mathbb{Z}}$

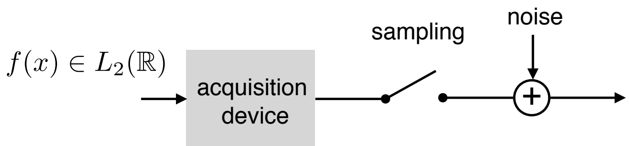


- Analysis: $\tilde{f}(k) = \langle \mathrm{sinc}(x-k), f(x) \rangle$
- Synthesis: $\tilde{f}(x) = \sum_{k\in Z} \tilde{f}(k)\, \mathrm{sinc}(x-k)$
- Orthogonal basis: $\langle \mathrm{sinc}(x-k), \mathrm{sinc}(x-l) \rangle = \delta_{k-l}$ [Hardy, 1941]

⇒ Orthogonal projection operator !

Generalized sampling: roadmap

- Nonideal acquisition system



Measurements:

$g[k] = (h * f)(x)|_{x=k} \quad + n[k]$

⇓

Reconstruction algorithm

⇓

signal coefficients

$\{c[k]\}_{k\in\mathbb{Z}}$

Goal: Specify φ and the reconstruction algorithm so that $\tilde{f}(x)$ is a good approximation of $f(x)$

Continuous-domain model

$\tilde{f}(x) = \sum_{k\in\mathbb{Z}} c[k]\varphi(x-k)$

⇔ *Riesz-basis property*

⇕ *Interpolation problem*

Discrete signal

$\{f[k]\}_{k\in\mathbb{Z}}$

SAMPLING PRELIMINARIES

- Function and sequence spaces
- Smoothness conditions and sampling
- Shift-invariant subspaces
- Equivalent basis functions

Continuous-domain signals

Mathematical representation: a function of the continuous variable $x \in \mathbb{R}$

- Lebesgue's space of finite-energy functions
  - $L_2(\mathbb{R}) = \left\{ f(x), x \in \mathbb{R} : \int_{x\in\mathbb{R}} |f(x)|^2 \mathrm{d}x < +\infty \right\}$
  - L_2-inner product: $\langle f, g \rangle = \int_{x\in\mathbb{R}} f(x) g^*(x) \mathrm{d}x$
  - L_2-norm: $\|f\|_{L_2} = \left(\int_{x\in\mathbb{R}} |f(x)|^2 \mathrm{d}x \right)^{1/2} = \sqrt{\langle f, f \rangle}$
- Fourier transform
  - Integral definition: $\hat{f}(\omega) = \int_{x\in\mathbb{R}} f(x) e^{-j\omega x} \mathrm{d}x$
  - Parseval relation: $\|f\|^2_{L_2} = \frac{1}{2\pi} \int_{\omega\in\mathbb{R}} |\hat{f}(\omega)|^2 \mathrm{d}\omega$

Discrete-domain signals

Mathematical representation: a sequence indexed by the discrete variable $k \in \mathbb{Z}$

- Space of finite-energy sequences
  - $\ell_2(\mathbb{Z}) = \left\{ a[k], k \in \mathbb{Z} : \sum_{k \in \mathbb{Z}} |a[k]|^2 < +\infty \right\}$
  - ℓ_2-norm: $\|a\|_{\ell_2} = \left(\sum_{k \in \mathbb{Z}} |a[k]|^2 \right)^{1/2}$

- Discrete-time Fourier transform
  - z-transform: $A(z) = \sum_{k \in \mathbb{Z}} a[k] z^{-k}$
  - Fourier transform: $A(e^{j\omega}) = \sum_{k \in \mathbb{Z}} a[k] e^{-j\omega k}$

Smoothness conditions and sampling

- Sobolev's space of order $s \in \mathbb{R}^+$

$$W_2^s(\mathbb{R}) = \left\{ f(x), x \in \mathbb{R} : \int_{\omega \in \mathbb{R}} (1 + |\omega|^{2s}) |\hat{f}(\omega)|^2 \mathrm{d}\omega < +\infty \right\}$$

f and all its derivatives up to (fractional) order s are in L_2

- Mathematical requirements for ideal sampling
  - The input signal $f(x)$ should be continuous
  - The samples $f[k] = f(x)|_{x=k}$ should be in ℓ_2

Theorem
Let $f(x) \in W_2^s$ with $s > \frac{1}{2}$. Then, the samples of f at the integers, $f[k] = f(x)|_{x=k}$, are in ℓ_2 and

$$F(e^{j\omega}) = \sum_{k \in \mathbb{Z}} f[k] e^{-j\omega k} = \sum_{n \in \mathbb{Z}} \hat{f}(\omega + 2\pi n) \qquad \text{a.e.}$$

Generalized (*almost everywhere*) version of Poisson's formula [Blu-U., 1999]

Shift-invariant spaces

Integer-shift-invariant subspace associated with a generating function φ (e.g., B-spline):

$$V(\varphi) = \left\{ f(x) = \sum_{k\in\mathbb{Z}} c[k]\varphi(x-k) : c \in \ell_2(\mathbb{Z}) \right\}$$

Generating function: $\varphi(x) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad \hat{\varphi}(\omega) = \int_{x\in\mathbb{R}} \varphi(x) e^{-j\omega x} \mathrm{d}x$

- Autocorrelation (or Gram) sequence

$$a_\varphi[k] \triangleq \langle \varphi(\cdot), \varphi(\cdot - k)\rangle \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad A_\varphi(e^{j\omega}) = \sum_{n\in\mathbb{Z}} |\hat{\varphi}(\omega + 2\pi n)|^2$$

- Riesz-basis condition

Positive-definite Gram sequence: $0 < A^2 \le \sum_{n\in\mathbb{Z}} A_\varphi(e^{j\omega}) \le B^2 < +\infty$

$$\Updownarrow$$

$$A \cdot \|c\|_{\ell_2} \le \underbrace{\left\| \sum_{k\in\mathbb{Z}} c[k]\varphi(x-k) \right\|_{L_2}}_{\|f\|_{L_2}} \le B \cdot \|c\|_{\ell_2}$$

Orthonormal basis $\Leftrightarrow$ $a_\varphi[k] = \delta_k$ $\Leftrightarrow$ $A_\varphi(e^{j\omega}) = 1$ $\Leftrightarrow$ $\|c\|_{\ell_2} = \|f\|_{L_2}$ (Parseval)

Example of sampling spaces

- Piecewise-constant functions

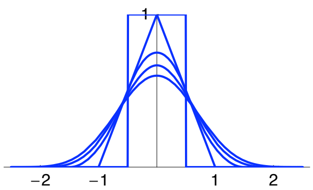
$\varphi(x) = \mathrm{rect}(x) = \beta^0(x)$ $\qquad a_\varphi[k] = \delta_k \quad \Leftrightarrow$ the basis is orthonormal

- bandlimited functions

$\varphi(x) = \mathrm{sinc}(x)$ $\qquad \sum_{n\in\mathbb{Z}} |\hat{\varphi}(\omega + 2\pi n)|^2 = 1 \quad \Leftrightarrow$ the basis is orthonormal

- Polynomial splines of degree n

$$\varphi(x) = \beta^n(x) = (\underbrace{\beta^0 * \beta^0 \cdots * \beta^0}_{(n+1) \text{ times}})(x)$$



Autocorrelation sequence: $a_{\beta^n}[k] = (\beta^n * \beta^n)(x)|_{x=k} = \beta^{2n+1}(k)$

Proposition. The B-spline of degree n, $\beta^n(x)$, generates a Riesz basis with lower and upper Riesz bounds $A = \inf_\omega\{A_{\beta^n}(e^{j\omega})\} \ge \left(\frac{2}{\pi}\right)^{n+1}$ and $B = \sup_\omega\{A_{\beta^n}(e^{j\omega})\} = 1$.

Equivalent and dual basis functions

- Equivalent basis functions: $\varphi_{\mathrm{eq}}(x) = \sum_{k\in\mathbb{Z}} p[k]\varphi(x-k)$

Proposition. Let φ be a valid (Riesz) generator of $V(\varphi) = \mathrm{span}\{\varphi(x-k)\}_{k\in\mathbb{Z}}$.
Then, φ_{eq} also generates a Riesz basis of $V(\varphi)$ iff.

$$0 < C_1 \leq |P(e^{j\omega})|^2 \leq C_2 < +\infty \qquad \text{(almost everywhere)}$$

- Dual basis function

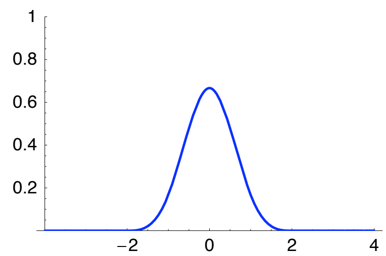
Unique function $\mathring{\varphi} \in V(\varphi)$ such that $\langle \varphi(x), \mathring{\varphi}(x-k)\rangle = \delta_k$ (biorthogonality)

Together, φ and $\mathring{\varphi}$ operate as if they were an orthogonal basis; i.e., the orthogonal projector of any function $f \in L_2$ onto $V(\varphi)$ is given by

$$\mathrm{P}_{V(\varphi)} f(x) = \sum_{k\in\mathbb{Z}} \underbrace{\langle f, \mathring{\varphi}(\cdot - k)\rangle}_{c[k]} \, \varphi(x-k)$$

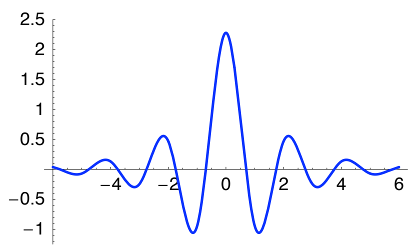
Example: four equivalent cubic-spline bases

- Cubic B-spline: $\varphi(x) = \beta^3(x)$



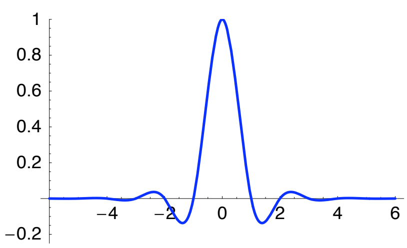
Compact support

- Dual spline: $\mathring{\varphi}(x)$



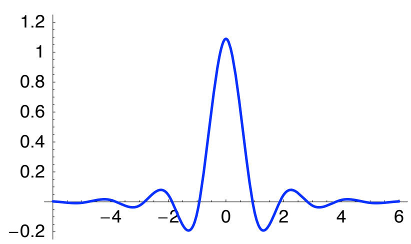
Biorthogonality: $\langle \varphi(x), \mathring{\varphi}(x-k)\rangle = \delta_k$

- Interpolating spline: $\varphi_{\mathrm{int}}(x)$



Interpolation: $\langle \varphi_{\mathrm{int}}(x), \delta(x-k)\rangle = \delta_k$

- Orthogonal spline: $\varphi_{\mathrm{ortho}}(x)$

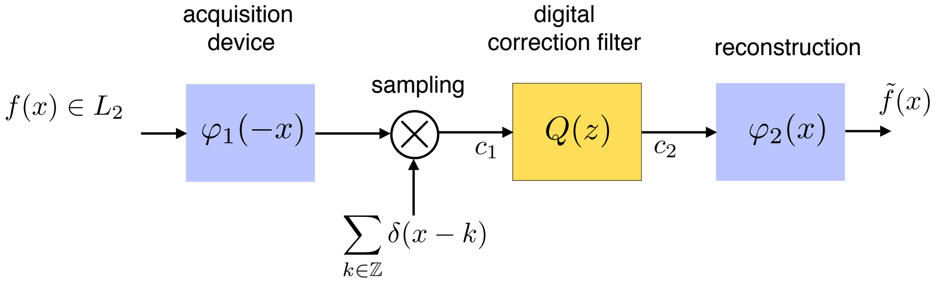


Orthogonality: $\langle \varphi_{\mathrm{ortho}}(x), \varphi_{\mathrm{ortho}}(x-k)\rangle = \delta_k$

SAMPLING REVISITED

- Generalized sampling system
- Generalized sampling theorem
- Consistent sampling: properties
- Performance analysis
- Applications

Generalized sampling system



- $\varphi_1(-x)$: prefilter (acquisition system)
- $\varphi_2(x)$: generating function (reconstruction subspace)

- Constraints
  - Consistent measurements: $\langle \tilde{f}, \varphi_1(\cdot - k)\rangle = c_1[k] = \langle f, \varphi_1(\cdot - k)\rangle, \ \forall k \in \mathbb{Z}$
  - Linearity and integer-shift invariance

➡ Digital filtering solution: $\tilde{f}(x) = \sum_{n\in\mathbb{Z}} \underbrace{(q * c_1)[k]}_{c_2[k]} \varphi_2(x-k)$

Generalized sampling theorem

Cross-correlation sequence: $a_{12}[k] = \langle \varphi_1(\cdot - k), \varphi_2(\cdot) \rangle \quad \xleftrightarrow{\mathcal{F}} \quad A_{12}(e^{j\omega})$

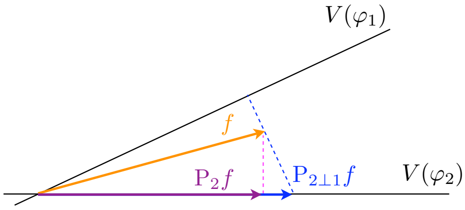
Consistent sampling theorem

Let $A_{12}(e^{j\omega}) \geq m > 0$. Then, there exists a unique solution $\tilde{f} \in V(\varphi_2)$ that is consistent with f in the sense that $c_1[k] = \langle f, \varphi_1(\cdot - k) \rangle = \langle \tilde{f}, \varphi_1(\cdot - k) \rangle$

$$\tilde{f}(x) = \mathrm{P}_{2\perp 1} f(x) = \sum_{n \in \mathbb{Z}} (q * c_1)[k] \varphi_2(x - k) \quad \text{with} \quad Q(z) = \frac{1}{\sum_{k \in \mathbb{Z}} a_{12}[k] z^{-k}}$$

- Geometric interpretation

$\tilde{f} = \mathrm{P}_{2\perp 1} f$ is the projection of f onto $V(\varphi_2)$ perpendicular to $V(\varphi_1)$.



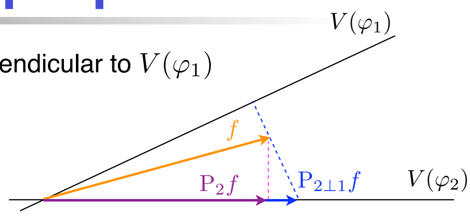
Orthogonality of error:

$$\langle f - \tilde{f}, \varphi_1(\cdot - k) \rangle = \underbrace{\langle f, \varphi_1(\cdot - k) \rangle}_{c_1[k]} - \underbrace{\langle \tilde{f}, \varphi_1(\cdot - k) \rangle}_{c_1[k]} = 0$$

(consistency)

Consistent sampling: properties

$\tilde{f} = \mathrm{P}_{2\perp 1} f$: oblique projection onto $V(\varphi_2)$ perpendicular to $V(\varphi_1)$



- Generalization of Shannon's theorem

  Every signal $f \in V(\varphi_2)$ can be reconstructed exactly

  - Flexibility and realism
    - φ_1 and φ_2 can be selected freely
    - They need not be biorthogonal (unlike wavelet pairs)

- Special case: least-squares approximation

  $\varphi_1 \in V(\varphi_2) \quad \Rightarrow \quad V(\varphi_1) = V(\varphi_2) \quad \Rightarrow \quad \mathrm{P}_{2\perp 1} = \mathrm{P}_2$ (orthogonal projection)

  Minimun-error approximation: $\tilde{f}(x) = \mathrm{P}_2 f(x) = \sum_{k \in \mathbb{Z}} \underbrace{\langle f, \mathring{\varphi}_2(\cdot - k) \rangle}_{(c_1 * q)[k]} \varphi_2(x - k)$

Application 1: interpolation revisited

- Interpolation constraint

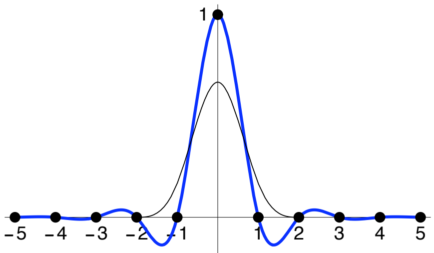
$$c_1[k] = f(x)|_{x=k} = \langle \delta(\cdot - k), f \rangle$$

- Interpolator = consistent ideal sampling system
  - Ideal sampler: $\varphi_1(x) = \delta(x)$
  - Reconstruction function: $\varphi_2(x) = \varphi(x)$
  - Cross-correlation: $a_{12}[k] = \langle \delta(\cdot - k), \varphi(\cdot) \rangle = \varphi(k)$

- Reconstruction/interpolation formula

$$Q_{\rm int}(z) = \frac{1}{\sum_{k\in\mathbb{Z}} \varphi(k) z^{-k}}$$

$$\begin{aligned} f(x) &= \sum_{k\in\mathbb{Z}} \overbrace{(f * q_{\rm int})[k]}^{c[k]} \; \varphi(x-k) \\ &= \sum_{k\in\mathbb{Z}} f[k] \; \varphi_{\rm int}(x-k) \end{aligned}$$



Example: cubic-spline interpolant

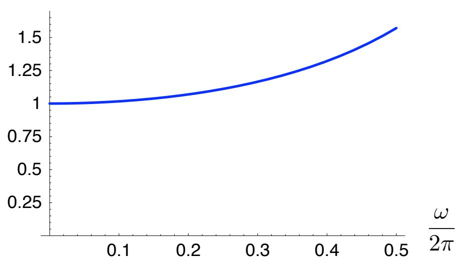
$$\varphi_{\rm int}(x) = \sum_{k\in\mathbb{Z}} q_{\rm int}[k] \; \varphi(x-k)$$

Application 2: consistent image display

- Problem specification
  - Ideal acquisition device: $\varphi_1(x,y) = \mathrm{sinc}(x) \cdot \mathrm{sinc}(y)$
  - LCD display: $\varphi_2(x,y) = \mathrm{rect}(x) \cdot \mathrm{rect}(y)$

- Separable image-enhancement filter

$$A_{12}(e^{j\omega}) = \sum_{n\in\mathbb{Z}} \hat{\varphi}_1^*(\omega + 2\pi n)\hat{\varphi}_2(\omega + 2\pi n) \quad \Rightarrow \quad Q(e^{j\omega}) = \frac{1}{\mathrm{sinc}\left(\frac{\omega}{2\pi}\right)}$$



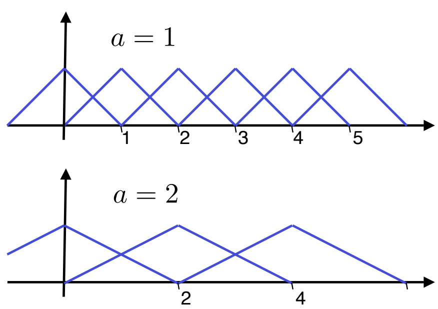
QUANTITATIVE APPROXIMATION THEORY

- Order of approximation
- Fourier-domain prediction of the L_2-error
- Strang-Fix conditions
- Spline case
- Asymptotic form of the error
- Optimized basis functions (MOMS)
- Comparison of interpolators

Order of approximation

- General "shift-invariant" space at scale a

$$V_a(\varphi) = \left\{ s_a(x) = \sum_{k\in\mathbb{Z}} c[k]\varphi\left(\frac{x}{a} - k\right) : c \in \ell_2 \right\}$$



- Projection operator

$$\forall f \in L_2, \qquad \mathrm{P}_a f = \arg \min_{s_a \in V_a} \|f - s_a\|_{L_2}$$

- Order of approximation

Definition

A scaling/generating function φ has order of approximation L iff.

$$\forall f \in W_2^L, \qquad \|f - \mathrm{P}_a f\|_{L_2} \le C \cdot a^L \cdot \|f^{(L)}\|_{L_2}$$

Fourier-domain prediction of the L_2-error

Theorem [Blu-U., 1999]
Let $\mathrm{P}_a f$ denote the orthogonal projection of f onto $V_a(\varphi)$ (at scale a). Then,

$$\forall f \in W_2^s, \qquad \|f - \mathrm{P}_a f\|_{L_2} = \left(\int_{-\infty}^{+\infty} |\hat{f}(\omega)|^2 E_\varphi(a\omega) \frac{d\omega}{2\pi} \right)^{1/2} + o(a^s)$$

where

$$E_\varphi(\omega) = 1 - \frac{|\hat{\varphi}(\omega)|^2}{\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\omega + 2\pi k)|^2}$$

Fourier-transform notation: $\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(x) e^{-j\omega x} \mathrm{d}x$

Strang-Fix conditions of order L

Let $\varphi(x)$ satisfy the Riesz-basis condition. Then, the following Strang-Fix conditions of order L are equivalent:

(1) $\hat{\varphi}(0) = 1$, and $\hat{\varphi}^{(n)}(2\pi k) = 0$ for $\begin{cases} k \neq 0 \\ n = 0 \ldots L-1 \end{cases}$

(2) $\varphi(x)$ reproduces the polynomials of degree $L-1$; i.e., there exist weights $p_n[k]$ such that

$x^n = \sum_{k\in\mathbb{Z}} p_n[k] \varphi(x-k)$, for $n = 0 \ldots L-1$

(3) $E_\varphi(\omega) = \frac{C_L^2}{(2L)!} \cdot \omega^{2L} + O(\omega^{2L+2})$

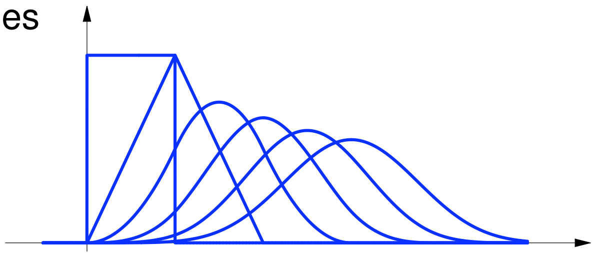
(4) $\forall f \in W_2^L, \qquad \|f - \mathrm{P}_a f\|_{L_2} = O(a^L)$

Polynomial splines

- Basis functions: causal B-splines

$$\beta_+^n(x) = \left(\beta_+^{n-1} * \beta_+^0\right)(x)$$

$$\beta_+^0(x) = \begin{cases} 1, & \text{for } 0 \leq x < 1 \\ 0, & \text{otherwise.} \end{cases}$$



- Fourier-domain formula

$$\hat{\beta}_+^n(\omega) = \left(\frac{1-e^{-j\omega}}{j\omega}\right)^{n+1}$$

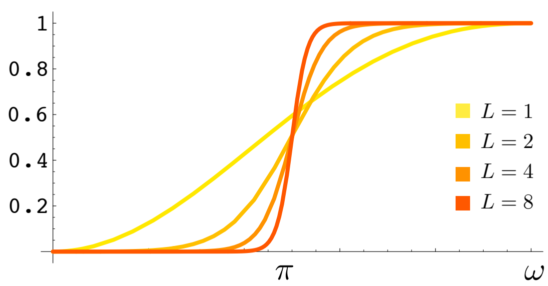
- Order of approximation

$$\hat{\beta}_+^n(2\pi k + \Delta\omega) = O(|\Delta\omega|^{n+1}) \text{ for } k \neq 0$$

$$\Longrightarrow \qquad \beta_+^n \text{ has order of approximation } L = n+1$$

Spline approximation

- Fourier approximation kernel



$$E_{\beta^n}(\omega) = \frac{\sum_{k\neq 0} |\hat{\beta}^n(\omega + 2\pi k)|^2}{\sum_{k\in\mathbb{Z}} |\hat{\beta}^n(\omega + 2\pi k)|^2}$$

Order: $L = n+1$

- Link with Riemann's zeta function

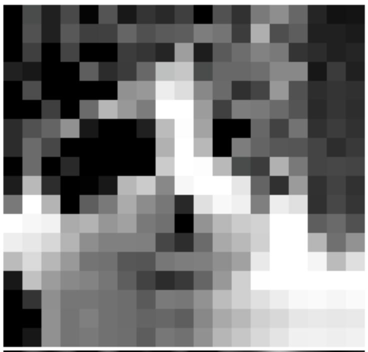
$$\boxed{\zeta(z) = \sum_{n=1}^{+\infty} n^{-z}}$$

$$\begin{aligned} E_{\beta^n}(\omega) &= |2\sin(\omega/2)|^{2n+2} \frac{\sum_{k\neq 0} \frac{1}{|\omega+2\pi k)|^{2n+2}}}{\sum_{k\in\mathbb{Z}} |\hat{\beta}^n(\omega + 2\pi k)|^2} \\ &= \frac{2\zeta(2n+2))}{(2\pi)^{2n+2}} \cdot \omega^{2n+2} + O(|\omega|^{2n+4}) \end{aligned}$$

Spline reconstruction of a PET-scan

Piecewise constant

$L = 1$



Cubic spline

$L = 4$

Asymptotic form of the error

Theorem [U.-Daubechies, 1997]

Let φ be an Lth order function. Then, asymptotically, as $a \to 0$,

$$\forall f \in W_2^L, \qquad \|f - \mathrm{P}_a f\|_{L_2} = C_L \cdot a^L \cdot \|f^{(L)}\|_{L_2}$$

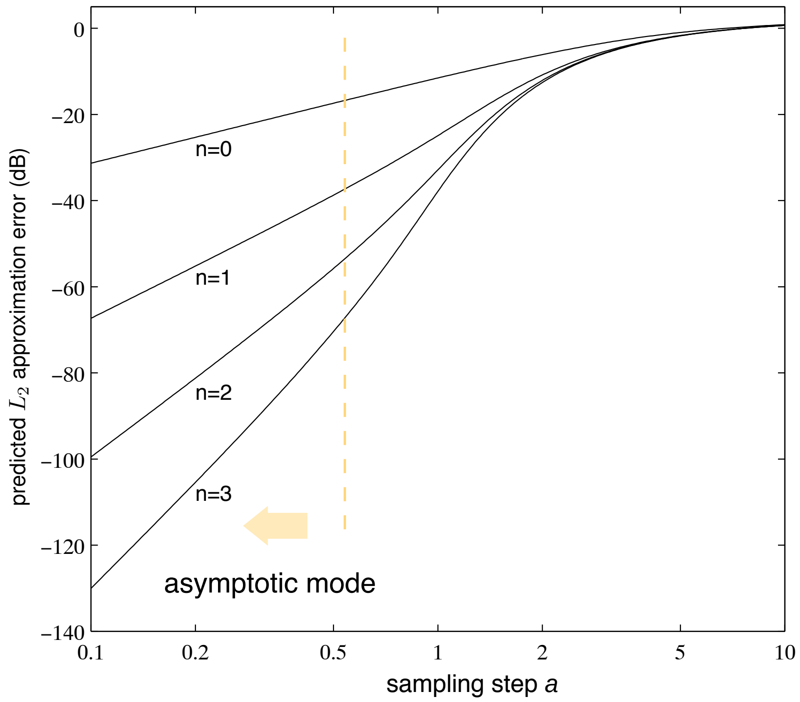
where

$$C_L = \frac{1}{L!}\sqrt{2\sum_{n=1}^{+\infty}\left|\hat{\varphi}^{(L)}(2\pi n)\right|^2} \qquad \left(= \sqrt{\frac{E_\varphi^{(2L)}(0)}{(2L)!}}\right)$$

- Special case: splines of order $L = n + 1$

$$C_{L,\text{splines}} = \frac{\sqrt{2\zeta(2L)}}{(2\pi)^L} = \sqrt{\frac{B_{2L}}{(2L)!}}$$ (Bernoulli number of order $2L$)

Characteristic decay of the error for splines



Least squares approximation of the function $f(x) = e^{-x^2/2}$

Optimized basis functions (MOMS)

- Motivation
  - Cost of prefiltering is negligible (in 2D and 3D)
  - Computational cost depends on kernel size W
  - Order of approximation is a strong determinant of quality

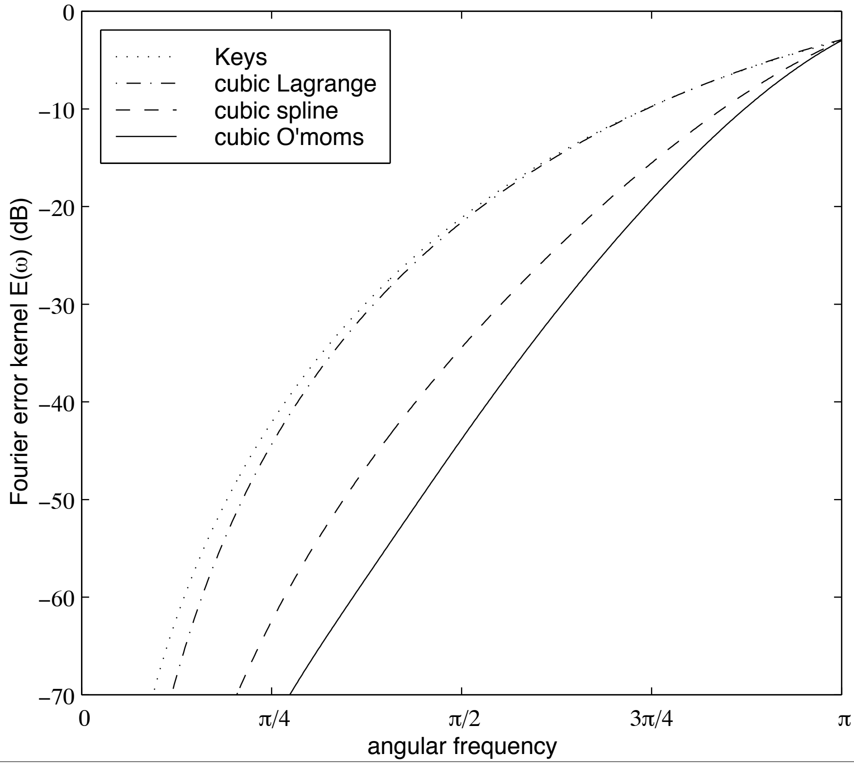
QUESTION: What are the basis functions with maximum order of approximation and minimum support ?

ANSWER: Shortest functions of order L (MOMS) $\varphi_{\text{moms}}(x) = \sum_{k=0}^{L-1} a_k \mathrm{D}^k \beta^{L-1}(x)$

- Most interesting MOMS
  - B-splines: smoothest ($\beta^{L-1} \in \dot{C}^{L-1}$) and only refinable MOMS
  - Shaum's piecewise-polynomial interpolants (no prefilter)
  - OMOMS: smallest approximation constant C_L

$$\varphi^3_{\text{opt}}(x) = \beta^3(x) + \frac{1}{42}\frac{\mathrm{d}^2\beta^3(x)}{\mathrm{d}x^2}$$

Comparisons of cubic interpolators of size *W*=4



INTERPOLATION IN THE PRESENCE OF NOISE

- Interpolation and regularization
- Smoothing splines
- General concept of an L-spline
- Optimal Wiener-like estimators

Spline-fitting with noisy data

- Context
  - Input data $\{f[k]\}_{k\in\mathbb{Z}}$ corrupted by noise
  - Model: continuously defined function $s(x)$
  - Data term: $\xi_{\mathrm{data}} = \sum_{k\in\mathbb{Z}} |f[k] - s(k)|^2$ (discrete domain)
  - Spline energy: $\xi_{\mathrm{spline}} = \|\mathrm{D}^m s\|^2_{L_2}$ (continuous domain)

- Possible formulations

$$\min_{s(x)\in L_2} \xi_{\mathrm{spline}} \quad \text{subject to } \xi_{\mathrm{data}} = C_1$$

A priori knowledge of noise variance

OR

$$\min_{s(x)\in L_2} \xi_{\mathrm{data}} \quad \text{subject to } \xi_{\mathrm{spline}} \leq C_2$$

A priori knowledge of signal class

$\Updownarrow$

$$\min_{s(x)\in L_2} \{\xi_{\mathrm{data}} + \lambda\, \xi_{\mathrm{spline}}\}$$

(Tikhonov-like regularization)

Lagrange multiplier

Regularized fit: smoothing splines

- B-spline representation: $s(x) = \sum_{k\in\mathbb{Z}} c[k]\beta^n(x-k)$

- Smoothing splines

Discrete, noisy input:

$f[k] = s(k) + n[k]$ $\longrightarrow$ Smoothing algorithm $\longrightarrow$ $c[k]$

Theorem: The solution (among all functions) of the smoothing spline problem

$$\min_{s(x)} \left\{ \sum_{k\in\mathbb{Z}} |f[k] - s(k)|^2 + \lambda \int_{-\infty}^{+\infty} |\mathrm{D}^m s(x)|^2 dx \right\}$$

is a cardinal spline of degree $2m-1$. Its coefficients $c[k] = h_\lambda * f[k]$ can be obtained by suitable recursive digital filtering of the input samples $f[k]$.

- Special case: the draftman's spline

The minimum-curvature interpolant is obtained by setting $m = 2$ and $\lambda \to 0$.

It is a cubic spline !

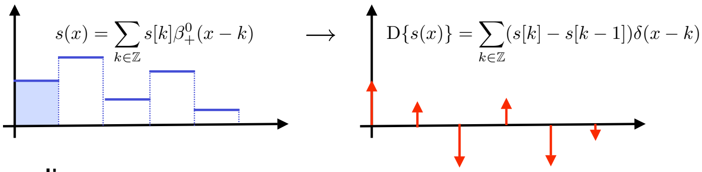
General concept of an L-spline

$\mathrm{L}\{\cdot\}$: differential operator (shift-invariant) $\delta(x)$: Dirac distribution

Definition

The function $s(x)$ is a ***cardinal* L-*spline*** (with knots at the integers) iff.

$$\mathrm{L}\{s(x)\} = \sum_{k\in\mathbb{Z}} a[k]\delta(x-k)$$



- Special cases
  - Piecewise-constant = D-splines
  - Polynomial splines = D^{n+1}-splines

    Justification:
    $\mathrm{D}^{n+1}\{\beta_+^n(x)\} = \Delta_+^{n+1}\{\delta(x)\} = \sum_{k\in\mathbb{Z}} d[k]\delta(x-k) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad D(e^{j\omega}) = (1-e^{-j\omega})^{n+1}$

Existence of B-spline-like bases

$\mathrm{L}\{\cdot\}$: generalized differential operator of order $s > \frac{1}{2}$

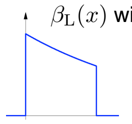
- Riesz-basis representation

  Cardinal L-splines generally admit a B-spline-like representation

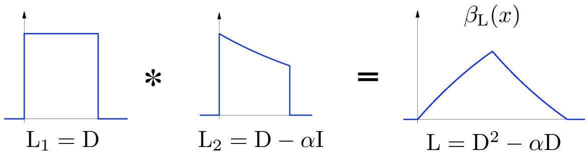
  $$s(x) = \sum_{k\in\mathbb{Z}} c[k]\beta_{\mathrm{L}}(x-k)$$

  $\beta_{\mathrm{L}}(x)$ with $\mathrm{L} = \mathrm{D} - \alpha\mathrm{I}$

  Example: first-order exponential B-spline



- Composition properties



  - Higher-order B-splines:
    $\beta_{\mathrm{L}_1}(x)$ and $\beta_{\mathrm{L}_2}(x)$ are B-spline generators for the cardinal L_1- and L_2-splines.
    Then, $\beta_{\mathrm{L}_1}(x) * \beta_{\mathrm{L}_2}(x)$ is a generator for the $(\mathrm{L}_1\mathrm{L}_2)$-splines.
  - Positive-definite operators: If $\beta_{\mathrm{L}}(x)$ generates a Riesz basis for the L-splines, then $\varphi(x) = \beta_{\mathrm{L}}(x) * \beta_{\mathrm{L}}(-x)$ generates a Riesz basis for the $(\mathrm{L}^*\mathrm{L})$-splines and the interpolation problem in $V(\varphi)$ is well posed.

Generalized smoothing splines

- Generalized spline energy: $\xi_{\text{spline}} = \|\mathrm{L}s\|_{L_2}^2$
- Generalized smoothing-spline fit

Theorem: The solution (among all functions) of the generalized smoothing problem

$$\min_{s(x)} \left\{ \sum_{k\in\mathbb{Z}} |f[k] - s(k)|^2 + \lambda \int_{-\infty}^{+\infty} |\mathrm{L}s(x)|^2 dx \right\}$$

is a cardinal $\mathrm{L}^*\mathrm{L}$-spline.

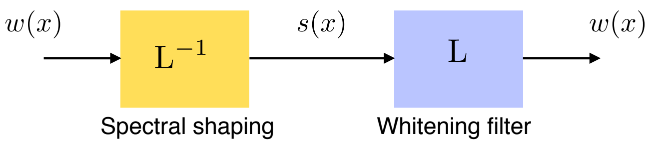
The solution has a B-spline representation $s_\lambda(x) = \sum_{k\in\mathbb{Z}} c[k]\varphi(x-k)$, the coefficients of which are obtained by suitable filtering of the input data (generalized smoothing-spline algorithm).

Stochastic signal models

- Wide-sense stationary process
  - Realization of the stochastic process: $s(x)$
  - Zero-mean: $E\{s(x)\} = 0$
  - Autocorrelation function: $E\{s(y) \cdot s(y-x)\} = c_s(x) \in L_2$
  - Spectral density function: $C_s(\omega) = \int_{x\in\mathbb{R}} c_s(x) e^{-j\omega x} \mathrm{d}x \in L_2$
- Stochastic differential equation

$\mathrm{L}\{s(x)\} = w(x)$ (driven by white Gaussian noise)

$c_w(x) = \sigma_0^2 \delta(x)$



$C_w(\omega) = \sigma_0^2$

$$C_s(\omega) = \frac{\sigma_0^2}{|\hat{L}(\omega)|^2}$$

MMSE estimation in the presence of noise

- Statistical hypotheses
  - Discrete measurements (signal + noise): $f[k] = s(k) + n[k]$
  - Signal autocorrelation: $c_s(x)$ such that $\mathrm{L}^*\mathrm{L}\{c_s(x)\} = \sigma_0^2 \cdot \delta(x)$
  - Discrete white noise with variance $\sigma^2 \Rightarrow c_n[k] = \sigma^2 \cdot \delta[k]$

- MMSE continuous-domain signal estimation

Theorem
Under the above assumptions, the linear Minimum-Mean Square Error Estimator of $s(x)$ at position $x = x_0$, given the measurements $\{f[k]\}_{k\in\mathbb{Z}}$, is $s_\lambda(x_0)$ with $\lambda = \frac{\sigma^2}{\sigma_0^2}$, where $s_\lambda(x)$ is the $\mathrm{L}^*\mathrm{L}$-smoothing-spline fit of $\{f[k]\}_{k\in\mathbb{Z}}$ given by the generalized smoothing-spline algorithm.

Remark: optimal overall estimators if one adds the assumption of Gaussianity

CONCLUSION

- Generalized sampling
  - Unifying Hilbert-space formulation: Riesz basis, etc.
  - Approximation point of view:
    projection operators (oblique vs. orthogonal)
  - Increased flexibility; closer to real-world systems
  - Generality: nonideal sampling, interpolation, etc...
- Quest for the “optimal” representation space
  - Not bandlimited ! (prohibitive cost, ringing, etc.)
  - Quantitative approximation theory: L_2-estimates, asymptotics
  - Optimized functions: MOMS
  - Signal-adapted design ?
- Interpolation/approximation in the presence of noise
  - Regularization theory: smoothing splines
  - Stochastic formulation: new, hybrid form of Wiener filter

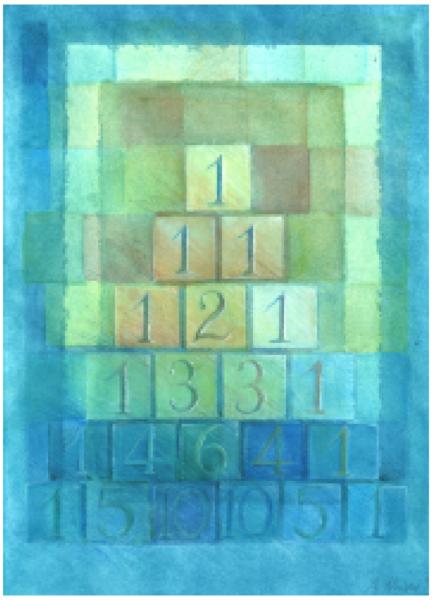
Acknowledgments

Many thanks to

- Dr. Thierry Blu
- Prof. Akram Aldroubi
- Prof. Ingrid Daubechies
- Annette Unser, Artist

\+ many other researchers,

and graduate students



- Software and demos at: http://bigwww.epfl.ch/

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