Reminder on signal detection in MRI

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Abstract

In this draft, we try to explain the common formulation of the signal in MRI. We start from the physics and explain the assumptions and tricks that are necessary. As a reference, we suggest the book by Haacke & Brown: *MRI Physical Principles and Sequence Design*

1 Original signal

We consider a distribution over space of rotating magnetization vectors $\mathbf{M}(\mathbf{r}, t)$ that correspond to the Free Induction Decay of an MR experiment. The z component that is along the static field is supposed to be constant (the time of acquisition is small compared to T_1). For a given location, the components in x and y are sines in quadrature. We introduce a complex notation to represent them: $\underline{M}(\mathbf{r}, t) = M_x(\mathbf{r}, t) + \mathrm{j}M_y(\mathbf{r}, t) = \underline{M}(\mathbf{r}, 0)\mathrm{e}^{-\mathrm{j}\int_0^t \omega(\mathbf{r},\tau)\,\mathrm{d}\tau}$.

2 Physics: coil sensitivity

To measure a signal that is representative of the magnetization, we use a coil. We precessing magnetization will produce a time-varying magnetic field and then an electromotive force (emf) in the coil.

Equivalent current First, the magnetization distribution admits an equivalent current distribution:

$$\mathbf{J_M}(\mathbf{r}, t) = \boldsymbol{\nabla} \times \mathbf{M}(\mathbf{r}, t). \tag{1}$$

Magnetic vector potential This current distribution generate a magnetic vector potential:

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} \frac{\mathbf{J_M}(\mathbf{r}', t)}{\|\mathbf{r} - \mathbf{r}'\|}\, \mathrm{d}^3\mathbf{r}'. \tag{2}$$

Here, we choose the Coulomb gauge ($\boldsymbol{\nabla} \cdot \mathbf{A} = 0$) and neglected propagation times (the propagation speed is more or less the speed of light).

Magnetic field The magnetic field is then given by $\mathbf{B}(\mathbf{r},t) = \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r},t)$.

Flux The flux of magnetic field through the coil surface $\mathcal{C}_\gamma$ is

$$\Phi_\gamma(t) = \int_{\mathcal{C}_\gamma} \mathbf{B}(\mathbf{r},t) \cdot \mathrm{d}^2 S = \oint_{\partial\mathcal{C}_\gamma} \mathbf{A}(\mathbf{r},t) \cdot \mathrm{d}\mathbf{r}, \tag{3}$$

where we used Stokes's theorem for the second term.

Using the relation $\boldsymbol{\nabla} \times (a\mathbf{v}) = (\boldsymbol{\nabla} a) \times \mathbf{v} + a \boldsymbol{\nabla} \times \mathbf{v}$, and the triple product rules, we can develop the flux expression:

$$\begin{aligned}
\Phi_\gamma(t) &= \oint_{\partial\mathcal{C}_\gamma} \frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} \frac{\mathbf{J}_\mathbf{M}(\mathbf{r}',t)}{\|\mathbf{r}-\mathbf{r}'\|} \mathrm{d}^3\mathbf{r}' \cdot \mathrm{d}\mathbf{r} = \frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} \oint_{\partial\mathcal{C}_\gamma} \frac{\boldsymbol{\nabla}_{\mathbf{r}'} \times \mathbf{M}(\mathbf{r}',t)}{\|\mathbf{r}-\mathbf{r}'\|} \cdot \mathrm{d}\mathbf{r}\ \mathrm{d}^3\mathbf{r}', &(4)\\
&= \frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} \oint_{\partial\mathcal{C}_\gamma} \left(\boldsymbol{\nabla}_{\mathbf{r}'} \times \frac{\mathbf{M}(\mathbf{r}',t)}{\|\mathbf{r}-\mathbf{r}'\|} - \left(\boldsymbol{\nabla}_{\mathbf{r}'} \frac{1}{\|\mathbf{r}-\mathbf{r}'\|} \right) \times \mathbf{M}(\mathbf{r}',t) \right) \cdot \mathrm{d}\mathbf{r}\ \mathrm{d}^3\mathbf{r}', &(5)\\
&= \frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} \oint_{\partial\mathcal{C}_\gamma} \left(\left(-\boldsymbol{\nabla}_{\mathbf{r}'} \frac{1}{\|\mathbf{r}-\mathbf{r}'\|} \right) \times \mathbf{M}(\mathbf{r}',t) \cdot \mathrm{d}\mathbf{r} \right) \mathrm{d}^3\mathbf{r}', &(6)\\
&= -\frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} \oint_{\partial\mathcal{C}_\gamma} \left(\mathbf{M}(\mathbf{r}',t) \cdot \frac{\mathbf{r}-\mathbf{r}'}{\|\mathbf{r}-\mathbf{r}'\|^3} \times \mathrm{d}\mathbf{r} \right) \mathrm{d}^3\mathbf{r}', &(7)\\
\Phi_\gamma(t) &= \int_{\mathbb{R}^3} \mathbf{M}(\mathbf{r}',t) \cdot \mathbf{B}^u_\gamma(\mathbf{r}')\ \mathrm{d}^3\mathbf{r}', &(8)
\end{aligned}$$

where we define

$$\mathbf{B}^u_\gamma(\mathbf{r}') = \frac{\mu_0}{4\pi} \oint_{\partial\mathcal{C}_\gamma} \frac{\mathrm{d}\mathbf{r} \times (\mathbf{r}-\mathbf{r}')}{\|\mathbf{r}-\mathbf{r}'\|^3}, \tag{9}$$

which corresponds exactly to the magnetic field that would be created at point $\mathbf{r}'$ by an unit steady current in the coil, according to the Biot-Savart law. The term $\boldsymbol{\nabla}_{\mathbf{r}'} \times \frac{\mathbf{M}(\mathbf{r}',t)}{\|\mathbf{r}-\mathbf{r}'\|}$ disappears because the sample (which is responsible for the magnetization) is assumed finite in space what allows to show that the integral over the space annihilates (we suggest to use the curl form of the divergence theorem to show it).

Electromotive force The emf induced in the coil is

$$e_\gamma(t) = -\frac{\mathrm{d}\Phi_\gamma}{\mathrm{d}t}(t) = -\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \left(M_x(\mathbf{r},t) B^u_{\gamma,x}(\mathbf{r}) + M_y(\mathbf{r},t) B^u_{\gamma,y}(\mathbf{r}) + M_z(\mathbf{r},t) B^u_{\gamma,z}(\mathbf{r}) \right) \mathrm{d}^3\mathbf{r}, \tag{10}$$

We introduce the notation $S_\gamma(\mathbf{r}) = B^u_{\gamma,x}(\mathbf{r}) - \mathrm{j} B^u_{\gamma,y}(\mathbf{r})$. This is the so-called coil sensitivity.

Since the z component of the magnetization is supposed to be constant we have

$$\begin{aligned}
e_\gamma(t) &\approx -\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \left(M_x(\mathbf{r},t) B^u_{\gamma,x}(\mathbf{r}) + M_y(\mathbf{r}',t) B^u_{\gamma,y}(\mathbf{r}) \right) \mathrm{d}^3\mathbf{r} &(11)\\
&\approx \mathrm{Re}\left(-\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \underline{M}(\mathbf{r},t) S_\gamma(\mathbf{r})\ \mathrm{d}^3\mathbf{r} \right) &(12)\\
&\approx \mathrm{Re}\left(\mathrm{j} \int_{\mathbb{R}^3} \omega(\mathbf{r},t) \underline{M}(\mathbf{r},0) \mathrm{e}^{-\mathrm{j} \int_0^t \omega(\mathbf{r},\tau)\, \mathrm{d}\tau} S_\gamma(\mathbf{r})\ \mathrm{d}^3\mathbf{r} \right) &(13)\\
e_\gamma(t) &\approx -\mathrm{Im}\left(\omega_0 \int_{\mathbb{R}^3} \underline{M}(\mathbf{r},0) \mathrm{e}^{-\mathrm{j} \int_0^t \omega(\mathbf{r},\tau)\, \mathrm{d}\tau} S_\gamma(\mathbf{r})\ \mathrm{d}^3\mathbf{r} \right) &(14)\\
& &(15)
\end{aligned}$$

We used the approximation $\omega(\mathbf{r},t) \approx \omega_0$, which is quite fair since ω_0 is huge compared to the variations of frequency[1].

[1] for protons of gyroscopic ratio $\gamma = 2.678\mathrm{e}^8 \mathrm{rad.T}^1.\mathrm{s}^{-1}$ with $B_0 = 1\mathrm{T}$, the Larmor frequency is $f_0 = 42.6\mathrm{MHz}$

3 Signal processing: demodulation

The coil signal is not directly used. Since its frequency is concentrated around ω_0, we would like to transpose it in low-frequencies. This operation is the phase-sensitive demodulation and is analogeous to the receiver in Quadrature Amplitude Modulation (QAM). We multiply the original signal by two sine at frequency $\Omega \approx \omega_0$ in quadrature and remove the high frequency components, thanks to an appropriate low-pass filtering. This detection results in to signals $s_{re}(t)$ and $s_{im}(t)$ that can be seen as real and imaginary parts of a single complex-valued signal $s_\gamma(t)$. Finally, the signal is linked with the magnetization

$$m_\gamma(t) \approx \omega_0 \int_{\mathbb{R}^3} \underline{M}(\mathbf{r}, 0) \mathrm{e}^{-\mathrm{j} \int_0^t \Delta\omega(\mathbf{r},\tau)\ \mathrm{d}\tau} S_\gamma(\mathbf{r})\ \mathrm{d}^3\mathbf{r}, \tag{16}$$

where $\Delta\omega(\mathbf{r}, \tau) = \omega(\mathbf{r}, \tau) - \omega_0$.

4 Assumptions summary

Description	Formulae	Comments
No propagation time for the magnetic field	$\delta t = \frac{\mathrm{FOV}}{c} \ll 1$	the sample has a small spatial extension
finite sample	$\int_{\mathbb{R}^3} \boldsymbol{\nabla}_{\mathbf{r}'} \times \frac{\mathbf{M}(\mathbf{r}',t)}{\\|\mathbf{r}-\mathbf{r}'\\|}\ \mathrm{d}^3\mathbf{r}' = \mathbf{0}$	exact
static z component of the magnetization	$\frac{\mathrm{d}M_z}{\mathrm{d}t}(\mathbf{r}, t) \approx 0$	small acquisition time compared to T_1
small variations in precessing frequency	$\lvert 1 - \frac{\omega(\mathbf{r},t)}{\omega_0} \rvert \ll 1$	small bandwidth approximation
good demodulation carrier	$\Omega \approx \omega_0$	fair