Description of the Richardson-Lucy Algorithm

Cédric Vonesch*,
* Ecole Polytechnique Fédérale de Lausanne (EPFL), Lausanne, Switzerland

***Abstract*—This note contains instructions on how to document the algorithm you used in the framework of the 3D Deconvolution Microscopy Challenge. These instructions appear in italic below.**

This document can also serve as a template since it contains a description of the Richardson-Lucy algorithm as applied to a block-Toeplitz forward operator.

I. Notations

List all the notations you will be using in subsequent sections and explain them briefly.

In this document we use the following matrix/vector notations:

- $\mathbf{x}$ is a vector corresponding to the ground-truth image;
- $\mathbf{A}$ is a matrix representing the forward operator;
- $\mathbf{b}$ is a constant vector representing the background signal;
- $\mathbf{y}$ is a random vector modeling the measurements;
- $\mathbf{x}_k$ is an estimate of $\mathbf{x}$ obtained after k iterations of the algorithm.

II. Algorithm

Use this section to give a concise description of the computational steps involved in your algorithm. Cite relevant work in the literature, especially the paper(s) where your algorithm was initially proposed.

The algorithm we describe here was proposed by Richardson [1] and Lucy [2]. It is an iterative procedure that consists in the repeated application of the following update rule:

$$\mathbf{x}_{k+1} = \mathrm{diag}(\mathbf{A}^T\mathbf{1})^{-1}\mathrm{diag}\big[\mathbf{A}^T\mathrm{diag}(\mathbf{A}\mathbf{x}_k + \mathbf{b})^{-1}\mathbf{y}\big]\mathbf{x}_k.$$

Note that if the matrix $\mathbf{A}$ were block-circulant and normalized such that $\mathbf{A}^T\mathbf{1} = \mathbf{1}$, the algorithm would simplify accordingly. However, in the framework of this challenge, $\mathbf{A}$ is a block-Toeplitz matrix and thus the factor $\mathrm{diag}(\mathbf{A}^T\mathbf{1})^{-1}$ is non-trivial.

III. Variational interpretation

Discuss in which sense your algorithm can be interpreted as an procedure for optimizing a cost functional.

The Richardson-Lucy algorithm is a maximum-likelihood algorithm that is based on a Poisson noise model. Specifically,

$$\mathbf{y} \sim \mathcal{P}(\mathbf{A}\mathbf{x} + \mathbf{b}),$$

where $\mathcal{P}(\boldsymbol{\lambda})$ is a Poisson-distributed random vector of mean $\boldsymbol{\lambda}$.

Include funding acknowledgments here.

The corresponding Poisson likelihood is

$$\begin{aligned} p(\mathbf{y}|\mathbf{x}) &= \exp\big(-(\mathbf{A}\mathbf{x} + \mathbf{b})^T\mathbf{1}\big) \\ &\quad \times \exp\big(\log(\mathbf{A}\mathbf{x} + \mathbf{b})^T\mathbf{y}\big) \times \prod_{n=1}^{N} 1/\mathbf{y}_n!, \end{aligned}$$

where the logarithm function is applied component-wise and N is the dimension of the measurement vector $\mathbf{y}$.

Maximizing $p(\mathbf{y}|\mathbf{x})$ with respect to $\mathbf{x}$ is equivalent to maximizing the log-likelihood function

$$L(\mathbf{x}) = \log\big(p(\mathbf{y}|\mathbf{x})\big).$$

The gradient of this function is

$$\nabla L(\mathbf{x}) = \mathbf{A}^T\mathrm{diag}(\mathbf{A}\mathbf{x} + \mathbf{b})^{-1}\mathbf{y} - \mathbf{A}^T\mathbf{1}.$$

Imposing that this quantity vanishes leads to the following multiplicative update rule of Section II.

IV. Choice of the parameters

In this section, we expect you to explain how you set the parameters of the algorithm in order to get the the best-possible result.

The only parameter that needs to be adjusted for the Richardson-Lucy algorithm is the number of iterations k.

One way to do this is to run the algorithm for a fixed number of iterations (say 100, 200 and 300) and then choose the result that is most pleasing visually.

One can also use standard stopping criteria, e.g., $\|\mathbf{x}_{k+1} - \mathbf{x}_k\| < \epsilon$ or $L(\mathbf{x}_k) > \delta$, where ϵ and δ are fixed thresholds.

References

[1] W. H. Richardson, "Bayesian-Based Iterative Method of Image Restoration", Journal of the Optical Society of America, Vol. 62(1), pp. 55-59 (1972).

[2] L. B. Lucy, "An iterative technique for the rectification of observed distributions", The Astronomical Journal, Vol. 79(1), pp. 55-59 (1974).