Analytical form of Shepp-Logan phantom for parallel MRI

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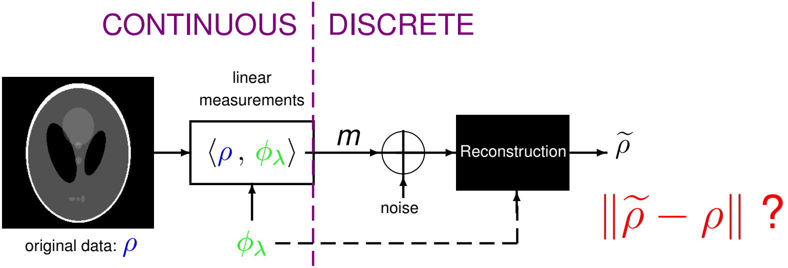
Outline

1. **Introduction**
   - Context
   - MRI case
   - Parallel MRI
2. **Theory**
   - Object
   - Sensitivity
   - Differentiation
3. **Experiments**
   - Sensitivity model
   - rasterization vs. analytic I
   - rasterization vs. analytic II
   - Example
4. **Conclusion**

Introduction: Context

Linear problem to invert:



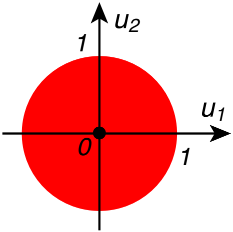
- Framework for the evaluation of reconstruction schemes
- No inverse crime!

Introduction: MRI case

In MRI, we have $\phi_\omega(\boldsymbol{r}) = e^{\mathrm{j}\omega.\boldsymbol{r}}$.

When ρ is a circular region:

$$m(\omega) = \iint_{\|\boldsymbol{r}\|\leq 1} e^{-\mathrm{j}\omega.\boldsymbol{r}} \mathrm{d}^2\boldsymbol{r}$$
$$= 2\pi J_1(\|\omega\|)/\|\omega\|.$$



R. Van de Walle, *et al.*
IEEE Transactions on Medical Imaging, 2000.

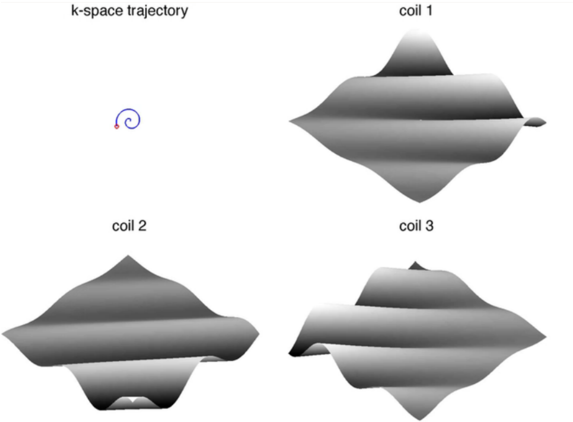
M. R. Smith, *et al.*
Int. Journal of Imaging Systems and Technology, 1997.

C. G. Koay, *et al.*
Magnetic Resonance in Medicine, 2007.

Introduction: Parallel MRI

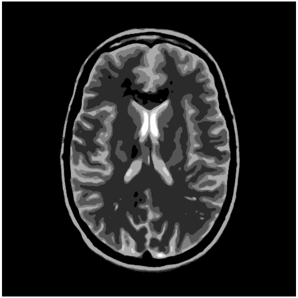
In parallel MRI:

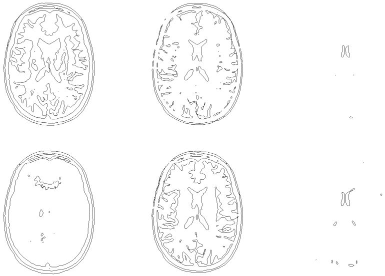
$$\phi_{\omega,c}(\boldsymbol{r}) = S(\boldsymbol{r})e^{j\omega.\boldsymbol{r}}$$



Theory: Object

$$\rho(\boldsymbol{r}) = \sum_{i=1}^{R} \rho_i \mathbb{1}_{\mathcal{A}_i}(\boldsymbol{r}).$$





In practice, we focus on regions defined by **ellipses.**

Theory: Sensitivity

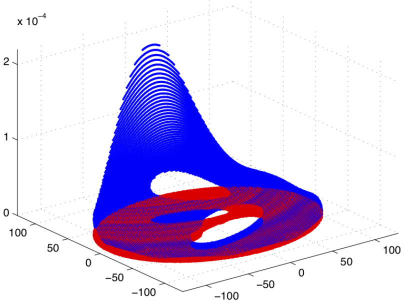
Physics:

$$S_{\text{phys}}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \oint_{coil} \frac{\mathrm{d}\boldsymbol{u} \times (\boldsymbol{u} - \boldsymbol{r})}{\|\boldsymbol{u} - \boldsymbol{r}\|^3},$$

Polynomial model:

$$S_{\text{model}}(\boldsymbol{r}) = \sum_{d=0}^{D} \sum_{|\boldsymbol{\alpha}| \leq d} s_{d,\boldsymbol{\alpha}} \boldsymbol{r}^{\boldsymbol{\alpha}},$$

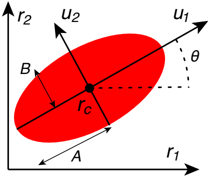
$$\forall \boldsymbol{r} \in \bigcup_{i=1}^{R} \mathcal{A}_i.$$

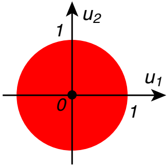


Theory: Sensitivity

Change of variables: $\boldsymbol{r} = \boldsymbol{r}_c + \boldsymbol{R}(\theta)\boldsymbol{D}\boldsymbol{u}$

$$S(\boldsymbol{r}) = \sum_{d=0}^{D} \sum_{|\boldsymbol{\alpha}|\leq d} s_{d,\boldsymbol{\alpha}} \boldsymbol{r}^{\boldsymbol{\alpha}}$$
$$= \sum_{d=0}^{D} \sum_{|\boldsymbol{\alpha}|\leq d} t_{d,\boldsymbol{\alpha}} \boldsymbol{u}^{\boldsymbol{\alpha}}.$$





Computation of $t_{d,\boldsymbol{\alpha}} \Longleftrightarrow$ solving $\boldsymbol{M}_s \boldsymbol{s} = \boldsymbol{M}_t \boldsymbol{t}$.

Then

$$m(c,\omega) = \sum_{i=1}^{R} \rho_i \sum_{d=0}^{D} \sum_{|\boldsymbol{\alpha}|\leq d} t_{i,d,\boldsymbol{\alpha}} \iint_{\|\boldsymbol{u}_i\|\leq 1} \boldsymbol{u}_i^{\boldsymbol{\alpha}} e^{-\mathrm{j}\boldsymbol{\omega}.\boldsymbol{r}} \mathrm{d}^2\boldsymbol{r}.$$

Theory: Differentiation

Definitions

$$G_n(\boldsymbol{v}) = J_n(\|\boldsymbol{v}\|) / \|\boldsymbol{v}\|^n \text{ and}$$

$$f_{\mathcal{A}}^{\boldsymbol{\alpha}}(\omega) = \iint_{\|\boldsymbol{u}\|\leq 1} \boldsymbol{u}^{\boldsymbol{\alpha}} e^{-\mathrm{j}\boldsymbol{\omega}.\boldsymbol{r}} \mathrm{d}^2\boldsymbol{r}, \quad \boldsymbol{\alpha} \in \mathbb{N}^2.$$

Proposition 1

$$\nabla G_n(\boldsymbol{v}) = -\boldsymbol{v} G_{n+1}(\boldsymbol{v}) \text{ and}$$

$$f_{\mathcal{A}}^{\boldsymbol{\alpha}}(\omega) = 2\pi |\boldsymbol{D}| \mathrm{j}^{|\boldsymbol{\alpha}|} e^{-\mathrm{j}\boldsymbol{\omega}.\boldsymbol{r}_c} \frac{\partial^{|\boldsymbol{\alpha}|} G_1}{\partial \boldsymbol{v}^{\boldsymbol{\alpha}}} \left(\boldsymbol{D}\boldsymbol{R}(-\theta)\omega\right).$$

Theory: Differentiation

Definitions

$$\begin{aligned} G_n(\boldsymbol{v}) &= J_n(\|\boldsymbol{v}\|)/\|\boldsymbol{v}\|^n \text{ and} \\ f_{\mathcal{A}}^{\boldsymbol{\alpha}}(\boldsymbol{\omega}) &= \iint_{\|\boldsymbol{u}\|\leq 1} \boldsymbol{u}^{\boldsymbol{\alpha}} e^{-\mathrm{j}\boldsymbol{\omega}.\boldsymbol{r}} \mathrm{d}^2\boldsymbol{r}, \quad \boldsymbol{\alpha} \in \mathbb{N}^2. \end{aligned}$$

Proposition 2

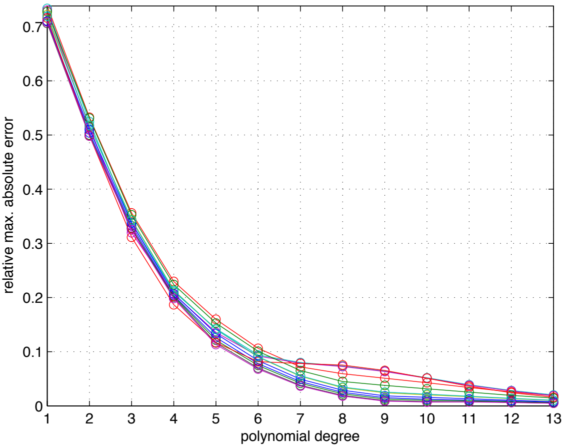
$\exists\, C_{\boldsymbol{i}}^{\boldsymbol{p},\boldsymbol{q}}$, $\boldsymbol{i} = \boldsymbol{0} \ldots \boldsymbol{p}$, such that

$$\frac{\partial^{|2\boldsymbol{p}+\boldsymbol{q}|} G_1}{\partial \boldsymbol{x}^{2\boldsymbol{p}+\boldsymbol{q}}}(\boldsymbol{x}) = \boldsymbol{x}^{\boldsymbol{q}} \sum_{\boldsymbol{i}=\boldsymbol{0}}^{\boldsymbol{p}} C_{\boldsymbol{i}}^{\boldsymbol{p},\boldsymbol{q}} \boldsymbol{x}^{2\boldsymbol{i}} G_{|\boldsymbol{p}+\boldsymbol{i}+\boldsymbol{q}|+1}(\boldsymbol{x}).$$

Experiments: Sensitivity model

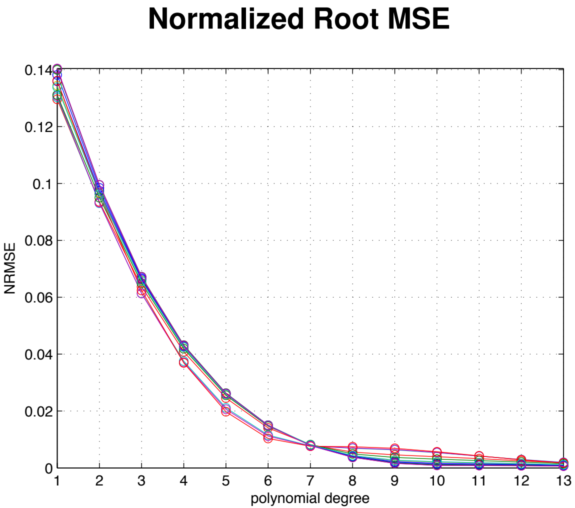
- Quality of the polynomial model for 12 head-coils

Maximal relative error



Experiments: Sensitivity model

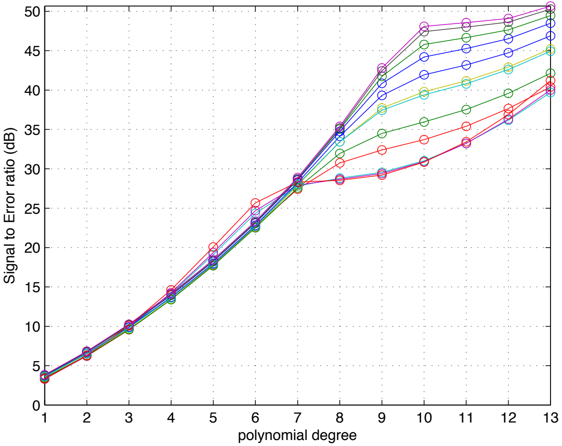
- Quality of the polynomial model for 12 head-coils



Experiments: Sensitivity model

- Quality of the polynomial model for 12 head-coils

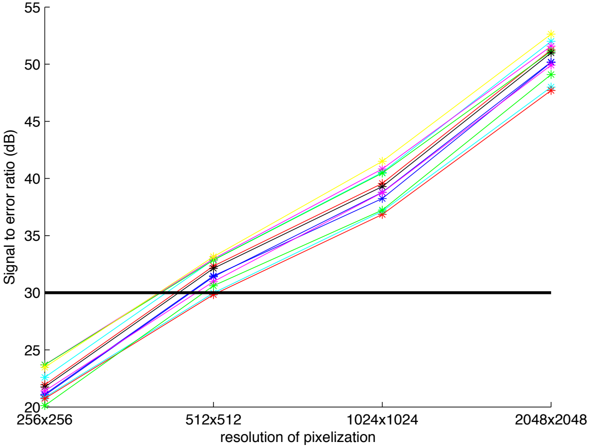
Signal to error ratio (dB)



Experiments: rasterization vs. analytic I

- Simulation of SL measurements on Cart. grid (128x128)

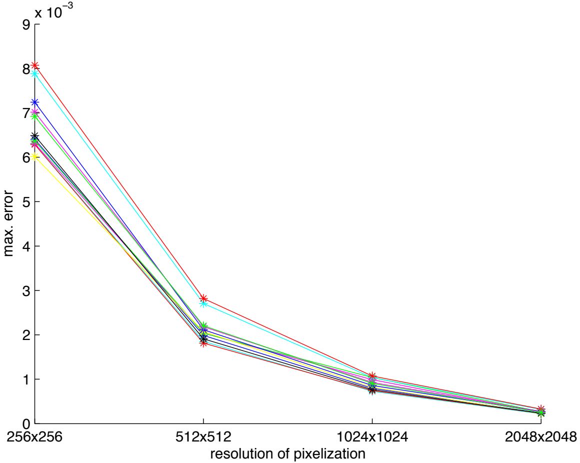
Signal to error ratio (dB)



Experiments: rasterization vs. analytic I

- Simulation of SL measurements on Cart. grid (128x128)

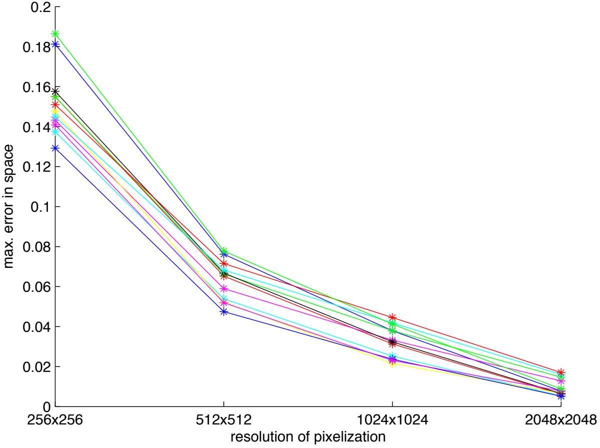
Maximal error (Fourier)



Experiments: rasterization vs. analytic I

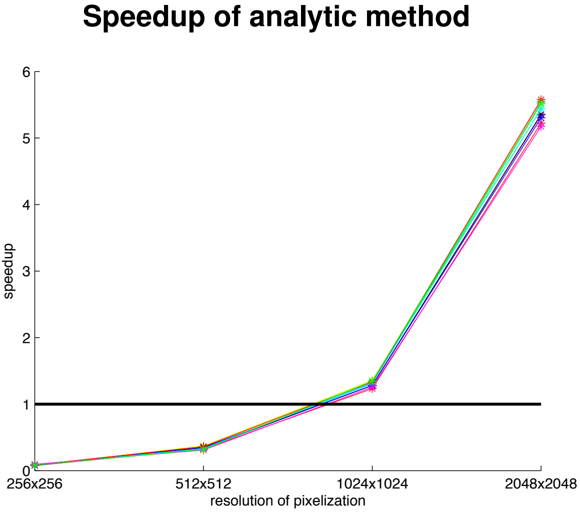
- Simulation of SL measurements on Cart. grid (128x128)

Maximal error (space)



Experiments: rasterization vs. analytic I

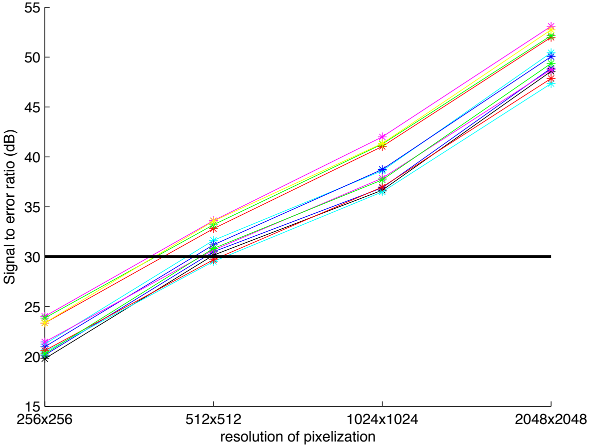
- Simulation of SL measurements on Cart. grid (128x128)



Experiments: rasterization vs. analytic II

- Simulation of SL measurements for a spiral (128x128)
  using NUFFT by J. Fessler

Signal to error ratio (dB)



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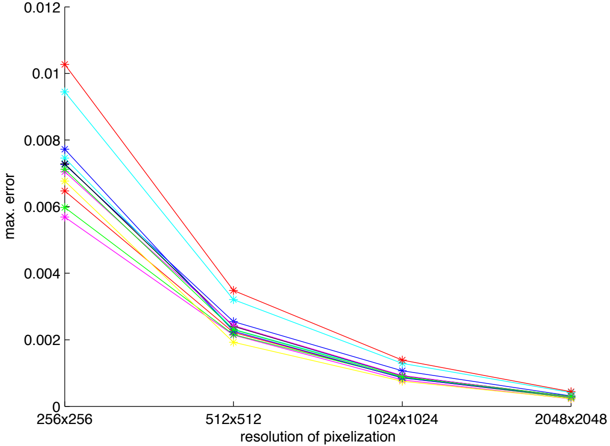

Experiments: rasterization vs. analytic II

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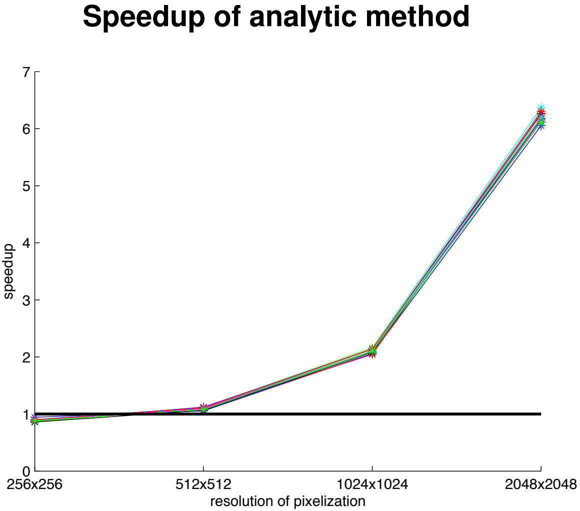

Experiments: Example

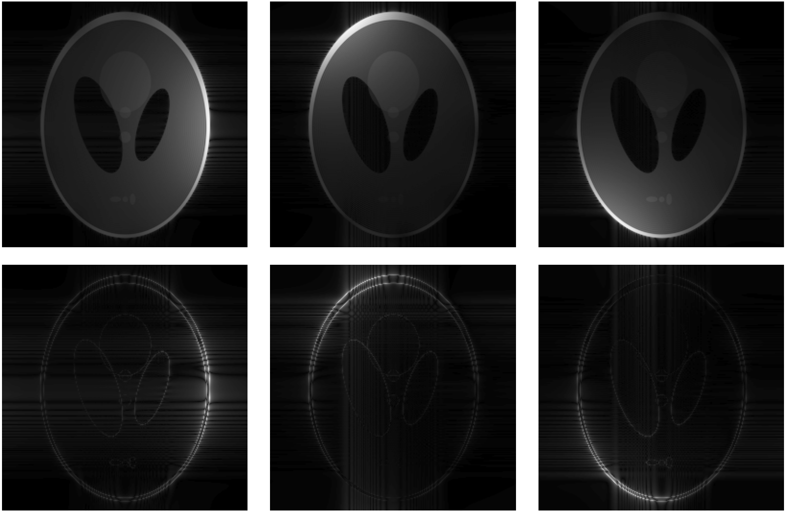

Figure: Reconstructed images after inverse Fourier transform.

Conclusion

- Analytical model of the famous Shepp-Logan phantom that is suitable for pMRI simulation.
- Reliable ground truth data for validation (no inverse crime)
- Rasterization: large errors for coarse resolutions
- Rasterization: inefficient for few measurements
- Matlab code for **your own reconstruction method**:

  http://bigwww.epfl.ch/algorithms

Outlook:

- Taking into account relaxation times
- Higher dimensions including time
- More shapes available for analytical computations

Thanks

Thanks for your attention.
Any questions?