MRI:
from Physics to Signal Processing

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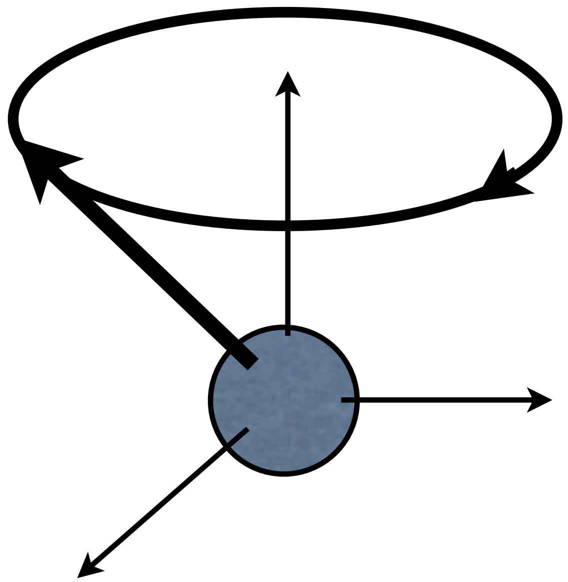
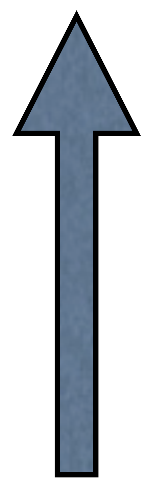
Physics

$$\omega_L = \gamma B_0$$

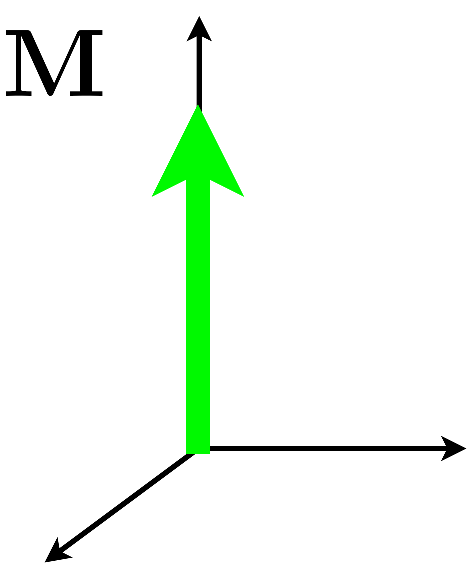


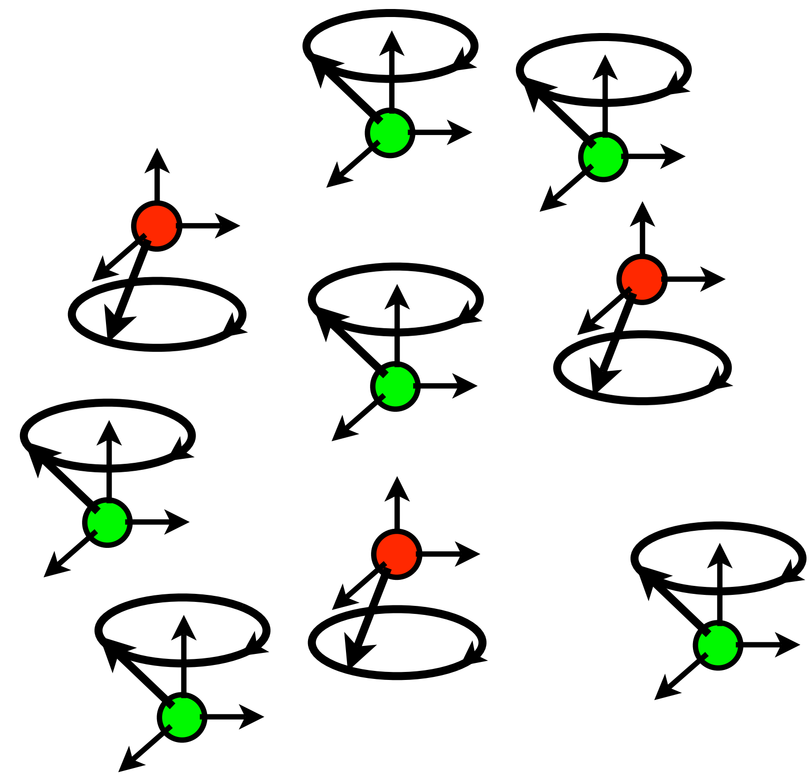
Spin's motion



$\mathbf{B_0}$

$\mathbf{M}$



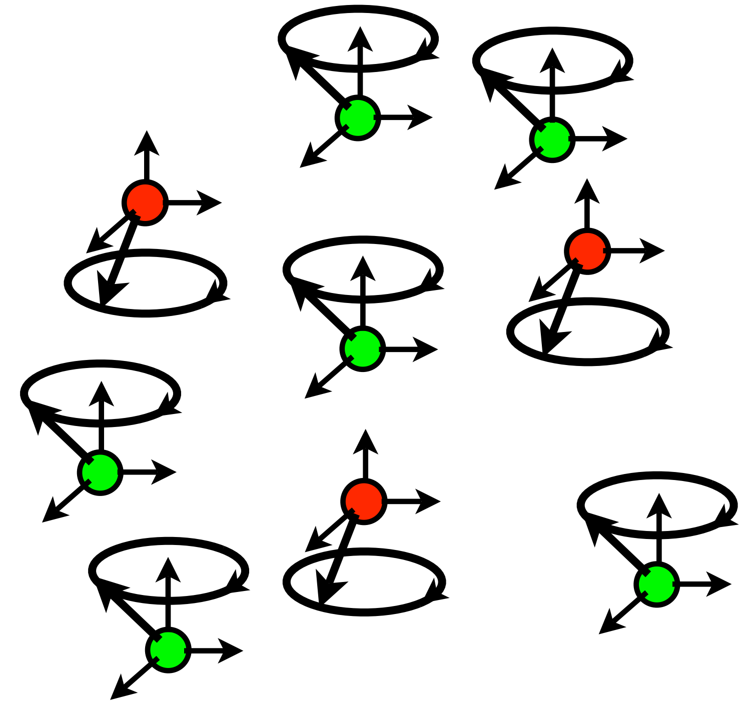


$$\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} = \gamma \mathbf{M} \times \mathbf{B}$$

Bloch equation

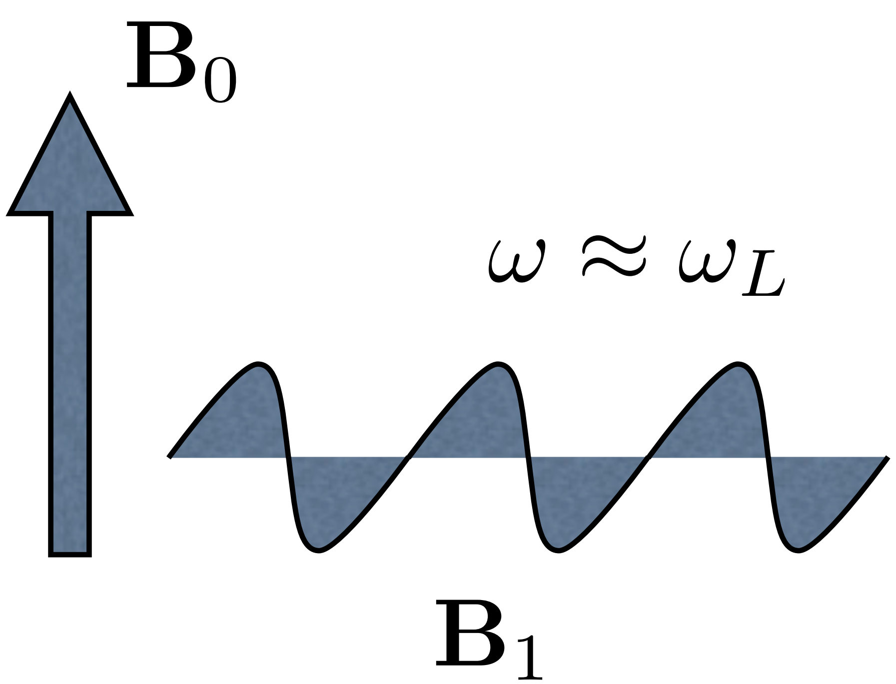
Resulting magnetization

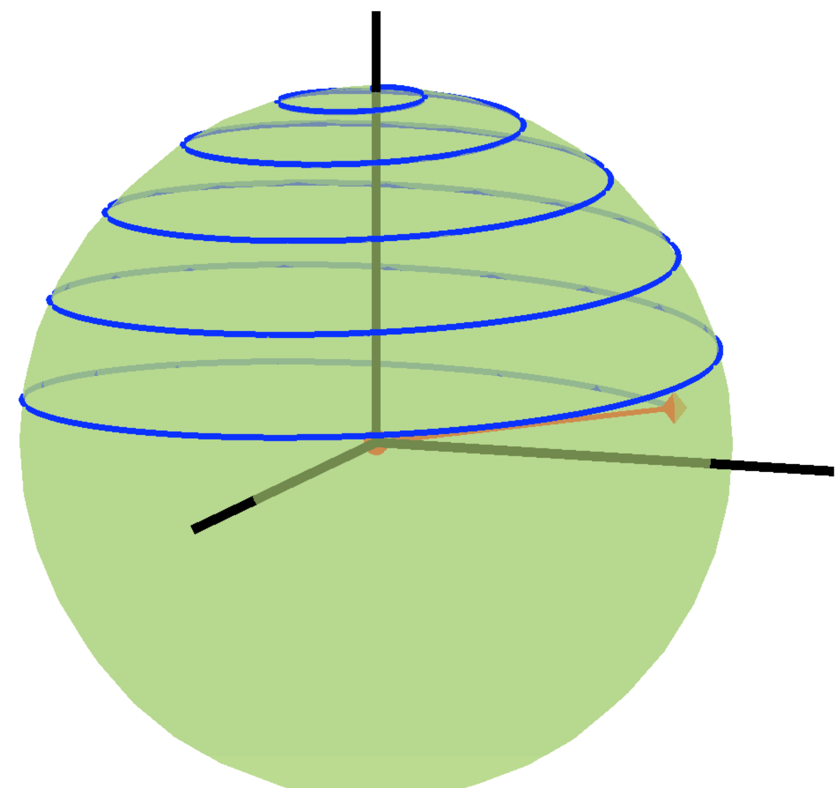
Physics: Excitation



Bloch equation

$$\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} = \gamma \mathbf{M} \times \mathbf{B}$$

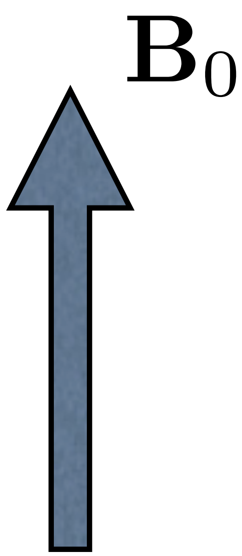


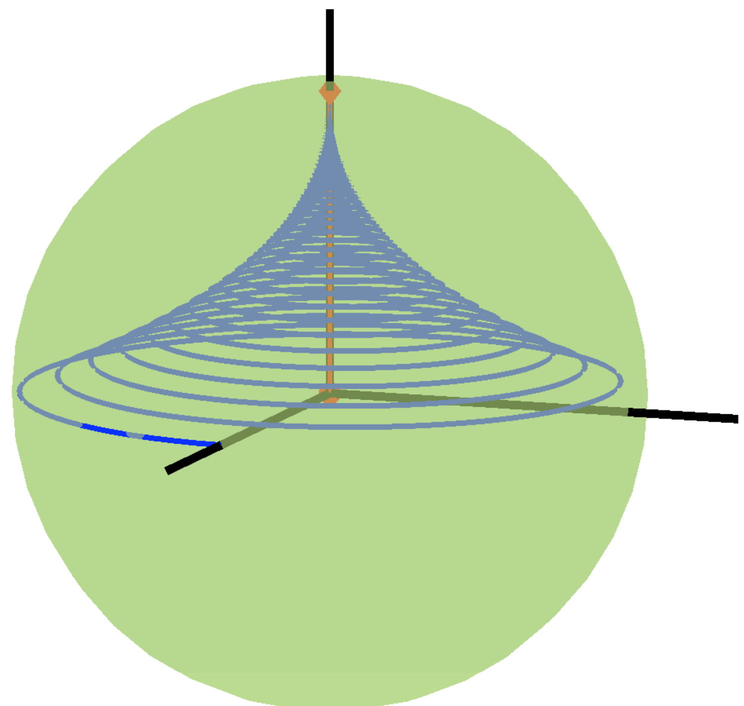


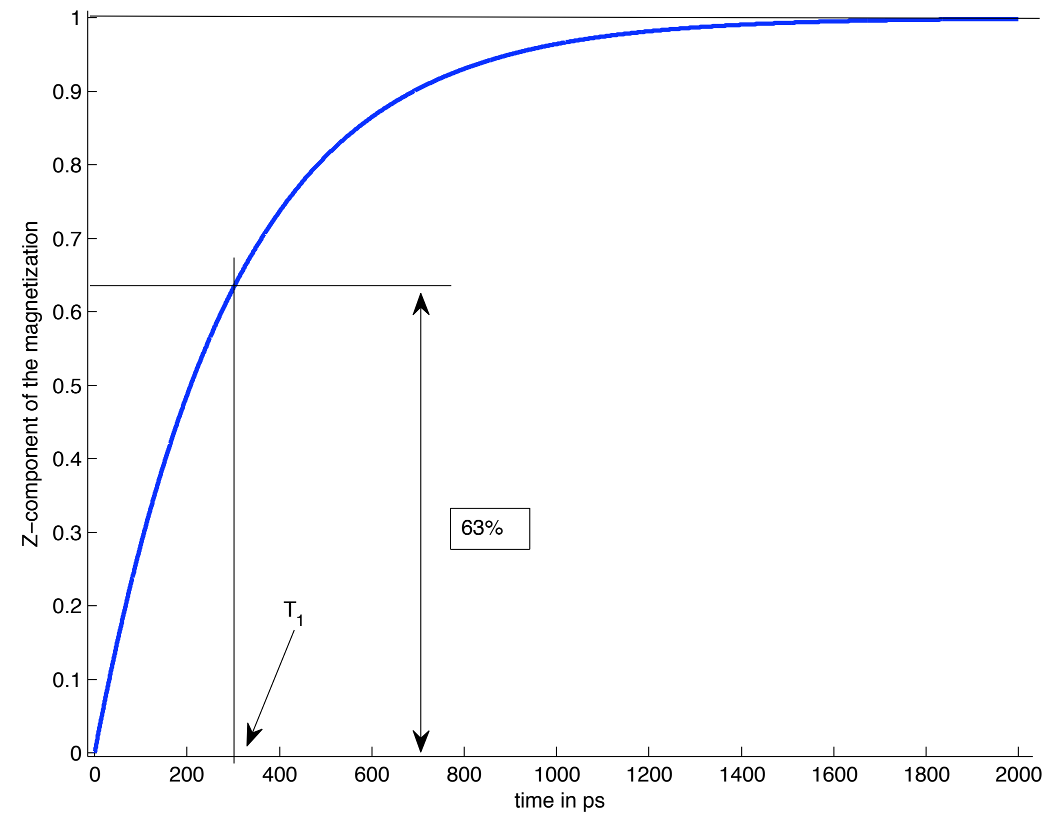
Transverse RF pulse

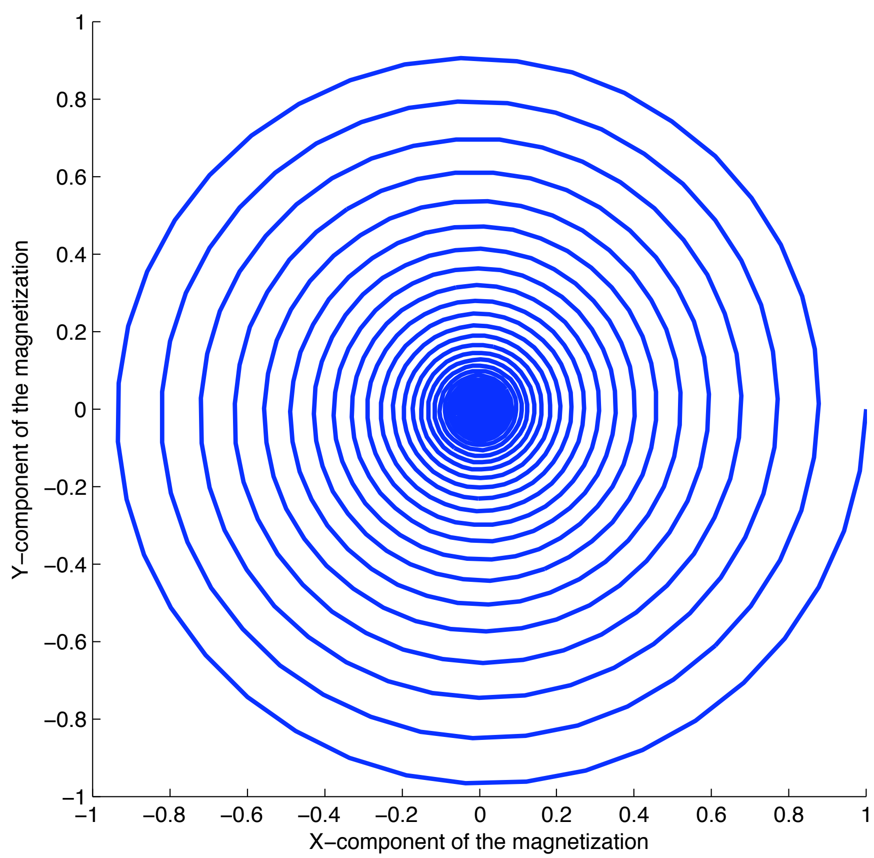
Resulting magnetization

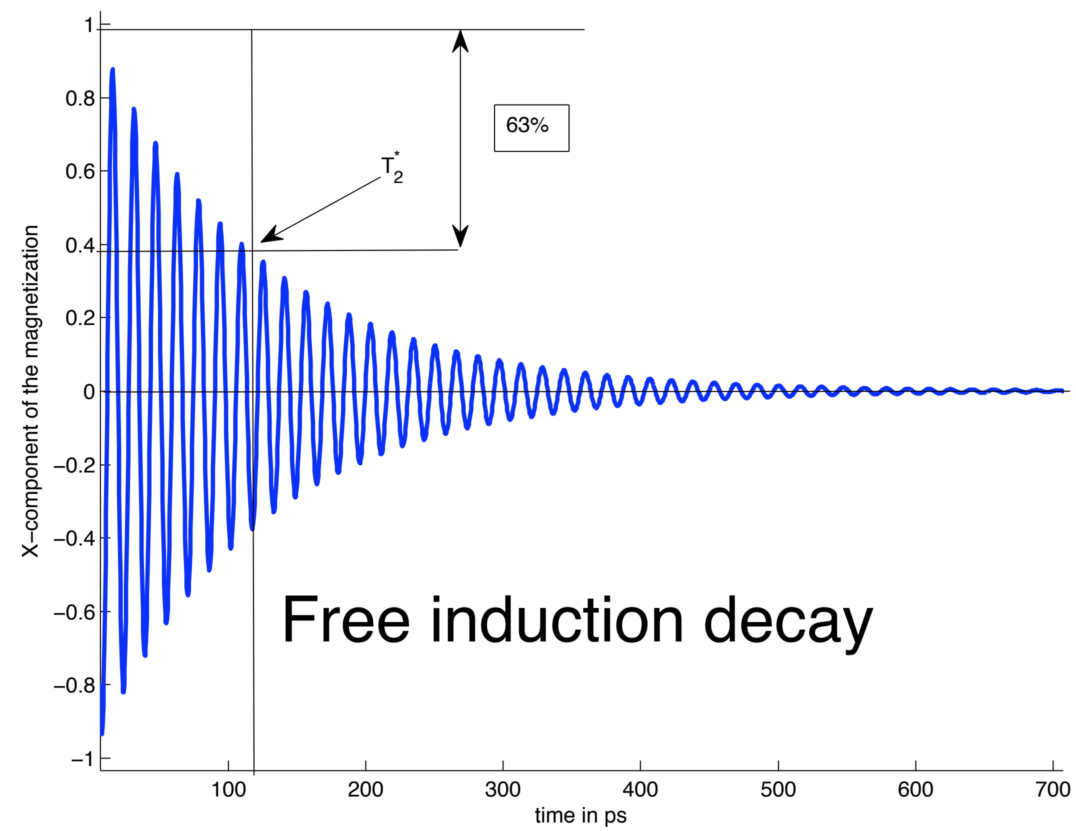
Physics: Relaxation











MR detection

Some formulae from electro-magnetism:

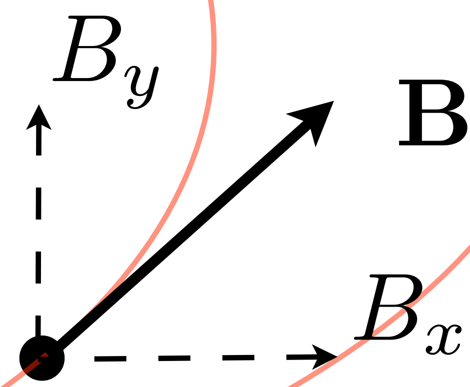
- Equivalent current distribution: $\mathbf{J}_{\mathbf{M}}(\mathbf{r},t) = \boldsymbol{\nabla} \times \mathbf{M}(\mathbf{r},t)$.
- Magnetic vector potential $\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} \frac{\mathbf{J}_{\mathbf{M}}(\mathbf{r}',t)}{\|\mathbf{r}-\mathbf{r}'\|} \, \mathrm{d}^3\mathbf{r}'$.
- Magnetic field $\mathbf{B}(\mathbf{r},t) = \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r},t)$
- Flux $\Phi_\gamma(t) = \int_{\mathcal{C}_\gamma} \mathbf{B}(\mathbf{r},t) \cdot \, \mathrm{d}^2S = \oint_{\partial\mathcal{C}_\gamma} \mathbf{A}(\mathbf{r},t) \cdot \, \mathrm{d}\mathbf{r}$,

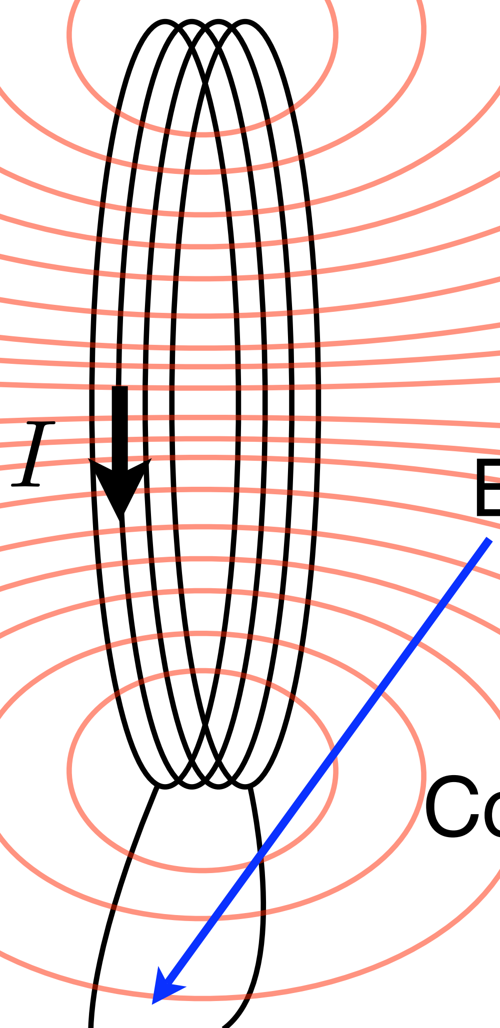
$$\Phi_\gamma(t) = \int_{\mathbb{R}^3} \mathbf{M}(\mathbf{r}',t) \cdot \mathbf{B}^u_\gamma(\mathbf{r}') \, \mathrm{d}^3\mathbf{r}',$$

where we define

$$\mathbf{B}^u_\gamma(\mathbf{r}') = \frac{\mu_0}{4\pi} \oint_{\partial\mathcal{C}_\gamma} \frac{\mathrm{d}\mathbf{r} \times (\mathbf{r}-\mathbf{r}')}{\|\mathbf{r}-\mathbf{r}'\|^3},$$

MR detection





Flux $\Phi_\gamma(t) = \int_{\mathbb{R}^3} \mathbf{M}(\mathbf{r}', t) \cdot \mathbf{B}^u_\gamma(\mathbf{r}')\, \mathrm{d}^3\mathbf{r}'$

Emf
$$e_\gamma(t) = -\frac{\mathrm{d}\Phi_\gamma}{\mathrm{d}t}(t)$$
$$\approx -\mathrm{Im}\left(\omega_0 \int_{\mathbb{R}^3} \underline{M}(\mathbf{r}, 0)\mathrm{e}^{-\mathrm{j}\int_0^t \omega(\mathbf{r},\tau)\,\mathrm{d}\tau} S_\gamma(\mathbf{r})\, \mathrm{d}^3\mathbf{r}\right)$$

Complex notations:

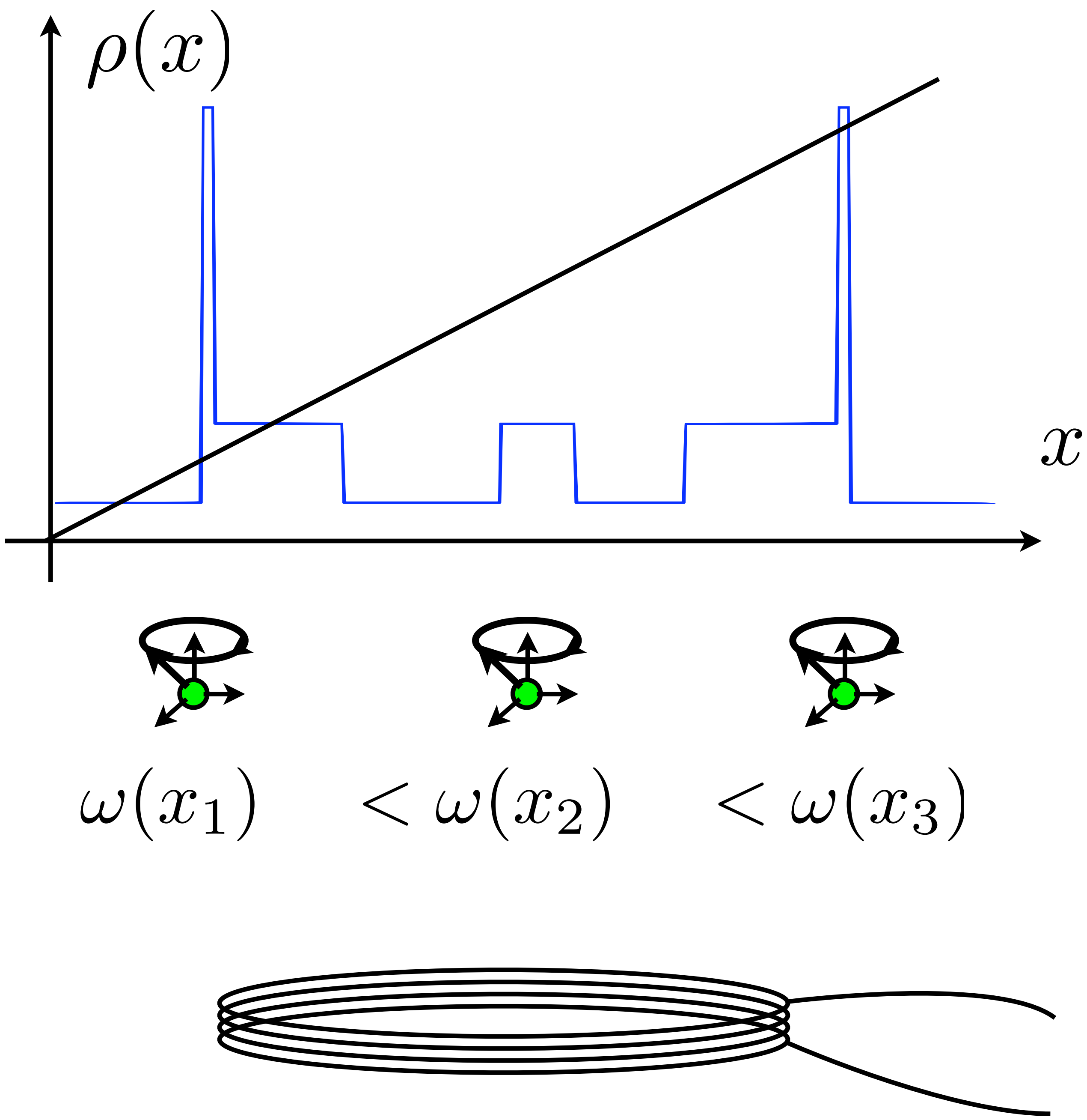
$$\underline{M}(\mathbf{r}, t) = M_x(\mathbf{r}, t) + \mathrm{j}M_y(\mathbf{r}, t) = \underline{M}(\mathbf{r}, 0)\mathrm{e}^{-\mathrm{j}\int_0^t \omega(\mathbf{r},\tau)\,\mathrm{d}\tau}$$
$$S_\gamma(\mathbf{r}) = B^u_{\gamma,x}(\mathbf{r}) - \mathrm{j}B^u_{\gamma,y}(\mathbf{r})$$

Synchronous demodulation

$$m_\gamma(t) \approx \omega_0 \int_{\mathbb{R}^3} \underline{M}(\mathbf{r}, 0)\mathrm{e}^{-\mathrm{j}\int_0^t \Delta\omega(\mathbf{r},\tau)\,\mathrm{d}\tau} S_\gamma(\mathbf{r})\, \mathrm{d}^3\mathbf{r}$$

Imaging: principle in 1D

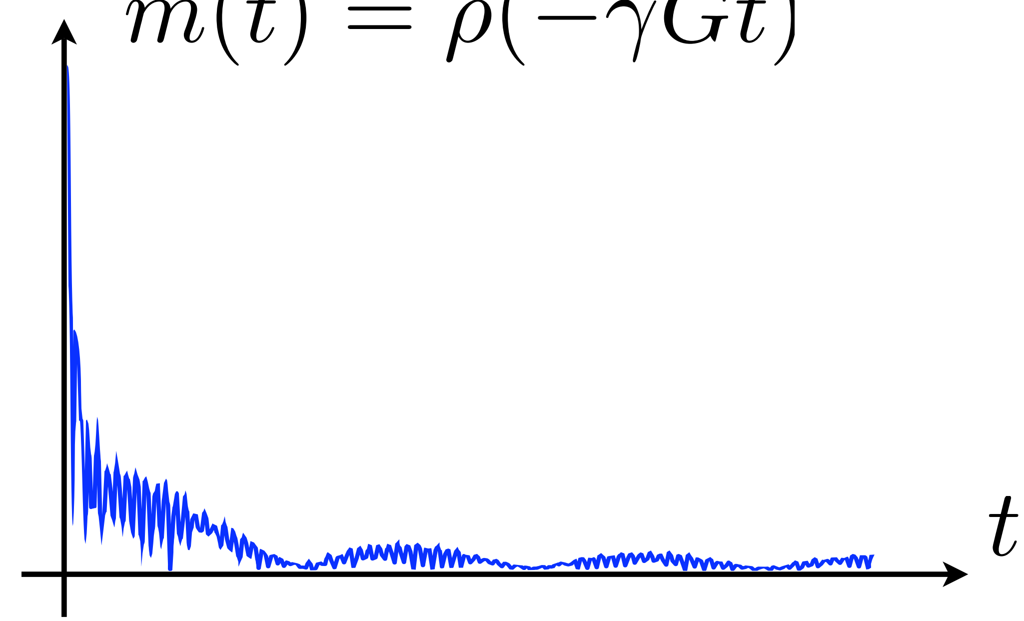
$B = B_0 + Gx \qquad \omega = \gamma B$



Space

$$V(t) = \int \rho(x) e^{j\omega t}\, dx = e^{j\gamma B_0 t} \underbrace{\int \rho(x) e^{j\gamma G t x}\, dx}_{m(t)}$$

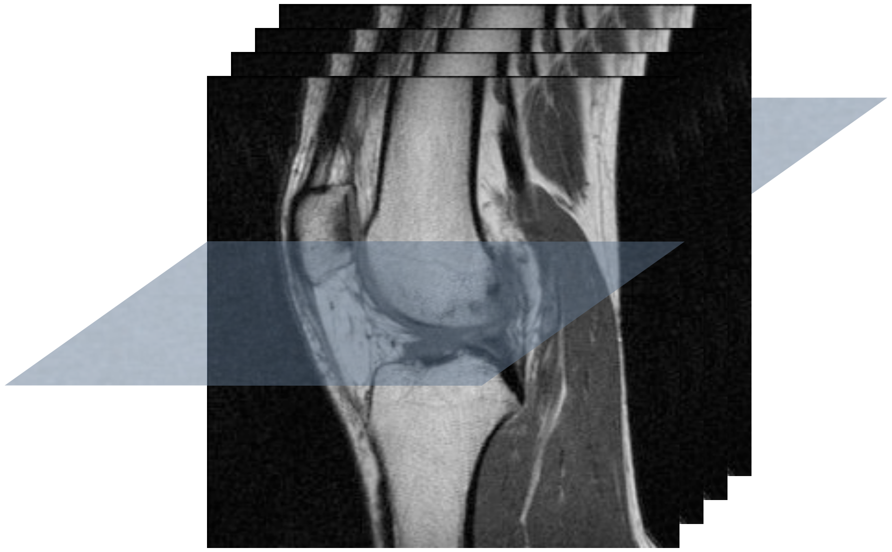
$m(t) = \widehat{\rho}(-\gamma G t)$

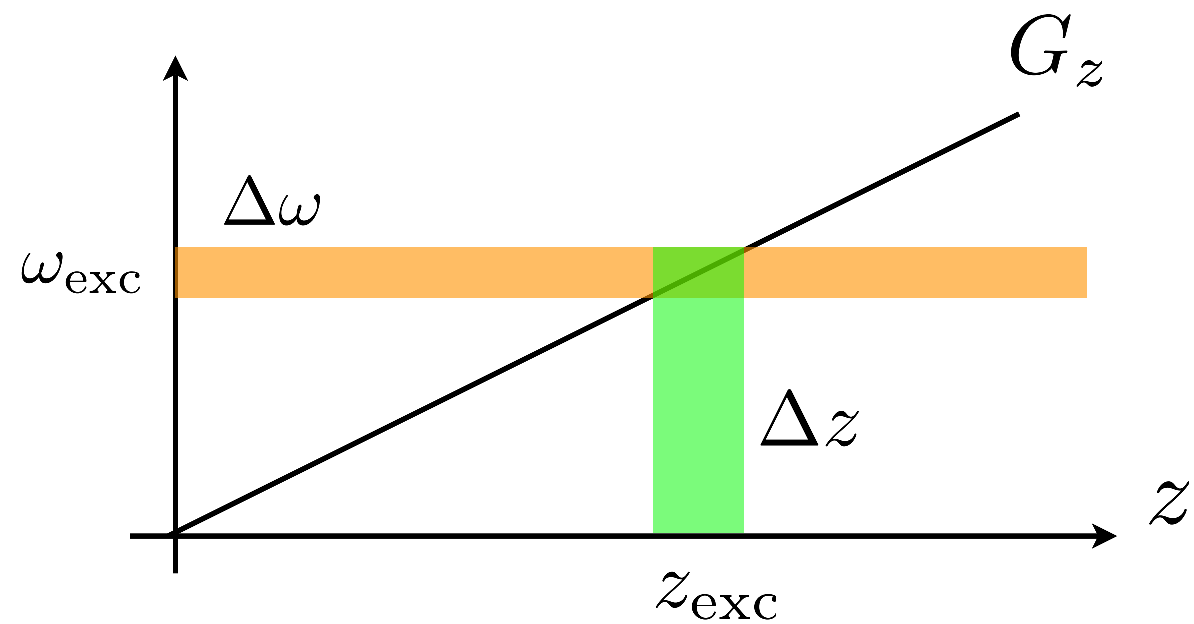


Frequency

Imaging: apply gradients!

- Slice selection

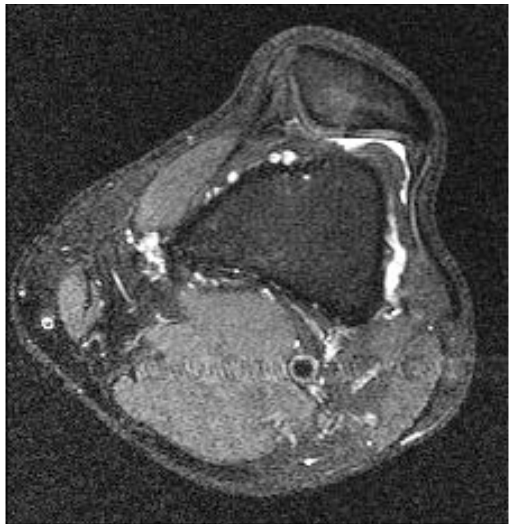


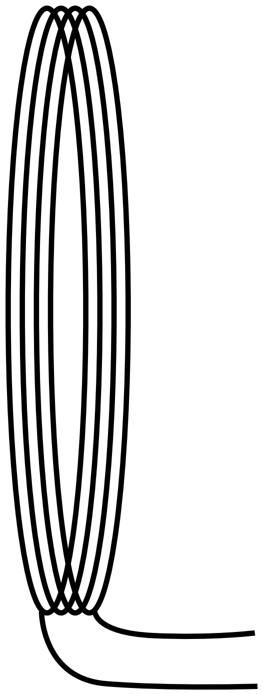


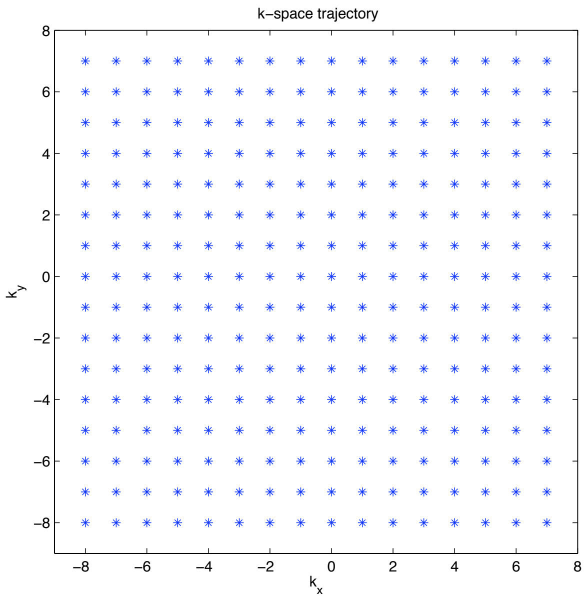
- Phase encoding

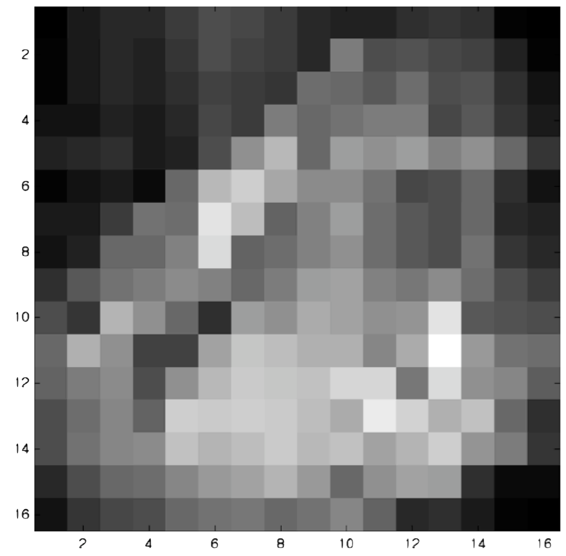
$$m\big(k_x(t), k_y(t)\big) = \iint \rho(x,y) \mathrm{e}^{\mathrm{j}2\pi(k_x(t)x + k_y(t)y)}\, \mathrm{d}x\, \mathrm{d}y$$

$$\text{with } k_x(t) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)\,\mathrm{d}\tau \text{ and } k_y(t) = \frac{\gamma}{2\pi}\int_0^t G_y(\tau)\,\mathrm{d}\tau$$



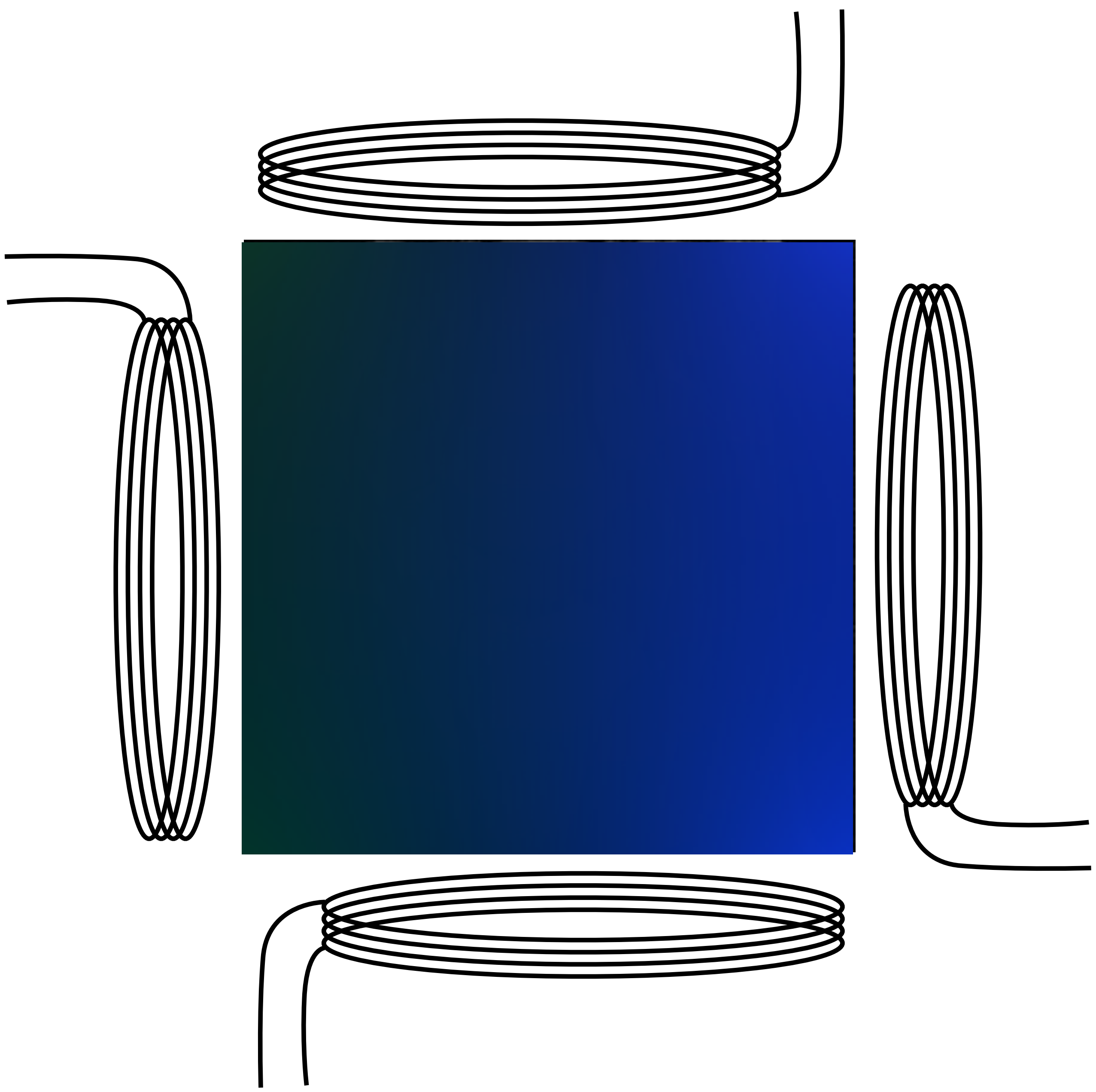






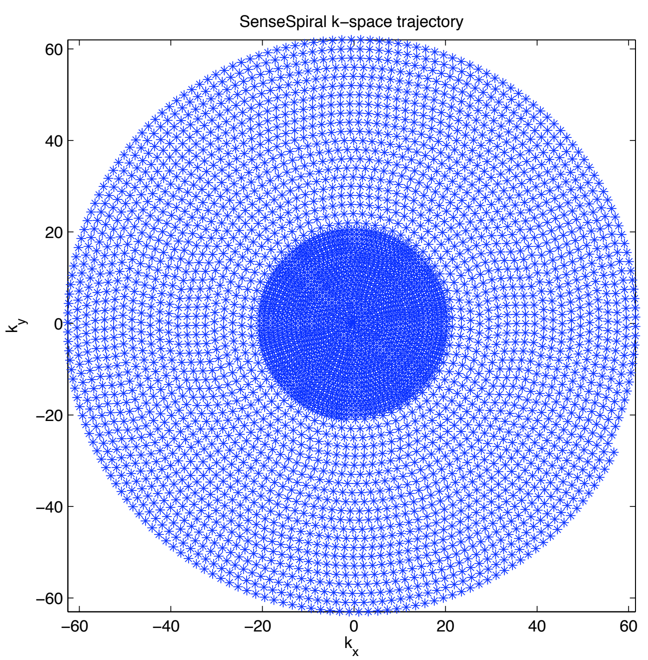
Parallel MRI: generalization

Array of non-homogeneous coils

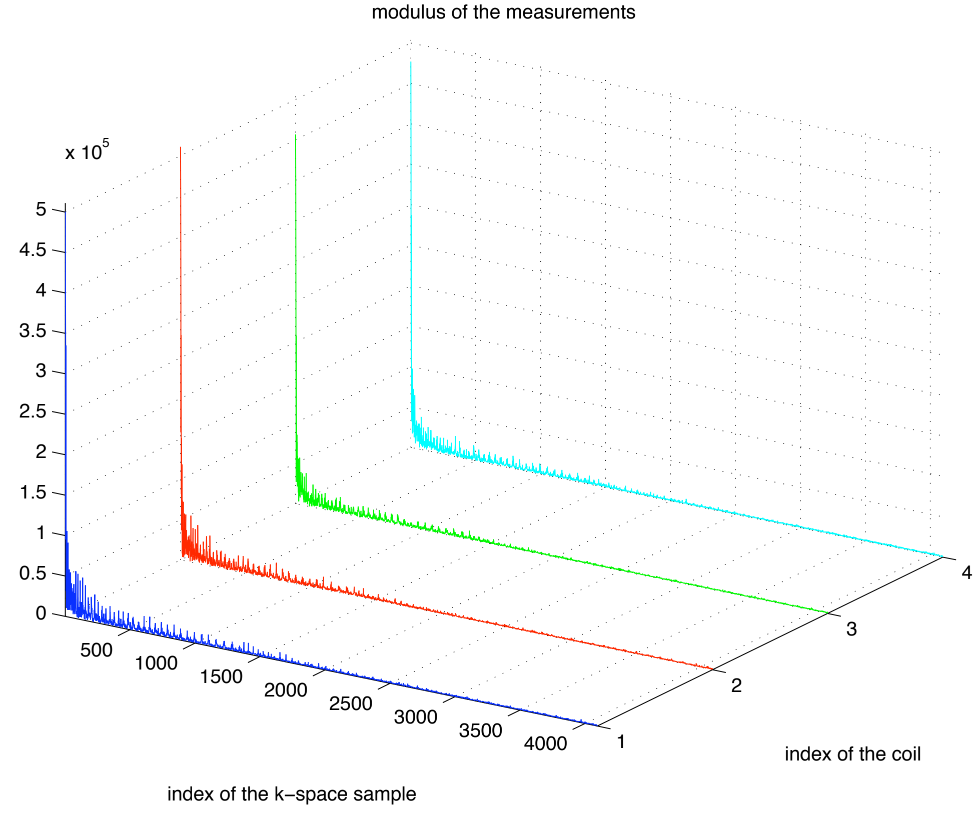


SENSitivity Encoding
[Pruessmann, 1999]

Non-cartesian and undersampled trajectory

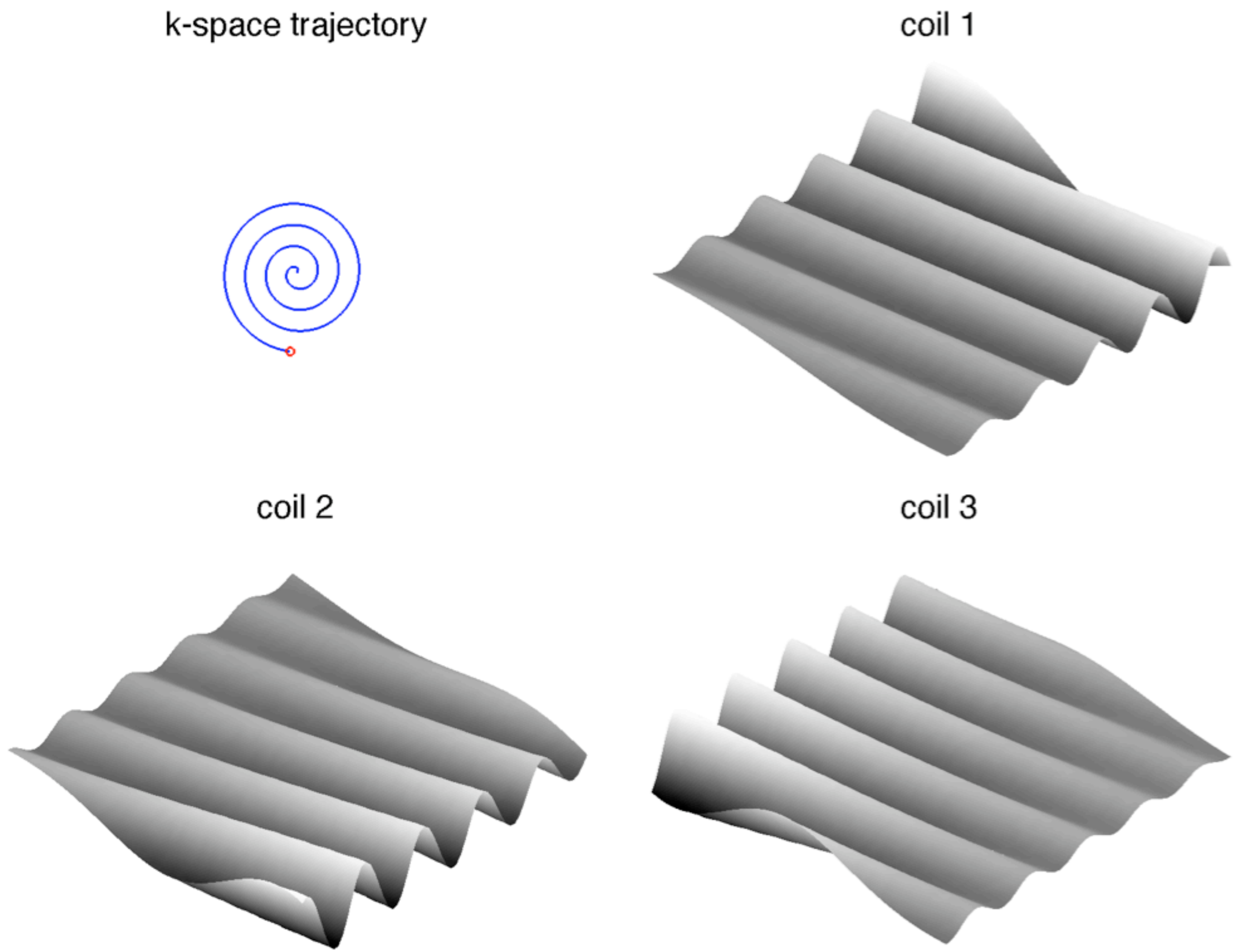


Phase encoding



SP point of view: projection

$$m_\gamma(\mathbf{k}) = \langle \varphi_{\gamma,\mathbf{k}} \, , \; \rho \rangle \quad \text{with} \quad \varphi_{\gamma,\mathbf{k}}(\mathbf{r}) = S_\gamma^*(\mathbf{r}) \mathrm{e}^{-2\mathrm{j}\pi \mathbf{k}\cdot\mathbf{r}}$$



Linear problem formalism

Continuous forward model:

$$m_\gamma(\mathbf{k}) = \int_{\mathbb{R}^2} S_\gamma(\mathbf{r})\rho(\mathbf{r})\mathrm{e}^{2\mathrm{j}\pi\langle \mathbf{k}\,,\,\mathbf{r}\rangle}\,\mathrm{d}\mathbf{r} + b_\gamma(\mathbf{k})$$

Discretization:

$$\mathbf{k}(t) \mapsto \mathbf{k}[n]$$

$$S_\gamma(\mathbf{r})\rho(\mathbf{r}) \approx \sum_{\mathbf{p}\in C_s} s_\gamma[\mathbf{p}]c[\mathbf{p}]\varphi(\mathbf{r}-\mathbf{p})$$

Discretized model:

$$m_\gamma[n] = \widehat{\varphi}(-2\pi\mathbf{k}_n)\sum_{\mathbf{p}\in C_s} s_\gamma[\mathbf{p}]c[\mathbf{p}]\mathrm{e}^{2\mathrm{j}\pi\langle \mathbf{k}_n\,,\,\mathbf{p}\rangle} + b_\gamma[n]$$

<u>Vector representation</u>:

$$\mathbf{m} = \mathbf{E}\mathbf{c} + \mathbf{b}$$

measurements

original signal

error to the model (noise)

First step to reconstruction

Reconstruction often requires the adjoint operator to be computed...

MRI adjoint: $\text{MRI}^H\{m_\gamma\}(\mathbf{r}) = S_\gamma^*(\mathbf{r}) \int_{\mathbb{R}} m_\gamma(t) \mathrm{e}^{2\mathrm{j}\pi\langle \mathbf{k}(t)\,,\,\mathbf{r}\rangle}\,\mathrm{d}t$

MRI measurement followed by adjoint:

$$\text{MRI}^H\text{MRI}\{\rho\}(\mathbf{r}) = S_\gamma^*(\mathbf{r})\left(g * S_\gamma \rho\right)(\mathbf{r})$$

with $\quad g(\mathbf{r}) = \int_{\mathbb{R}} \mathrm{e}^{2\mathrm{j}\pi\langle \mathbf{k}(t)\,,\,\mathbf{r}\rangle}\,\mathrm{d}t$

and $\quad \widehat{g}(\boldsymbol{\omega}) = \int_{\mathbb{R}} \delta(\boldsymbol{\omega} + 2\pi\mathbf{k}(t))\,\mathrm{d}t$

Questions?

Thanks for your attention!

Any question?