Dear Dr. Liebling,

I am pleased to inform you that you were selected to receive the 2004 Research Award of the Swiss Society of Biomedical Engineering for your thesis work “On Fresnelets, interference fringes, and digital holography”. The award will be presented during the general assembly of the SSBE, September 3, Zurich, Switzerland.

Please, let us know if:
1) you will be present to receive the award,
2) you would be willing to give a 10 minutes presentation of the work during the general assembly.

The award comes with a cash prize of 1000.- CHF. Would you please send your banking information to the treasurer of the SSBE, Uli Diermann (Email: uli.diermann@bfh.ch), so that he can transfer the cash prize to your account?

I congratulate you on your achievement.

With best regards,

Michael Unser, Professor
Chairman of the SSBE Award Committee

cc: Ralph Mueller, president of the SSBE; Uli Diermann, treasurer
Invariant signals

- Natural signals/images often exhibit some degree of invariance (at least locally, if not globally)
  - Stationarity, texture: T-invariance
  - Isotropy (no preferred orientation): R-invariance
  - Self-similarity, fractality: S-invariance (Pentland 1984; Mumford, 2001)

OUTLINE

- Splines and T-invariant operators
  - Green functions as elementary building blocks
  - Existence of a local B-spline basis
  - Link with stochastic processes
- Imposing affine (TSR) invariance
  - Fractional Laplace operator & polyharmonic splines
  - Fractal processes
- Laplacian-like, quasi-isotropic wavelets
  - Polyharmonic spline wavelet bases
  - Analysis of fractal processes
- The Marr wavelet
  - Complex Laplace/gradient operator
  - Steerable complex wavelets
  - Wavelet primal sketch
  - Directional wavelet analysis
General concept of an L-spline

L\{\cdot\}: differential operator (translation-invariant)
\delta(x) = \prod_{i=1}^{d} \delta(x_i): multidimensional Dirac distribution

Definition
The continuous-domain function \( s(x) \) is a **cardinal L-spline** iff.

\[
L\{s\}(x) = \sum_{k \in \mathbb{Z}^d} a[k] \delta(x - k)
\]

- Cardinality: the knots (or spline singularities) are on the (multi-)integers
- Generalization: includes polynomial splines as particular case (\( L = \frac{d^N}{dx^N} \))

Example: piecewise-constant splines

- Spline-defining operators
  
  Continuous-domain derivative:
  \[
  D = \frac{d}{dx} \longleftrightarrow j\omega
  \]
  Discrete derivative:
  \[
  \Delta_+\{\cdot\} \longleftrightarrow 1 - e^{-j\omega}
  \]

- Piecewise-constant or D-spline

\[
s(x) = \sum_{k \in \mathbb{Z}} s[k] \beta_0^0(x - k)
\]

\[
D\{s\}(x) = \sum_{k \in \mathbb{Z}} a[k] \delta(x - k)
\]

- B-spline function

\[
\beta_0^0(x) = \Delta_+ D^{-1}\{\delta\}(x) \longleftrightarrow \frac{1 - e^{-j\omega}}{j\omega}
\]
Splines and Green’s functions

Definition

\( \rho(x) \) is a Green function of the shift-invariant operator \( L \) iff \( L\{\rho\} = \delta \)

\[
\rho(x) \xrightarrow{L\{\cdot\}} \delta(x) \quad \Rightarrow \quad \delta(x) \xrightarrow{L^{-1}\{\cdot\}} \rho(x)
\]

(\(+\) null-space component?)

Cardinal \( L \)-spline:

\[
L\{s\}(x) = \sum_{k \in \mathbb{Z}^d} a[k] \delta(x - k)
\]

Formal integration

\[
\sum_{k \in \mathbb{Z}^d} a[k] \delta(x - k) \xrightarrow{L^{-1}\{\cdot\}} s(x) = \sum_{k \in \mathbb{Z}^d} a[k] \rho(x - k)
\]

\[
\Rightarrow \quad V_L = \text{span}\{\rho(x - k)\}_{k \in \mathbb{Z}^d}
\]

Example of spline synthesis

\[
L = \frac{d}{dx} = D \quad \Rightarrow \quad L^{-1}: \text{integrator}
\]

\[
\delta(x) \quad \xrightarrow{L^{-1}\{\cdot\}} \quad \rho(x)
\]

Green function = Impulse response

Translation invariance

\[
\delta(x - x_0) \quad \xrightarrow{L^{-1}\{\cdot\}} \quad \rho(x - x_0)
\]

Linearity

\[
\sum_{k \in \mathbb{Z}} a[k] \delta(x - k) \quad \xrightarrow{L^{-1}\{\cdot\}} \quad s(x) = \sum_{k \in \mathbb{Z}} a[k] \rho(x - k)
\]
Existence of a local, shift-invariant basis?

- Space of cardinal L-splines
  \[ V_L = \left\{ s(x) : L\{s\}(x) = \sum_{k \in \mathbb{Z}^d} a[k] \delta(x - k) \right\} \cap L_2(\mathbb{R}^d) \]

- Generalized B-spline representation
  A “localized” function \( \varphi(x) \in V_L \) is called generalized B-spline if it generates a Riesz basis of \( V_L \); i.e., iff there exists \( (A > 0, B < \infty) \) s.t.
  \[
  A \cdot \|c\|_{\ell_2(\mathbb{Z}^d)} \leq \left\| \sum_{k \in \mathbb{Z}^d} c[k] \varphi(x - k) \right\|_{L_2(\mathbb{R}^d)} \leq B \cdot \|c\|_{\ell_2(\mathbb{Z}^d)}
  \]

  \[
  V_L = \left\{ s(x) = \sum_{k \in \mathbb{Z}^d} c[k] \varphi(x - k) : x \in \mathbb{R}^d, c \in \ell_2(\mathbb{Z}^d) \right\}
  \]

Link with stochastic processes

Splines are in direct correspondence with stochastic processes (stationary or fractals) that are solution of the same partial differential equation, but with a random driving term.

- Defining operator equation: \( L\{s(\cdot)\}(x) = r(x) \)

  - Specific driving terms
    - \( r(x) = \delta(x) \quad \Rightarrow \quad s(x) = L^{-1}\{\delta\}(x) \) : Green function
    - \( r(x) = \sum_{k \in \mathbb{Z}^d} a[k] \delta(x - k) \quad \Rightarrow \quad s(x) : \text{Cardinal L-spline} \)
    - \( r(x) \) : white Gaussian noise \( \Rightarrow \) \( s(x) \) : generalized stochastic process

  non-empty null space of \( L \), boundary conditions

References: stationary proc. (U.-Blu, IEEE-SP 2006), fractals (Blu-U., IEEE-SP 2007)
Example: Brownian motion vs. spline synthesis

\[ L = \frac{d}{dx} \Rightarrow L^{-1}: \text{integrator} \]

white noise or stream of Diracs

\[ L^{-1}\{\cdot\} \]

Brownian motion

Cardinal spline (Schoenberg, 1946)

---

**IMPOSING SCALE INVARINANCE**

- Affine-invariant operators
- Polyharmonic splines
- Associated fractal random fields: fBms
Scale- and rotation-invariant operators

**Definition:** An operator $L$ is affine-invariant (or SR-invariant) iff.

$$\forall s(x), \ L\{s(\cdot)\}(R_{\theta}x/a) = C_a \cdot L\{s(R_{\theta} \cdot /a)\}(x)$$

where $R_{\theta}$ is an arbitrary $d \times d$ unitary matrix and $C_a$ a constant.

- **Invariance theorem**

  The complete family of real, scale- and rotation-invariant convolution operators is given by the fractional Laplacians

  $$(-\Delta)^{\frac{\gamma}{2}} \quad \leftrightarrow \quad \|\omega\|^\gamma$$

- **Invariant Green functions (a.k.a. RBF)**

  $$(\text{Duchon, 1979})$$

  $$\rho(x) = \begin{cases} 
  \|x\|^{\gamma-d} \log \|x\|, & \text{if } \gamma - d \text{ is even} \\
  \|x\|^{\gamma-d}, & \text{otherwise}
  \end{cases}$$
Polyharmonic splines

Spline functions associated with fractional Laplace operator \((-\Delta)^{\gamma/2}\)

- Distributional definition
  
  \[ s(x) \text{ is a cardinal polyharmonic spline of order } \gamma \text{ iff.} \]
  
  \[ (-\Delta)^{\gamma/2}s(x) = \sum_{k \in \mathbb{Z}^d} d[k] \delta(x - k) \]

- Explicit Shannon-like characterization

  \[ \mathcal{V}_0 = \left\{ s(x) = \sum_{k \in \mathbb{Z}^d} s[k] \phi_\gamma(x - k) \right\} \]

  \[ \phi_\gamma(x): \text{ Unique polyharmonic spline interpolator s.t. } \phi_\gamma(k) = \delta_k \]

  \[ \mathcal{F} \quad \phi_\gamma(\omega) = \frac{1}{1 + \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \left( \frac{\|\omega\|}{\|\omega + 2\pi k\|} \right)^\gamma} \]

Construction of polyharmonic B-splines

Laplacian operator: \(\Delta \leftrightarrow -\|\omega\|^2\)

Discrete Laplacian: \(\Delta_d \leftrightarrow -\sum_{i=1}^{d} 4 \sin^2(\omega_i/2) \hat{\Delta} - \|2 \sin(\omega/2)\|^2\)

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- Polyharmonic B-splines (Rabut, 1992)

  Discrete operator: localization filter \(Q(\cos \omega)\)

  \[
  \|2 \sin(\omega/2)\|^\gamma \quad \mathcal{F}^{-1} \quad \beta_\gamma(x)
  \]

  Continuous-domain operator: \(\hat{L}(\omega)\)
Polyharmonic B-splines properties

- Stable representation of polyharmonic splines (Riesz basis)
  \[ V_{(-\Delta)^\frac{\gamma}{2}} = \left\{ s(x) = \sum_{k \in \mathbb{Z}^d} c[k] \beta_\gamma(x - k) : c[k] \in \ell_2(\mathbb{Z}^d) \right\} \text{ Condition: } \gamma > \frac{d}{2} \]

- Two-scale relation: \( \beta_\gamma(x/2) = \sum_{k \in \mathbb{Z}^d} h_\gamma[k] \beta_\gamma(x - k) \)

- Order of approximation \( \gamma \) (possibly fractional)

- Reproduction of polynomials
  The polyharmonic B-splines \( \{ \varphi_\gamma(x - k) \}_{k \in \mathbb{Z}^d} \) reproduce the polynomials of degree \( n = \lceil \gamma - 1 \rceil \). In particular,
  \[ \sum_{k \in \mathbb{Z}^d} \varphi_\gamma(x - k) = 1 \]  
  (partition of unity)  
  (Rabut, 1992; Van De Ville, 2005)

Associated random field: multi-D fBm

Formalism: Gelfand’s theory of generalized stochastic processes

White noise \[ (-\Delta)^{\frac{\gamma}{2}} \]

fractional Brownian field

\[ (-\Delta)^{\frac{\gamma}{2}} \]

fractional integrator (appropriate boundary conditions)

Whitening (fractional Laplacian)

Hurst exponent: \( H = \gamma - \frac{d}{2} \)

(Tafti et al., IEEE-IP 2009)
LAPLACIAN-LIKE WAVELET BASES

- Operator-like wavelet design
- Fractional Laplacian-like wavelet basis
- Improving shift-invariance and isotropy
- Wavelet analysis of fractal processes (multidimensional generalization of pioneering work of Flandrin and Abry)

Multiresolution analysis of $L_2(\mathbb{R}^d)$

- Multiresolution basis functions: $\phi_{i,k}(x) = 2^{-id/2} \phi \left( \frac{x - 2^i k}{2^i} \right)$
- Subspace at resolution $i$: $V_i = \text{span} \{ \phi_{i,k} \}_{k \in \mathbb{Z}^d}$

Two-scale relation $\Rightarrow V_i \subset V_j$, for $i \geq j$

Partition of unity $\Leftarrow \bigcup_{i \in \mathbb{Z}} V_i = L_2(\mathbb{R}^d)$
General operator-like wavelet design

Search for a single wavelet that generates a basis of $L_2(\mathbb{R}^d)$ and that is a multi-scale version of the operator $L$; i.e., $\psi = L^* \phi$ where $\phi$ is a suitable smoothing kernel

- General operator-based construction
  - Basic space $V_0$ generated by the integer shifts of the Green function $\rho$ of $L$:
    \[ V_0 = \text{span}\{\rho(x-k)\}_{k \in \mathbb{Z}^d} \text{ with } L\rho = \delta \]
  - Orthogonality between $V_0$ and $W_0 = \text{span}\{\psi(x - \frac{1}{2}k)\}_{k \in \mathbb{Z}^d \setminus 2\mathbb{Z}^d}$
    \[ \langle \psi(\cdot - x_0), \rho(\cdot - k) \rangle = \langle \phi, L\rho(\cdot - k + x_0) \rangle = \langle \phi, \delta(\cdot - k + x_0) \rangle = \phi(k - x_0) = 0 \]
    (can be enforced via a judicious choice of $\phi$ (interpolator) and $x_0$)
  - Works in arbitrary dimensions and for any dilation matrix $D$

---

Fractional Laplacian-like wavelet basis

$\psi_\gamma(x) = (-\Delta)^{\frac{\gamma}{2}} \phi_{2\gamma}(Dx)$

$\phi_{2\gamma}(x)$: polyharmonic spline interpolator of order $2\gamma > 1$

$D$: admissible dilation matrix

- Wavelet basis functions: $\psi_{(i,k)}(x) \triangleq |\det(D)|^{1/2} \psi_\gamma(D^T x - D^{-1} k)$
- $\{\psi_{(i,k)}\}_{(i \in \mathbb{Z}, k \in \mathbb{Z}^d \setminus \mathbb{D}^2)}$ forms a semi-orthogonal basis of $L_2(\mathbb{R}^2)$
  \[ \forall f \in L_2(\mathbb{R}^2), \quad f = \sum_{i \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d \setminus \mathbb{D}^2} \langle f, \psi_{(i,k)} \rangle \tilde{\psi}_{(i,k)} = \sum_{i \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d \setminus \mathbb{D}^2} \langle f, \tilde{\psi}_{(i,k)} \rangle \psi_{(i,k)} \]
  where $\{\tilde{\psi}_{(i,k)}\}$ is the dual wavelet basis of $\{\psi_{(i,k)}\}$
- The wavelets $\psi_{(i,k)}$ and $\tilde{\psi}_{(i,k)}$ have $\lceil \gamma \rceil$ vanishing moments
- The wavelet analysis implements a multiscale version of the Laplace operator and is perfectly reversible (one-to-one transform)
- The wavelet transform has a fast filterbank algorithm (based on FFT)

[Van De Ville, IEEE-IP, 2005]
Laplacian-like wavelet decomposition

- Nonredundant transform
  \[ f(x) \]

\[ d_i[k] = \langle f, \psi_{i,k} \rangle \]

Dyadic sampling pattern

first decomposition level (one-to-one)

\[ \beta_\gamma(D^{-1}x - k) \]

scaling functions (dilated by 2)

\[ \psi_\gamma(D^{-1}x - k) \]

wavelets (dilated by 2)

Improving shift-invariance and isotropy

- Wavelet subspace at resolution \( i \)
  \[ \mathcal{W}_i = \text{span} \{ \psi_{i,k} \}_{k \in \mathbb{Z}^2 \setminus \mathbb{DZ}^2} \]

- Augmented wavelet subspace at resolution \( i \)
  \[ \mathcal{W}^+_i = \text{span} \{ \psi_{i,k} \}_{k \in \mathbb{Z}^2} = \text{span} \{ \psi_{\text{iso},i,k} \}_{k \in \mathbb{Z}^2} \]

- Admissible polyharmonic spline wavelets

  - Operator-like generator: \[ \psi_\gamma(x) = (-\Delta)^{\gamma/2} \phi_{2\gamma}(Dx) \]

  - More isotropic wavelet: \[ \psi_{\text{iso}}(x) = (-\Delta)^{\gamma/2} \beta_{2\gamma}(Dx) \]

  - "Quasi-isotropic" polyharmonic B-spline \[ \beta_{2\gamma}(x) \to C_{\gamma} \exp(-\|x\|^2/(\gamma/6)) \]

Wavelet sampling patterns:

- Non-redundant (basis)
- Mildly redundant (frame)

[Van De Ville, 2005]
### Building Mexican-Hat-like wavelets

\[ \psi_{\gamma}(x) = (-\Delta)^{\gamma/2} \phi_{2\gamma}(2x) \]

\[ \psi_{\text{iso}}(x) = (-\Delta)^{\gamma/2} \beta_{2\gamma}(2x) \]

Gaussian-like smoothing kernel: \( \beta_{2\gamma}(2x) \)

### Mexican-Hat multiresolution analysis

- Pyramid decomposition: redundancy \( 4/3 \)

\[ \psi_{\text{iso}}(x) = (-\Delta)^{\gamma/2} \beta_{2\gamma}(Dx) \]

Dyadic sampling pattern

First decomposition level:

\[ \varphi(D^{-1}(x - k)) \quad \psi(D^{-1}(x - k)) \]

Scaling functions

Wavelets

(redundant by \( 4/3 \))

(U.-Van De Ville, *IEEE-IP* 2008)
Wavelet analysis of fBm: whitening revisited

- Operator-like behavior of wavelet
  - Analysis wavelet: \( \psi_\gamma = (\Delta)^{\frac{d}{2}} \phi(x) = (\Delta)^{\frac{d}{2} + \frac{d}{2}} \psi'_\gamma(x) \)
  - Reduced-order wavelet: \( \psi'_\gamma(x) = (-\Delta)^{-\frac{d}{2}} \phi(x) \) with \( \gamma' = \gamma - (H + \frac{d}{2}) > 0 \)

- Stationarizing effect of wavelet analysis
  - Analysis of fractional Brownian field with exponent \( H \):
    \[
    \langle B_H, \psi_\gamma (\frac{\cdot - x_0}{a}) \rangle \propto \langle (-\Delta)^{\frac{d}{2} + \frac{d}{2}} B_H, \psi'_\gamma (\frac{\cdot - x_0}{a}) \rangle = \langle W, \psi'_\gamma (\frac{\cdot - x_0}{a}) \rangle
    \]
  - Equivalent spectral noise shaping: \( S_{\text{wave}}(e^{j\omega}) = \sum_{n \in \mathbb{Z}} |\hat{\psi}'_\gamma(\omega + 2\pi n)|^2 \)
    \( \Rightarrow \) Extent of wavelet-domain whitening depends on flatness of \( S_{\text{wave}}(e^{j\omega}) \)
  - "Whitening" effect is the same at all scales up to a proportionality factor
    \( \Rightarrow \) fractal exponent can be deduced from the log-log plot of the variance

(Tafti et al., IEEE-IP 2009)

Wavelet analysis of fractional Brownian fields

Theoretical scaling law: \( \text{Var}\{w_a[k]\} = \sigma_0^2 \cdot a^{(2H+d)} \)

log-log plot of variance

H_est = 0.31
Fractals in bioimaging: fibrous tissue

DDSM: University of Florida

(Digital Database for Screening Mammography)

(Laine, 1993; Li et al., 1997)

Wavelet analysis of mammograms

Fractal dimension: $D = 1 + d - H = 2.56$ with $d = 2$ (topological dimension)
Brain as a biofractal

(Bullmore, 1994)

Courtesy R. Mueller ETHZ

Wavelet analysis of fMRI data

Brain: courtesy of Jan Kybic

Fractal dimension: \( D = 1 + d - H = 2.65 \) with \( d = 2 \) (topological dimension)
THE MARR WAVELET

- Laplace/gradient operator
- Steerable Marr wavelets
- Wavelet primal sketch
- Directional wavelet analysis
**Complex TRS-invariant operators in 2D**

- **Invariance theorem**
  The complete family of complex, translation-, scale- and rotation-invariant 2D operators is given by the fractional complex Laplace-gradient operators

\[
L_{\gamma, N} = (-\Delta)^{\frac{\gamma-N}{2}} \left( \frac{\partial}{\partial x_1} + j \frac{\partial}{\partial x_2} \right)^N 
\]

with \( N \in \mathbb{N} \) and \( \gamma \geq N \in \mathbb{R} \)

- **Key property: steerability**

\[
L_{\gamma, N}\{\delta\}(R_{\theta}x) = e^{jN\theta} L_{\gamma, N}\{\delta\}(x)
\]

**Simplifying the maths: Unitary Riesz mapping**

- **Complex Laplace-gradient operator**

\[
L_{\gamma, N} = (-\Delta)^{\frac{\gamma-N}{2}} \left( \frac{\partial}{\partial x_1} + j \frac{\partial}{\partial x_2} \right)^N = (-\Delta)^{\frac{\gamma-N}{2}} \mathcal{R}^N
\]

where \( \mathcal{R} = L_{\frac{1}{2}, 1} \leftrightarrow \left( \frac{j\omega_1 - \omega_2}{\|\omega\|} \right) \)

- **Property of Riesz operator \( \mathcal{R} \)**
  - \( \mathcal{R} \) is shift- and scale-invariant
  - \( \mathcal{R} \) is rotation covariant (a.k.a. steerable)
  - \( \mathcal{R} \) is unitary
    in particular, \( \mathcal{R} \) will map a Laplace-like wavelet basis into a complex Marr-like wavelet basis
Complex Laplace-gradient wavelet basis

\[
\psi'_\gamma(x) = (-\Delta)^{-\frac{\gamma}{2}} \left( \frac{\partial}{\partial x_1} + j \frac{\partial}{\partial x_2} \right) \phi_{2\gamma}(Dx) = R\psi_\gamma(x)
\]

\(\phi_{2\gamma}(x)\): polyharmonic spline interpolator of order \(2\gamma > 1\)

\(D\): admissible dilation matrix

- Wavelet basis functions: \(\psi'_{(i,k)}(x) = R\psi'_{(i,k)}(x) = |\det(D)|^{i/2}\psi'_\gamma(D^i x - D^{-1} k)\)

- \(\{\psi'_{(i,k)}\}_{i\in\mathbb{Z}, k\in\mathbb{Z}^2\setminus D\mathbb{Z}^2}\) forms a complex semi-orthogonal basis of \(L_2(\mathbb{R}^2)\)

\[
\forall f \in L_2(\mathbb{R}^2), \quad f = \sum_{i\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^2\setminus D\mathbb{Z}^2} \langle f, \psi'_{(i,k)} \rangle \hat{\psi}'_{(i,k)} = \sum_{i\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^2\setminus D\mathbb{Z}^2} \langle f, \hat{\psi}'_{(i,k)} \rangle \psi'_{(i,k)}
\]

where \(\{\hat{\psi}'_{(i,k)}\}\) is the dual wavelet basis of \(\{\psi'_{(i,k)}\}\)

- The wavelet analysis implements a multiscale version of the Gradient-Laplace (or Marr) operator and is perfectly reversible (one-to-one transform)

- The wavelet transform has a fast filterbank algorithm

[Van De Ville-U., IEEE-IP, 2008]

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Wavelet frequency responses

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<th>Laplacian-like / Mexican hat</th>
<th>Marr pyramid (steerable)</th>
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<td>(\hat{\psi}_{\text{iso},i}(\omega))</td>
<td>(\hat{\psi}_{\text{Re},i}(\omega))</td>
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<tr>
<td>(\hat{\psi}_{\text{Im},i}(\omega))</td>
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</table>

Unitary mapping (Riesz transform)

\(\gamma = 4\)
Marr wavelet pyramid

- Steerable pyramid-like decomposition: redundancy $2 \times \frac{4}{3}$

$$\psi_{3\mu}(x) = \frac{\partial}{\partial x} \Delta \beta_2(2x)$$

$$\psi_{\iota 2\gamma}(x) = \frac{\partial}{\partial y} \Delta \beta_2(2x)$$

Basic dyadic sampling cell

Steerable pyramid-like decomposition: redundancy $2 \times 4 \times 3$

Overcomplete by $\frac{1}{3}$

Wavelet basis

Processing in early vision - primal sketch

- Wavelet primal sketch
  - blurring — smoothing kernel $\phi$
  - Laplacian filtering — $\Delta$
  - zero-crossings and orientation — $\nabla$
  - segment detection and grouping — Canny edge detection scheme
Edge detection in wavelet domain

- Edge map (using Canny’s edge detector)
  - Key visual information (Marr’s theory of vision)

Similar to Mallat’s representation from wavelet modulus maxima [Mallat-Zhong, 1992]

... but much less redundant!

Iterative reconstruction

- Reconstruction from information on edge map only
  - Better than 30dB PSNR

31.4dB

[Van De Ville-U., IEEE-IP, 2008]
Directional wavelet analysis: Fingerprint

Wavelet-domain structure tensor

\[ \gamma = 2, \sigma = 2 \]

Marr wavelet pyramid - discussion

- Comparison against state-of-the-art

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<th>Marr wavelet pyramid</th>
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[Simoncelli, Freeman, 1995] [Kingsbury, 2001] [Selesnick et al, 2005]
CONCLUSION

- Unifying operator-based paradigm
  - Operator identification based on invariance principles (TSR)
  - Specification of corresponding spline and wavelet families
  - Characterization of stochastic processes (fractals)

- Isotropic and steerable wavelet transforms
  - Riesz basis, analytical formulae
  - Mildly redundant frame extension for improved TR invariance
  - Fractal and/or directional analyses
  - Fast filterbank algorithm (fully reversible)

- Marr wavelet pyramid
  - Multiresolution Marr-type analysis; wavelet primal sketch
  - Reconstruction from multiscale edge map

- Implementation will be available very soon (Matlab)

References


EPFL’s Biomedical Imaging Group

Preprints and demos: [http://bigwww.epfl.ch/](http://bigwww.epfl.ch/)