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Asymptotic properties of least squares spline filters and application to multi-scale decomposition of signals.

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ABSTRACT

We use B-spline functions to define a family of sequence-spaces $\mathbb{S}_m^n$ included in the finite energy and discrete space l_2. We derive invariant filters that operate on finite energy signals to output their least square approximations in $\mathbb{S}_m^n$. We obtain results on the convergence of the various filters to the ideal discrete lowpass filter providing the link with Shannon's sampling theorem. As an application, we derive pyramidal representations of signals that can be implemented with fast algorithms and compare these representations with the Gaussian/Laplacian pyramid which is widely used in signal processing and computer vision.

INTRODUCTION

Typical application of B-spline functions in signal processing have been in magnification or minification [1], image coding and reconstruction [2]. Most of the early work dealt with exact interpolation problems where the interpolant agrees precisely with the signal samples. Less attention has been devoted to non-exact interpolation techniques which are relevant to noisy signals and the problem of under-sampling for minimum error data compression. An exception to this is the work by R. Hummel who constructs a general theory about sampling, reconstruction, and optimal filtering that is based on least square approximation [3]. More recently, Mallat [4], used polynomial spline functions to construct some examples of multi-resolution pyramids using a wavelet representation.

BEST APPROXIMATION IN THE SPACES $\mathbb{S}_m^n$

We define discrete spaces $\mathbb{S}_m^n$ included in the space of square summable discrete sequences l_2:

$$\mathbb{S}_m^n := \left\{ v:\ v(k)=\sum_{i=-\infty}^{+\infty} c(i)\, b_m^n(k-mi)\ ,\ \forall k\in \mathbb{Z},\ c\in l_2 \right\}, \qquad (1)$$

where $b_m^n(k) := \beta^n(k/m)$ and where $\beta^n(x)$ is the B-spline function of order n [5]. The signal $v(k)$ is a sampled polynomial spline of order n with a knot spacing of m. Thus, n represent a smoothness constraint while m is a scale index representing the coarseness of the signals in $\mathbb{S}_m^n$. Moreover, the spaces $\mathbb{S}_m^n$ are closed sub-spaces of $l_2=\mathbb{S}_0^n$ and have the property that for n odd, if $m2=k\, m1$ (m1,m2,k positive integer) then $\mathbb{S}_{m1}^n \supset \mathbb{S}_{m2}^n$. This property, however, does not hold for n even.

The least square approximation of a signal $s(k)$ in $\mathbb{S}_m^n$ is obtained by filtering as described in the following theorem.

Theorem 1. The least square approximation in $\mathbb{S}_m^n$ of a signal $s\in l_2$ is given by:

$$\mathfrak{s} = h_m^n * [\,\mathfrak{s}_m]\uparrow_m \qquad (2)$$

$$\mathfrak{s}_m = [\mathring{h}_m^n * s]\downarrow_m \qquad (3)$$

where h_m^n is a discrete spline interpolator with an expansion factor m (i.e. $\mathfrak{s}_m = [\mathfrak{s}]\downarrow_m$) [6] and where $\mathring{h}_m^n$ is the optimal pre-filter needed before the discrete cardinal-spline interpolator h_m^n. Their expressions are given by:

$$h_m^n := b_m^n * [(b_1^n)^{-1}]\uparrow_m \qquad (4)$$

$$\mathring{h}_m^n := [([b_m^n * b_m^n]\downarrow_m)^{-1}]\uparrow_m * [b_1^n]\uparrow_m * b_m^n \qquad (5)$$

In effect, the approximation is obtained using a pre-filtering followed by an under-sampling and then an interpolation as illustrated in Fig. 1. The proof of the theorem is given at the end of this paper.

CONVERGENCE PROPERTIES

The convergence properties of the pre-filter $\mathring{h}_m^n$ and the interpolator h_m^n which are stated in the following theorems and corollaries provide the link between the discrete version of the classical Whittaker-Kotel'nikov-Shannon sampling theorem [7] and the least square approximation in $\mathbb{S}_m^n$.

Theorem 2. The Fourier transforms of the pre-filter $\mathring{h}_m^n$ converge pointwise a.e. to an ideal discrete lowpass filter as n tends to infinity :

$$\lim_{n\to\infty} \mathring{H}_m^n(f) = \mathrm{Rect}_m(f) = \begin{cases} 1 & |f| < 1/2m \\ 1/2 & |f| = 1/2m \\ 0 & 1/2m < |f| < 1 \end{cases} \quad (6)$$

Moreover, $\mathring{H}_m^n$ converge to $\mathrm{Rect}_m(f)$ in $\mathbb{L}_2(-1/2, +1/2)$ as n goes to infinity.

Theorem 3. For n odd, the Fourier transforms of the interpolators $H_m^n(f)$ converge pointwise a.e. to an ideal discrete lowpass filter with gain m as n tends to infinity :

$$\lim_{n\to\infty} H_m^n(f) = m\,\mathrm{Rect}_m(f)$$

Moreover, H_m^n converge to m $\mathrm{Rect}_m(f)$ in $\mathbb{L}_2(-1/2, +1/2)$ as n goes to infinity.

Corollary 1. The impulse responses $\mathring{h}_m^n$ converge in l_2 to the discrete sinc filter as n tends to infinity.

Corollary 2. For n odd, the interpolator h_m^n converges in l_2 to the ideal sinc interpolator with gain m, as n tends to infinity.

Figure 2 illustrates the convergence of the Fourier transforms of $\mathring{h}_m^n$ and h_m^n to the ideal discrete low-pass filters.

MULTI-SCALE REPRESENTATIONS

A multiresolution-pyramid representation consists of several versions of the signal at different resolution levels in which the low resolution levels are described by fewer samples than the high resolution counterparts [4, 8]. They are commonly obtained by iteratively applying a filter and a down-sampler to produce the pyramid layers. As an application of our results, we derive the spline pyramid that minimizes the loss of information occuring when a discrete signal is approximated by a coarse resolution one. Using equs. 1-5, we obtain the representations of a signal s(k) with a factor of compression between two consecutive levels equal to 2 ($m=2^j$):

$$\begin{cases} \tilde{s}_{(j+1)} = k_{2j}^n * \tilde{y}_{(j+1)} \\ \tilde{y}_{(j+1)} = [\mathring{h}_2^n * \tilde{x}_{(j)}]_{\downarrow 2} \\ \tilde{x}_{(j)} = o_{2j}^n * \tilde{s}_{(j)} \\ \tilde{s}_{(0)} = s \end{cases} \quad (7)$$

where k_{2j}^n and o_{2j}^n are convolution operators and are given by:

$$k_{2j}^n = (t_{2j+1}^n)^{-1} * t_2^n \quad (8)$$

$$o_{2j}^n = (b_1^n)^{-1} * (b_2^n)^{-1} * \left[b_{2j}^n * b_{2j+1}^n\right]_{\downarrow 2j} \quad (9)$$

and where t_m^n is given by (14). A drawback to this multi-resolution representation is that the filters k_{2j}^n and o_{2j}^n of the first and third equation in (7) depend on the resolution level (j). On the other hand, the second equation of (7) is independent of the resolution level and is precisely the one that defines the first pyramid level for the representation of the signal $\tilde{x}_{(j)}$. This observation suggests the alternative step-wise optimal multi-resolution representation:

$$\begin{cases} \check{s}_{(j+1)} = [\mathring{h}_2^n * \check{s}_{(j)}]_{\downarrow 2} \\ \check{s}_{(0)} = s \end{cases} \quad (10)$$

The question of how the step-wise optimal algorithm (10) compares with the optimal algorithm (7) is partially answered by the following theorem.

Theorem 4. The Fourier transforms of the filters k_m^n and o_m^n converge pointwise a.e. to a discrete pass-all filter as n tends to infinity :

$$\lim_{n\to\infty} K_m^n(f) = 1 \quad \text{a.e. } f \in (-1/2, 1/2)$$

$$\lim_{n\to\infty} O_m^n(f) = 1 \quad \text{a.e. } f \in (-1/2, 1/2)$$

Moreover, K_m^n and O_m^n tend to 1 in $\mathbb{L}_2(-1/2, +1/2)$ as n goes to infinity.

AN EXPERIMENT

As an experiment, we use the "Women" image (Fig. 3) to compare the step-wise optimal spline pyramid (10) (SOSP) with the Laplacian pyramid (LP) of Burt and Adelson developed for compact image coding [8]. Fig. 3 shows the difference-image pyramid for the SOSP representations and the LP with the same intensity scaling to facilitate the comparison. Each level in the difference-image pyramid consists of the difference between the image at a given level and its interpolated version at the next level. For this experiment we have chosen the values n=3 and j=1,2,3. Table 1 gives the signal to noise ratios SNR [9] associated with the full resolution approximation $h^n_{2^j}*[\mathfrak{s}_{(j)}]_{\uparrow 2^j}$ together with the standard deviation or root mean square error (RMS), the entropy and the range of the difference-image. The SNRs for the representation obtained by the SOSP algorithm are better than the ones obtained by the Laplacian pyramid. As a matter of fact, the SNRs for the step-wise optimal representations at a given level (i) are comparable to the ones at level (i-1) for the LP representation. This improvement can be advantageously applied to progressive image transmission and compact image coding.

PROOF OF THEOREM 1

Since $\mathbb{S}^n_m$ is a closed subspace of the Hilbert space l_2, the least square approximation $\mathfrak{s}$ is given by the orthogonal projection of s onto $\mathbb{S}^n_m$. Hence, the error $\mathfrak{s}$ - s is orthogonal to $\mathbb{S}^n_m$. In particular, because of the definition (1), the error is orthogonal to $b^n_m(k)$ and all of its shifted versions with shift factors that are integer multiples of m:

$$\left((\mathfrak{s} - s)(k)\ ,\ b^n_m(k-lm)\right)_2 = 0\ ,\ \forall l \in \mathbb{Z} \qquad (11)$$

where $(.,.)_2$ denotes the usual l_2 inner product and where

$$\mathfrak{s}(k)=\sum_{i=-\infty}^{+\infty}\mathfrak{c}(i)b^n_m(k-mi) \text{ is in } \mathbb{S}^n_m.$$

Using the linearity property of the inner product and the fact that $\mathfrak{s}$ is in $\mathbb{S}^n_m$, we rewrite (11) to get:

$$\left(s,b^n_m(k-lm)\right)_2 =$$

$$\sum_{i=-\infty}^{+\infty}\mathfrak{c}(i)\left(b^n_m(k-im),b^n_m(k-lm)\right)_2,\ \forall l \in \mathbb{Z} \qquad (12)$$

Equation (12) can be expressed as the convolution equation:

$$[s*b^n_m]_{\downarrow m}(l) = (\mathfrak{c}*[b^n_m*b^n_m]_{\downarrow m})(l) \qquad (13)$$

The operator $t^n_m(l)$:

$$t^n_m(l) := [b^n_m*b^n_m]_{\downarrow m}(l) \qquad (14)$$

has finitely many non-zero values and defines a bounded linear operator from l_2 into itself. It can be shown that $(t^n_m)^{-1}(l)$ exists and decays exponentially fast as $|l|\to\infty$ [10]. Thus, $(t^n_m)^{-1}(l)$ is absolutely summable and hence defines a bounded linear operator from l_2 into itself. Therefore, the operator $t^n_m(l)$ has the bounded inverse $(t^n_m)^{-1}$ that we use together with (13) to obtain the approximation $\mathfrak{s}$:

$$\mathfrak{s} = b^n_m*[\mathfrak{c}]_{\uparrow m}$$

$$= b^n_m*\left[(t^n_m)^{-1}*[s*b^n_m]_{\downarrow m}\right]_{\uparrow m}$$

$$= b^n_m*\left[[[(t^n_m)^{-1}]_{\uparrow m}*b^n_m*s]_{\downarrow m}\right]_{\uparrow m} \qquad (15)$$

where in the last equality of (15), we have used the functional equality:

$$a*[b]_{\downarrow m} = [[a]_{\uparrow m}*b]_{\downarrow m} \qquad (16)$$

Using the functional equality (16), the fact that

$$a*[b*c]_{\uparrow m} = a*[b]_{\uparrow m}*[c]_{\uparrow m} \qquad (17)$$

and the identity $(b^n_1*(b^n_1)^{-1})(k) = \delta_0(k)$ (where the existence of $(b^n_1)^{-1}$ follows from [10]), we manipulate (15) so as to exhibit a term that is a pure interpolator given by (4), we obtain:

$$\mathfrak{s} = b^n_m*\left[[[(t^n_m)^{-1}]_{\uparrow m}*[b^n_1]_{\uparrow m}*b^n_m*s]_{\downarrow m}*(b^n_1)^{-1}\right]_{\uparrow m}$$

$$= b^n_m*[(b^n_1)^{-1}]_{\uparrow m}*\left[[[(t^n_m)^{-1}]_{\uparrow m}*[b^n_1]_{\uparrow m}*b^n_m*s]_{\downarrow m}\right]_{\uparrow m}$$

$$= h^n_m*[\,[\mathring{h}^n_m*s]_{\downarrow m}]_{\uparrow m} \qquad (18)$$

where $\mathring{h}^n_m$ is given by (5).

Similarly to Hummel [3], we can interpret $\mathring{h}^n_m$ to be the optimal pre-filter needed before the interpolator h^n_m.

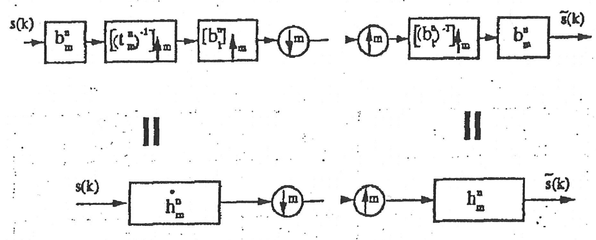

Fig. 1: Schematic representation of the operation for the least square approximation in $\mathbb{S}^n_m$.

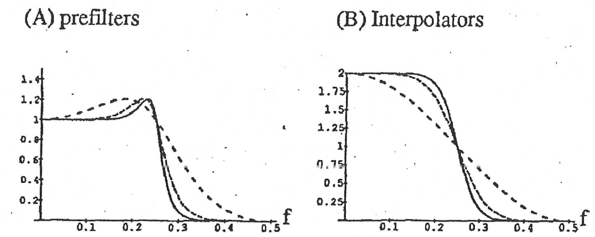

Fig. 2: Fourier transforms of some least square spline filters. (A) Prefilters $\mathring{H}^1_2(f)$ (- - -), $\mathring{H}^3_2(f)$ (-----) and $\mathring{H}^5_2(f)$ (Continuous line). (B) interpolators $H^1_2(f)$ (- - -), $H^3_2(f)$ (-----) and $H^5_2(f)$ (Continuous line).

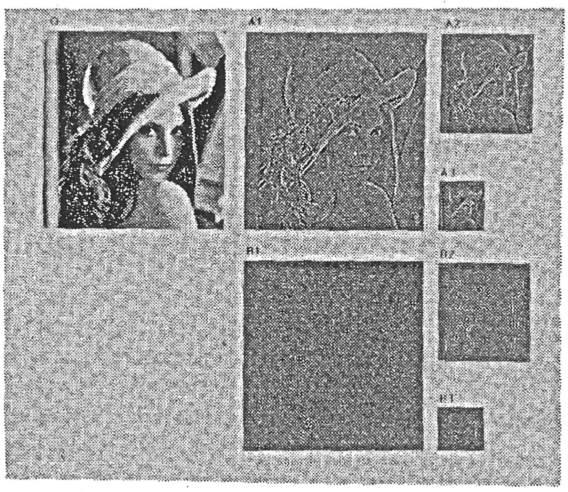

Fig. 3 : Error images between two consecutive levels of the SOSP pyramid and of the Laplacian pyramids for the "Women" image O. (A1-A3): error/difference images of the Laplacian pyramid. (B1-B3): error/difference images of the SOSP pyramid.

Pyramid level	Range	RMS	Entropy	SNR (dB)
LP-1	(-80, 85)	11.75	5.10	23.70
LP-2	(-69, 64)	12.27	5.39	19.44
LP-3	(-60, 52)	14.98	5.85	16.48
SOSP-1	(-57, 78)	6.66	4.35	28.63
SOSP-2	(-79, 76)	12.24	5.32	23.00
SOSP-3	(-73, 109)	16.21	5.90	19.50

Table 1 : Comparison of performance measured at successive pyramid levels for the "Women" image.

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BIOGRAPHY

Akram Aldroubi (member of AMS '83) was born in Homs, Syria in 1958. He received the M.S. in Electrical Engineering From the Swiss Federal Institute of Technology in Lausanne, Switzerland in 1982, the M.S. and the Ph.D. in Mathematics in 1984 and 1987 respectively, from Carnegie-Mellon University. He is currently Staff Fellow at the Biomedical Engineering and Instrumentation Program, National Institutes of Health.

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