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Cardinal spline filters: Stability and convergence to the ideal sinc interpolator

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Abstract. In this paper, we provide an interpretation of polynomial spline interpolation as a continuous filtering process. We prove that the frequency responses of the cardinal spline filters converge to the ideal lowpass filter in all L_p-norms with $1 \leqslant p < +\infty$ as the order of the spline tends to infinity. We provide estimates for the resolution errors and the interpolation errors of the various filters. We also derive an upper bound for the error associated with the reconstruction of bandlimited signals using polynomial splines.

Zusammenfassung. In dieser Arbeit wird eine Interpretation der Polynom-Spline-Interpolation als kontinuierlicher Filterungsprozeß vorgestellt. Es wird bewiesen, daß die Frequenzgänge von Cardinal-Spline-Filtern in allen L_p-Normen mit $1 \leqslant p < \infty$ gegen den idealen Tiefpaß konvergieren, wenn die Ordnung der Splines gegen unendlich geht. Wir stellen Abschätzungen für die Auflösungsfehler und Interpolationsfehler für verschiedene Filter vor. Es wird außerdem eine obere Grenze für den Fehler im Zusammenhang mit der Rekonstruktion von bandbegrenzten Signalen mit Hilfe von Polynom-Splines hergeleitet.

Resumé. Dans cet article, nous proposons une interprétation de l'interpolation B-spline en terme de filtres continus. Nous prouvons que les réponses fréquentielles des filtres splines cardinaux convergent vers le filtre passe-bas idéal, et ceci dans toutes les normes L_p avec $1 \leqslant p < \infty$, quand l'ordre des splines tend vers l'infini. Nous donnons l'estimation des erreurs de résolution et d'interpolation des differents filtres splines. Nous obtenons une limite superieure de l'erreur associée à la reconstruction de signaux à bande limitée par interpolation spline.

Keywords. Shannon's sampling theorem, cardinal spline filters, spline interpolation, bandlimited signal reconstruction, convergence rates.

1. Introduction

In signal processing, the classical interpolation method for bandlimited signals (which follows from the Whittaker–Kotel'nikov–Shannon sampling theorem) is usually described as a convolution with a sinc function which corresponds to filtering with an ideal lowpass filter [5]. An alternative interpolation technique is the method of polynomial splines which is used in a variety of applications [9, 15, 17]. The purpose of this paper is to relate these concepts by (i) showing that the polynomial spline interpolation, for equally spaced data points, can also be interpreted as a shift invariant continuous filtering process, and (ii) studying the asymptotic properties of these filters. In the limit, the polynomial spline method tends to the classical sinc interpolation, a property that can be traced back to [14]. We will look at this issue from a new perspective by studying the rates at which

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the spline interpolation filters converge to the ideal lowpass filter in the frequency domain. We will also investigate the rates at which the spline interpolation of the samples of a bandlimited signal converge to the signal when n tends to infinity.

Polynomial spline interpolation problems can be effectively resolved by finding the coefficients of the B-spline basis functions. The determination of these coefficients for equally spaced data points is referred to as the cardinal spline interpolation problem [12, 13]. Hou and Andrews pioneered the use of cubic spline interpolation in signal processing applications. They showed that the reconstruction part of the procedure is equivalent to applying a digital filter to the sequence of B-spline coefficients [4]. However, polynomial splines have not been very popular in the signal processing community for two main reasons. First, the algorithm proposed by Hou and Andrews for the determination of B-spline coefficients requires explicit and computationally expensive matrix operations. Second, there is a current incorrect belief that high order spline interpolation causes signal degradation [7, 8, 11]; this issue is further discussed in [16]. There are also a number of theoretical results in the mathematical literature that indicate that this cannot be the case. For instance, Schoenberg showed that the polynomial spline that interpolates a bandlimited function tends uniformly to the function itself as the order of the spline goes to infinity [14]. Marsden et al. also proved that the corresponding approximation error tends to zero in all L_p-norms as the order of the spline goes to infinity [6]. De Boor et al. have established similar convergence results in higher dimensions [2, 3]; see also [10].

It is only recently that the discrete version of the B-spline interpolation problem (e.g. zooming or signal magnification) has been recognized to be equivalent to a shift invariant filtering operation. Toraichi et al. derived the impulse response of the discrete quadratic spline interpolator [15]. They suggested a truncated approximation using a finite impulse response filter. More recently, Unser et al. considered the general case of discrete B-spline interpolators of any order and provided simple mechanisms for the determination of their frequency and impulse responses. More important, they recognized the recursive structure of these operators and derived fast algorithms for the direct and indirect B-spline transforms [16]. They also found through numerical computation that the frequency response of the cardinal spline interpolator of order n approaches an ideal lowpass filter for increasing values of n.

In this paper, we will build on these recent ideas and provide an interpretation of polynomial spline interpolation as a continuous filtering process. In Section 2, we derive the frequency responses of the cardinal spline interpolation filters and give a proof of their stability. We present the convergence results of the cardinal spline filters to the ideal lowpass filter in Section 3.1, and obtain error estimates for the resolution error and the interpolation error in Section 3.2. In Section 3.3, we get the rates at which the interpolants converge to their corresponding bandlimited signals as the order of the splines tends to infinity. A practical implication of these results is that high order spline interpolation can only improve signal reconstruction and that it is possible to determine an a priori value of the order of the spline for a given error tolerance.

2. B-spline interpolation

2.1. The discrete direct B-spline filters

The continuous B-spline representation of order n of a discrete signal $g(k)$ is obtained by finding the function $g^n(x)$ of the form

$$g^n(x) = \sum_{i=-\infty}^{i=+\infty} y[i]\beta^n(x-i), \tag{2.1}$$

such that

$$g^n(k) = g(k) \quad \forall k \in \mathbb{Z}. \tag{2.2}$$

The function β^n is the centered B-spline of order n; it can be generated iteratively by repeated convolution of a B-spline of order 0:

$$\beta^n(x) = \beta^0 * \beta^{n-1}(x), \tag{2.3}$$

where $\beta^0(x)$ is the characteristic function in the interval $[-\frac{1}{2}, \frac{1}{2})$, i.e.,

$$\beta^0(x)=\begin{cases}1, & x\in[-\frac{1}{2}, \frac{1}{2}),\\ 0, & \text{elsewhere.}\end{cases} \tag{2.4}$$

Both $g^n(x)$ and $\beta^n(x)$ are piecewise polynomial functions of degree n of class $C^{n-1}(-\infty, \infty)$. The polynomial segments are defined between all knot points and are joined together so that the interpolating function and its derivatives are continuous up to order $n-1$. A general procedure providing fast algorithms for finding the expansion coefficients $y[i]$ in (2.1) has been presented in [16]. The principle relies on the use of the discrete direct B-spline filter of order n which provides an efficient mean to solve the system of equations (2.1)–(2.2). The transfer function (z-transform) of this operator is given by

$$S^n(z)=(B_1^n(z))^{-1}=\left(\sum_{k=-[n/2]}^{[n/2]} b^n(k)z^{-k}\right)^{-1}. \tag{2.5}$$

We note that the filter coefficients can be derived iteratively for successive values of n through the recursive equations

$$\begin{aligned} b^n(k) &:= \beta^n(k)\\ &= n^{-1}(k+\tfrac{1}{2}(n+1))c^{n-1}(k)\\ &\quad + n^{-1}(\tfrac{1}{2}(n+1)-k)c^{n-1}(k-1), \end{aligned} \tag{2.6}$$

$$\begin{aligned} c^n(k) &:= \beta^n(k+\tfrac{1}{2})\\ &= n^{-1}(k+\tfrac{1}{2}(n+2))b^{n-1}(k+1)\\ &\quad + n^{-1}(\tfrac{1}{2}n-k)b^{n-1}(k+1), \end{aligned} \tag{2.7}$$

where, in (2.6) and (2.7), $k\in\mathbb{Z}$.

We have previously discussed the issue of efficient implementation of these filters and studied the operators up to $n=5$ [16]. For higher order splines, however, the question remains whether these filters are stable or not and this point will be treated in Section 2.3. Using (2.6) and (2.7), we have computed the discrete B-spline $b^n(k)$, $n=1,\ldots,7$. For reference, we give in Table 1 the transfer functions $S^n(z)$ of the corresponding direct B-spline filters together with their poles, which can be seen to be all negative and not to lie on the unit circle.

Table 1
Transfer functions and poles of direct B-spline filters for $n=0$ to 7

n	$S^n(z)$	Poles
0	1	–
1	1	
2	$\dfrac{8}{z+6+z^{-1}}$	$\{z_1=-3+2\sqrt{2},\ z_2=z_1^{-1}\}$
3	$\dfrac{6}{z+4+z^{-1}}$	$\{z_1=-2+\sqrt{3},\ z_2=z_1^{-1}\}$
4	$\dfrac{384}{z^2+76z+230+76z^{-1}+z^{-2}}$	$\{z_1=-0.361341,\ z_2=-0.0137254,\ z_3=z_1^{-1},\ z_4=z_2^{-1}\}$
5	$\dfrac{120}{z^2+26z+66+26z^{-1}+z^{-2}}$	$\{z_1=-0.430575,\ z_2=-0.0430963,\ z_3=z_1^{-1},\ z_4=z_2^{-1}\}$
6	$\dfrac{46080}{z^3+722z^2+10543z+23548+10543z^{-1}+722z^{-2}+z^{-3}}$	$\{z_1=-0.488295,\ z_2=-0.0816793,\ z_3=-0.00141415,$ $z_4=z_1^{-1},\ z_5=z_2^{-1},\ z_6=z_3^{-1}\}$
7	$\dfrac{5040}{z^3+120z^2+1191z+2416+1191z^{-1}+120z^{-2}+z^{-3}}$	$\{z_1=-0.53528,\ z_2=-0.122555,\ z_3=-0.009144869,$ $z_4=z_1^{-1},\ z_5=z_2^{-1},\ z_6=z_3^{-1}\}$

2.2. Interpolation as a filtering operation

In this section, we present an interpretation of polynomial spline interpolation as a continuous filtering operation of the initial sampled signal values. First, we note that (2.1) can be written as a convolution:

$$g^n(x)=(\beta^n * y_\delta)(x), \tag{2.8}$$

where we use the notation r_δ to represent the tempered distribution consisting of the train of weighted dirac delta impulses given by

$$r_\delta(x)=\sum_{k=-\infty}^{+\infty} r(k)\delta(x-k). \tag{2.9}$$

Since the B-splines are derived from the initial sampled values $g_\delta(x)$ through a shift-invariant filtering operation (the direct B-spline filter), we have that

$$y_\delta(x)=(s_\delta^n * g_\delta)(x), \tag{2.10}$$

where $s_\delta^n(x)$ is the continuous representation of the impulse response of the discrete filter defined by (2.5). It is then straightforward to express the interpolation function in terms of initial sample values:

$$g^n(x)=(\beta^n * s_\delta^n * g_\delta)(x)=(\eta^n * g_\delta)(x), \tag{2.11}$$

where $\eta^n(x)=(\beta^n * s_\delta^n)(x)$ is the continuous impulse response of the cardinal spline filter of order n which we call the cardinal spline of order n. In the approximation theory literature, this function is also commonly referred to as the fundamental spline. This process is illustrated by Fig. 1.

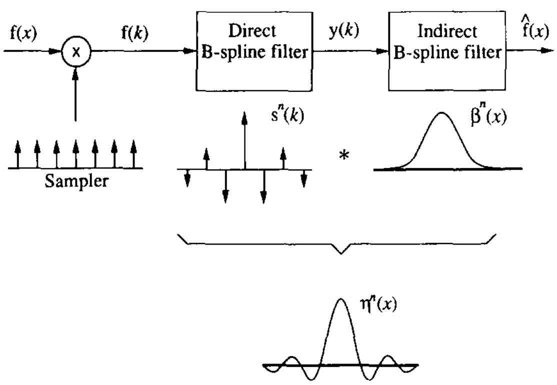

Fig. 1. Continuous filtering interpretation of a B-spline interpolation of order n.

From (2.3), we find that the Fourier transform of the B-spline of order n is given by

$$B^n(f)=\int_{-\infty}^{+\infty} \beta^n(x)\,\mathrm{e}^{-\mathrm{j}2\pi fx}\,\mathrm{d}x=\left(\frac{\sin(\pi f)}{\pi f}\right)^{n+1}. \tag{2.12}$$

The frequency response of $s_\delta^n(x)$ is obtained from (2.5) by simply replacing z by $\mathrm{e}^{\mathrm{j}2\pi f}$:

$$\begin{aligned} S^n(\mathrm{e}^{\mathrm{j}2\pi f}) &= \int_{-\infty}^{+\infty} s_\delta^n(x)\,\mathrm{e}^{-\mathrm{j}2\pi fx}\,\mathrm{d}x \\ &= \left(\beta^n(0)+\sum_{k=1}^{[n/2]} 2\beta^n(k)\cos(2\pi f)\right)^{-1}. \end{aligned} \tag{2.13}$$

It follows immediately that the frequency response of the nth order B-spline interpolator is

$$\begin{aligned} H^n(f) &= \int_{-\infty}^{+\infty} \eta^n(x)\,\mathrm{e}^{-\mathrm{j}2\pi fx}\,\mathrm{d}x \\ &= \frac{\left(\dfrac{\sin(\pi f)}{\pi f}\right)^{n+1}}{\beta^n(0)+\sum_{k=1}^{[n/2]} 2\beta^n(k)\cos(2\pi f)}. \end{aligned} \tag{2.14}$$

Alternatively, we may also consider that $s_\delta^n(x)$ is the inverse of the indirect B-spline filter whose impulse response $b^n(k)$ is obtained by sampling the continuous B-spline function $\beta^n(x)$. In the Fourier domain, sampling corresponds to the sum of a periodic repetition of the spectrum of the continuous function, which yields

$$\begin{aligned} B_1^n(f) &= \int_{-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} \beta^n(k)\delta(x-k)\,\mathrm{e}^{-\mathrm{j}2\pi fx}\,\mathrm{d}x \\ &= \sum_{i=-\infty}^{i=+\infty} \left(\frac{\sin(\pi(f-i))}{\pi(f-i)}\right)^{n+1} \end{aligned} \tag{2.15}$$

Therefore, we have also that

$$H^n(f)=\frac{\left(\dfrac{\sin(\pi f)}{\pi f}\right)^{n+1}}{\sum_{i=-\infty}^{i=+\infty} \left(\dfrac{\sin(\pi(f-i))}{\pi(f-i)}\right)^{n+1}}, \tag{2.16}$$

which is equivalent to (2.14). For illustration purposes, Fig. 2 displays $\beta^3(x)$, $B^3(f)$, $\eta^3(x)$ and

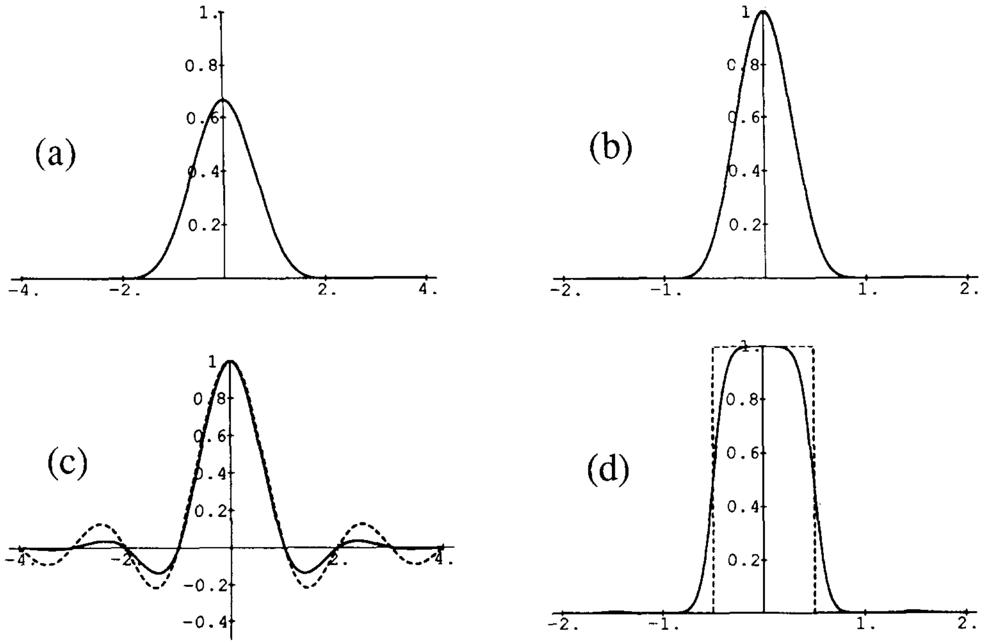

Fig. 2. Signals associated with the cubic spline interpolator. (a) Cubic B-spline, (b) Fourier transform of the cubic B-spline, (c) cardinal cubic spline, (d) Fourier transform of the cardinal cubic spline. The responses in dotted lines correspond to the ideal interpolator for bandlimited signals.

$H^3(f)$ which are the main functions that arise in the context of cubic spline interpolation. It can be seen that the cardinal spline $\eta^3(x)$ resembles a sinc function. The asymptotic behavior of the filters $H^n(f)$ will be examined systematically in Sections 3.1 and 3.2.

2.3. Stability of direct spline filters

The stability of the discrete direct B-spline filters given by (2.5) can be deduced from the general existence theorem of Schoenberg [12, Theorem 1]. We have chosen here to present a simple and direct proof of this result that is solely based on their transfer function. These filters are symmetrical and their stability follows directly from the following proposition.

PROPOSITION 1. The z-transform of the discrete direct B*-spline filters has no pole on the unit circle.*

PROOF. We only need to prove that $B_1^n(f)\neq 0$, $\forall f\in[0, \frac{1}{2}]$. The expression for $B_1^n(f)$ in (2.15) can be simplified to yield

$$B_1^n(f)=\sum_{i=-\infty}^{i=+\infty}(-1)^{(n+1)i}\left(\frac{\sin(\pi f)}{\pi(f-i)}\right)^{n+1}. \tag{2.17}$$

For n odd the terms of the series (2.17) are all positive, and hence $B_1^n(f)$ is strictly positive.

For n even we rewrite (2.17) as

$$B_1^n(f)=\left(\frac{\sin(\pi f)}{\pi f}\right)^{n+1}+\left(\frac{\sin(\pi f)}{\pi}\right)^{n+1} \times\sum_{i=1}^{i=+\infty}(-1)^i\left(\frac{1}{(f+i)^{n+1}}-\frac{1}{(i-f)^{n+1}}\right). \tag{2.18}$$

For $f\in[0, \frac{1}{2}]$ the series in the second term of the right-hand side of (2.18) is alternating. The absolute value of its terms are monotonically decreasing to zero and its first term is positive, hence the series is strictly positive. This together with the fact that $\sin(\pi f)$ is non-negative and $\sin(\pi f)/\pi f$ is positive for $f \in [0, \frac{1}{2}]$ imply that $B_1^n(f)$ is strictly positive. □

2.4. Extensions in higher dimensions

Although all our results are presented for the one-dimensional case, they are directly applicable to higher dimensions through the use of tensor product polynomial splines [9]. The corresponding basis functions are simply obtained from the product of one-dimensional functions of the individual index variables. For image processing applications, this means that the corresponding interpolation algorithms are separable and can be implemented by successive one-dimensional processing along the rows and columns.

There are also extensions using non-tensor product splines [1], but the computations and mathematical derivations tend to be more involved. De Boor et al. give a comprehensive treatment of bivariate cardinal spline interpolation on a three-direction mesh [2]. In particular, they prove that the two-dimensional box-spline analogue of the function $B_1^n(f)$ in (2.18) is strictly positive, from which they conclude that the corresponding interpolation problem is well posed. These authors also provide convergence results for 2D box-splines that are analogous to some of our results for the univariate case described in Theorems 1 and 3, below. In 1D, the situation is comparatively simpler and we will obtain a much wider variety of norm estimates, and hence stronger convergence results.

3. Convergence properties

The purpose of this section is to first show that the continuous cardinal spline filters $H^n(f)$ approach the ideal filter as n goes to infinity. We then provide explicit estimates and convergence rates for the resolution error, approximation error (Section 3.2). Finally, we determine the rates at which bandlimited signals are recovered from the polynomial spline interpolation of their samples (Section 3.3).

3.1. Convergence to the ideal filter

As a preliminary, we start by the following three Lemmas.

LEMMA 1. The functions $H^n(f)$ converge pointwise to 1 $\forall f \in (0, \frac{1}{2})$. *Moreover, $H^n(f)$ converge to* 1 *in $L_p(0, \frac{1}{2})$ as n goes to infinity $\forall p \in [1, \infty)$.*

LEMMA 2. For n odd, the functions $H^n(f)$ converge pointwise to 0 $\forall f \in (\frac{1}{2}, \infty)$. *Moreover, $H^n(f)$ converges to* 0 *in $L_p(\frac{1}{2}, \infty)$ as n goes to infinity $\forall p \in [1, \infty)$.*

LEMMA 3. For n even, the functions $H^n(f)$ converge pointwise to 0 $\forall f \in (\frac{1}{2}, \infty)$. *Moreover, $H^n(f)$ converge to* 0 *in $L_p(\frac{1}{2}, \infty)$ as n goes to infinity $\forall p \in [1, \infty)$.*

As a corollary of these three Lemmas, we immediately get the following theorem.

THEOREM 1. The Fourier transforms of the cardinal spline interpolators $H^n(f)$ converge pointwise to an ideal lowpass filter as n tends to infinity:

$$\lim_{n\to\infty} H^n(f) = \mathrm{Rect}(f) = \begin{cases} 1, & |f| < \frac{1}{2}, \\ \frac{1}{2}, & |f| = \frac{1}{2}, \\ 0, & |f| > \frac{1}{2}. \end{cases}$$

Moreover, $H^n(f)$ converges to $\mathrm{Rect}(f)$ *in $L_p(-\infty, +\infty)$ as n goes to infinity $\forall p \in [1, \infty)$.*

It should be noted that the convergences in the L_p-norms are stronger than the pointwise convergence. The reason is that the L_p convergence result can be interpreted in both the time and frequency domains. For example, L_2 convergence in the frequency domain implies L_2 convergence of the corresponding time functions in the time domain (Parseval's identity). In fact, the result of Theorem 1 implies that the cardinal spline interpolators η^n

converge to the ideal sinc interpolator in L_q-norms for all q in $[2, +\infty)$.

PROOF OF LEMMA 1. Using (2.17), (2.16) can be simplified to yield

$$H^n(f) = \begin{cases} 0, & f \text{ integer and } f \neq 0, \\ 1, & f = 0, \\ \left(\sum_{i=-\infty}^{i=+\infty} (-1)^{(n+1)i} \left(1 - \frac{i}{f}\right)^{-n-1} \right)^{-1}, & \\ \text{elsewhere.} & \end{cases} \tag{3.1}$$

We rewrite the series in (3.1) as

$$D^n(f) = 1 + U^n(f). \tag{3.2}$$

where

$$U^n(f) = \begin{cases} \sum_{i=1}^{i=+\infty} ((i/f+1)^{-n-1} + (i/f-1)^{-n-1}), \\ n \text{ odd}, \\ \sum_{i=1}^{i=+\infty} (-1)^i ((i/f+1)^{-n-1} - (i/f-1)^{-n-1}), \\ n \text{ even}. \end{cases} \tag{3.3}$$

For n even and $f \in (0, \frac{1}{2})$, the series in (3.3) is alternating, the absolute values of its terms are monotonically decreasing to zero, and its first term is positive. Hence the series takes a strictly positive value. For n odd and $f \in (0, \frac{1}{2})$, the series is also positive since all of its terms are positive. Using these two facts, we can estimate $|H^n(f) - 1|$ by

$$|H^n(f) - 1| = \frac{|U^n(f)|}{|1 + U^n(f)|} \leq U^n(f). \tag{3.4}$$

For $f \in (0, \frac{1}{2})$ and $1 \leq n$, we consider the two integrals:

$$I^+ = \int_1^{\infty} (1 + x/f)^{-n-1} \, dx = \frac{f}{n} (1/f + 1)^{-n}, \tag{3.5}$$

$$I^- = \int_1^{\infty} (x/f - 1)^{-n-1} \, dx = \frac{f}{n} (1/f - 1)^{-n}. \tag{3.6}$$

With the help of (3.5) and (3.6), we use a convexity argument to estimate the value of $U^n(f)$ by

$$0 < U^n(f) \leq (1/f + 1)^{-n-1} + (1/f - 1)^{-n-1} + I^+ + I^-. \tag{3.7}$$

For $f \in (0, \frac{1}{2})$, all the terms of the right-hand side of (3.7) converge to zero as n goes to infinity which proves the pointwise convergence. To prove the second part of the lemma we only need to look at the L_p-norms of the terms on the right-hand side of (3.7). We raise the first term to the power p and integrate to get

$$\int_0^{1/2} (1/f + 1)^{-(n+1)p} \, df \leq \int_0^{1/2} 3^{-(n+1)p} \, df \leq 3^{-(np+p)} (2)^{-1}. \tag{3.8}$$

Similar estimates for the other terms yield

$$\int_0^{1/2} (1/f - 1)^{-(n+1)p} \, df \leq \int_0^{1/2} (2f)^{(n+1)p} \, df \leq 2^{-1} (np + p + 1)^{-1}, \tag{3.9}$$

$$\int_0^{1/2} |I^+|^p \, df \leq \frac{3^{-np} 2^{-(p+1)}}{n^p (p+1)}, \tag{3.10}$$

$$\int_0^{1/2} |I^-|^p \, df \leq 2^{-1} (np + p + 1)^{-1}. \tag{3.11}$$

From the above estimates, we immediately deduce the second part of the lemma. □

PROOF OF LEMMA 2. For n odd and for $f \in (\frac{1}{2}, 1)$, all the terms of the series (3.3) are positive, hence we can estimate $|H^n(f)|$ from above by

$$|H^n(f)| \leq \frac{1}{1 + (1/f - 1)^{-n-1}} \leq \frac{(2(1-f))^{n+1}}{2}. \tag{3.12}$$

Since $f \in (\frac{1}{2}, 1)$, the right-hand side of (3.12) goes to zero as n tends to infinity. Thus $|H^n(f)|$ tends pointwise to zero.

For $1<k<f<k+1$ and n odd, we can estimate $|H^n(f)|$ from above by

$$|H^n(f)| \leqslant \mathrm{Max}\left(\frac{(k+1-f)^{n+1}}{(f)^{n+1}}, \frac{(k+1-f)^{n+1}}{(f)^{n+1}}\right) \leqslant \frac{1}{(2f)^{n+1}}. \tag{3.13}$$

Since $f>1$, $H^n(f)$ tends pointwise to zero as n tends to infinity.

To estimate the L_p-norm of $H^n(f)$, we use the inequalities (3.12) and (3.13) to get

$$\int_{1/2}^{\infty} |H^n(f)|^p \,\mathrm{d}f \leqslant \int_{1/2}^{1} (1/f-1)^{(n+1)p} \,\mathrm{d}f + \int_{1}^{\infty} (2f)^{-(n+1)p} \,\mathrm{d}f \leqslant 2^{-(p+1)}(np+p+1)^{-1} + 2^{-(n+1)p}(np+p-1)^{-1}. \tag{3.14}$$

From inequality (3.14) the second part of the lemma follows. □

PROOF OF LEMMA 3. For n even, we rewrite $U^n(f)$ in (3.2) as the sum of two series as follows:

$$U^n(f) = U_1^n(f) + U_2^n(f), \tag{3.15}$$

where U_1^n and U_2^n are given by

$$U_1^n(f) = \sum_{i=1}^{i=+\infty} (-1)^i (i/f+1)^{-n-1}, \tag{3.16}$$

$$U_2^n(f) = \sum_{i=1}^{i=+\infty} (-1)^{i+1} (i/f-1)^{-n-1}. \tag{3.17}$$

We note that $1+U_1^n(f)$ is an alternating series with the absolute value of its terms monotonically decreasing to zero, hence it is positive and bounded above by 1 $\forall f>0$. Moreover for $f\in(\frac{1}{2}, 1)$ the alternating series $U_2^n(f)$ is also positive. Using these two facts we can estimate $|H^n(f)|$ for $f\in(\frac{1}{2}, 1)$ by

$$|H^n(f)| \leqslant \frac{1}{|U_2^n(f)|}. \tag{3.18}$$

Using the alternating and decaying properties of the series $U_2^n(f)$, we can estimate it from below by

$$|U_2^n(f)| \geqslant (1/f-1)^{-n-1} - (2/f-1)^{-n-1} \geqslant (1/f-1)^{-n-1}\left(1-\frac{(1-f)^{n+1}}{(2-f)^{n+1}}\right) \geqslant \tfrac{2}{3}(1/f-1)^{-n-1}. \tag{3.19}$$

Using this inequality in (3.18) we get

$$|H^n(f)| \leqslant \tfrac{3}{2}(1/f-1)^{n+1} \quad \forall f\in(\tfrac{1}{2}, 1). \tag{3.20}$$

Inequality (3.20) immediately yields the pointwise convergence of $H^n(f)$ for $f\in(\frac{1}{2}, 1)$.

For an integer k with $1<k<f<k+1$, we estimate $|H^n(f)|$ from above by

$$|H^n(f)| \leqslant \frac{1}{|U_2^n(f)|\,|a-1|}, \tag{3.21}$$

where a is given by

$$a = \frac{|1+U_1^n(f)|}{|U_2^n(f)|}. \tag{3.22}$$

As before, $1+U_1^n(f)$ is positive and bounded above by 1 $\forall f>0$. The series $U_2^n(f)$ can be decomposed into the sum of two series as follows:

$$U_2^n(f) = \sum_{i=1}^{i=k} (-1)^{i+1}(i/f-1)^{-n-1} + \sum_{i=k+1}^{i=\infty} (-1)^{i+1}(i/f-1)^{-n-1}. \tag{3.23}$$

For $f\in(1, 2)$, we can estimate $U_2^n(f)$ using the decomposition (3.23) by

$$|U_2^n(f)| \geqslant (1-1/f)^{-n-1} + (2/f-1)^{-n-1} - (3/f-1)^{-n-1} \geqslant \frac{f^{n+1}}{(2-f)^{n+1}}\left(1-\frac{(2-f)^{n+1}}{(3-f)^{n+1}}\right) + \frac{f^{n+1}}{(f-1)^{n+1}} \geqslant \tfrac{3}{2}f^{n+1}. \tag{3.24}$$

For an integer k with $2<k<f<k+1$, we estimate $U_2^n(f)$ using the decomposition (3.23) to get

$$|U_2^n(f)| \geqslant \left(1-\frac{k}{f}\right)^{-n-1} - \left(1-\frac{k-1}{f}\right)^{-n-1} + \left(\frac{k+1}{f}-1\right)^{-n-1} - \left(\frac{k+2}{f}-1\right)^{-n-1}$$
$$\geqslant \frac{f^{n+1}}{(f-k)^{n+1}}\left(1-\frac{(f-k)^{n+1}}{(f+1-k)^{n+1}}\right) + \frac{f^{n+1}}{(k+1-f)^{n+1}}\left(1-\frac{(k+1-f)^{n+1}}{(k+2-f)^{n+1}}\right)$$
$$\geqslant f^{n+1}. \tag{3.25}$$

Using (3.21) and the last two inequalities we can estimate $|H^n(f)|$ by

$$|H^n(f)| \leqslant \frac{3}{(f)^{n+1}}, \quad \text{for } 1<k<f<k+1 \text{ and } k \text{ integer.} \tag{3.26}$$

Inequality (3.26) implies the pointwise convergence of $H^n(f)$ for $f>1$. To complete the proof of this lemma, we use (3.20) and (3.26) to get

$$\int_{1/2}^{\infty} |H^n(f)|^p \, \mathrm{d}f \leqslant 3\int_{1/2}^{1} (1/f-1)^{(n+1)p} \, \mathrm{d}f + 3\int_{1}^{\infty} (f)^{-(n+1)p} \, \mathrm{d}f \leqslant \frac{6}{np}. \tag{3.27}$$

□

3.2. *Rates of convergence to the ideal lowpass filter*

The rates of convergence of the cardinal spline filter to the ideal lowpass filter can be used to provide a quantitative manner to determine the appropriate spline order needed to match a maximal error tolerance. In a manner similar to that of Pratt [8, pp. 118–119], we define two types of errors due to this non-ideal interpolation: The resolution error Δ_{R} and the interpolation error Δ_{H} which are given by

$$\Delta_{\mathrm{R}}(n,p) = \|H^n(f)-1\|^p_{L_p(-1/2,1/2)}, \tag{3.28}$$

$$\Delta_{\mathrm{H}}(n,p) = \|H^n(f)\|^p_{L_p(\mathbb{R}/(-1/2,1/2))}. \tag{3.29}$$

If the cardinal spline filters are used to reconstruct a bandlimited signal from its Nyquist samples, then Δ_{R} measures the attenuation effect on the signal frequencies (the bandpass region of the filter). The interpolation error Δ_{H} measures the high frequency modes that are added to the signal. In this section, we will provide estimates in the case for which n is odd. Similar estimates can be obtained for the case n even.

We start by defining the following functions:

$$A1(n,p) = 3^{-(np+p)}(2)^{-1},$$
$$A2(n,p) = 2^{-(p+1)}(np+p+1)^{-1},$$
$$A3(n,p) = \frac{3^{-np}2^{-(p+1)}}{n^p(p+1)},$$
$$A4(n,p) = 2^{-2p}n^{-p}(np+p+1)^{-1} + p2^{-p}n^{-p} \times (np+p+1)^{-1}(np+p+2)^{-1}, \tag{3.30}$$
$$A5(n,p) = 2^{-(np+p)}(np+p-1)^{-1},$$
$$A_{\mathrm{R}}(n,p) = 2((A1)^{1/p}+(A2)^{1/p} + (A3)^{1/p}+(A4)^{1/p})^p = \mathrm{O}((2^p pn)^{-1}),$$
$$A_{\mathrm{H}}(n,p) = 2(A5(n,p)+A2(n,p)) = \mathrm{O}((2^p pn)^{-1}).$$

We collect the convergence rates for the two type of errors in the following theorem.

THEOREM 2. The resolution error Δ_{R} and the interpolation error Δ_{H} can be estimated by

$$\Delta_{\mathrm{R}}(n,p) \leqslant A_{\mathrm{R}}(n,p), \tag{3.31}$$

$$\Delta_{\mathrm{H}}(n,p) \leqslant A_{\mathrm{H}}(n,p). \tag{3.32}$$

The dominant term in both A_{R} and A_{H} is $A2(n,p)$; the other terms are negligible for sufficiently large values of n. Thus, the interpolation and resolution errors are of the order of $(2^p pn)^{-1}$ as n tends to infinity. Using the results of Theorem 2, a simple computation shows that, for the spline of order 5 and for $p=2$, the resolution error is no more than 2.6% while the interpolation error is no more than 2%. Given that errors of this magnitude are usually acceptable, the use of the cardinal spline filter of order five should be sufficient for most applications.

From (3.31) and (3.32), it is easy to see that the two types of errors tend to zero as n goes to infinity which means that a bandlimited signal can ultimately be reconstructed from its samples by cardinal spline interpolation. This result is in contradiction with the one presented in [8, pp. 116–120]. The reason for this discrepancy can be easily explained as follows.

To represent a discrete signal $g(k)$, Pratt in [8] uses the B-spline function of order n to get a continuous signal given by

$$\hat{g}^n(x)=\sum_{i=-\infty}^{i=+\infty} g[i]\beta^n(x-i).$$

It is easy to see from the above equation that $\hat{g}^n(k)$ will not be equal to $g(k)$ unless $n=0$ or $n=1$. In order to satisfy (2.1), the B-spline function β^n in the equation above should be replaced by the cardinal spline η^n (see (2.11)).

If this is not done, a continuous signal and the continuous representation of its sampled values will ultimately lose their resemblance for sufficiently large values of n. This explains why the author in [8] correctly finds, in his approach, that the resolution error grows as n gets large. Obviously, this effect will not appear if (2.11) is used instead.

To get error bounds for the cardinal spline of order n in the time domain, we only need to use the boundedness property of the Fourier operator (Titchmarsh's inequality) to obtain

$$\|\eta^n(x)-\operatorname{sinc}(x)\|_{L_q(\mathbb{R})} \leqslant (A_{\mathrm{R}}(n,(1-q)/q)+A_{\mathrm{H}}(n,(1-q)/q))^{(1-q)/q}, \tag{3.33}$$

where the above estimate is valid as long as $2\leqslant q\leqslant\infty$.

PROOF OF THEOREM 2. We start from (3.4) to get for n odd:

$$|H^n(f)-1|=\frac{|U^n(f)|}{|1+U^n(f)|} \leqslant (1/f+1)^{-n-1}+I^+ + t1+t2, \tag{3.34}$$

where I^+ is as before, and $t1$ and $t2$ are given by

$$t1=\frac{(1/f-1)^{-n-1}}{1+(1/f-1)^{-n-1}},\qquad t2=\frac{I^-}{1+(1/f-1)^{-n-1}}. \tag{3.35}$$

We estimate $t1$ by

$$t1=\frac{f^{n+1}}{(1-f)^{n+1}+f^{n+1}}\leqslant\frac{(2f)^{n+1}}{2}, \tag{3.36}$$

which yields the bound

$$\int_0^{1/2}(t1)^p\,\mathrm{d}f\leqslant 2^{-(p+1)}(np+p+1)^{-1}. \tag{3.37}$$

A similar estimate for $t2$ gives

$$t2=\frac{f^{n+1}(1-f)}{n((1-f)^{n+1}+f^{n+1})}\leqslant\frac{(2f)^{n+1}}{2n}(1-f), \tag{3.38}$$

from which, using integration by parts, we get the estimate

$$\int_0^{1/2}(t2)^p\,\mathrm{d}f\leqslant A4(n,p). \tag{3.39}$$

From (3.8), (3.10), (3.37) and (3.39) we obtain the first inequality of the theorem. The second bound is directly obtained from (3.14). □

3.3. Convergence properties for bandlimited signals

In this section, we use the convergence results of the previous section to estimate the rate at which the spline interpolation of the samples of a bandlimited signal converges to the signal when n tends to infinity. In all the following we assume n to be odd, although similar results can be derived for n even.

We start by defining the following functions:

$$E_1(n,p,r,s)=2(A_{\rm H}(n,pr)/2)^{1/r}+(A_{\rm R}(n,pr))^{1/r} + \frac{(3)^{-p(n+1)+1}}{(np+p-1)}+2(3)^{-np-p}. \qquad (3.40)$$

THEOREM 3. Let $g(x)$ be a function and $G(f)$ be its Fourier transform. Assume that the support of G is in $[-\frac{1}{2},\frac{1}{2}]$. Let $G_n(f)$ be the Fourier transform of the cardinal spline interpolation of the Nyquist samples of g. We have the following error bounds:

$$\|G(f)-G_n(f)\|^p_{L_p(\mathbb{R})} \leqslant E_1(n,p,s/s-1,s)\|G(f)\|^p_{L_{ps}} \quad \forall s>1. \qquad (3.41)$$

This theorem gives an upper bound on the interpolation error for bandlimited functions. Using (3.30) and (3.40), it is easily seen that the dominant term in E_1 is of the order of $(2^{ps}psn)^{-(1-1/s)}$. Taking the limit as n tends to infinity, E_1 converges to zero. This implies that the L_p-norm of the interpolation error between a bandlimited function and its spline approximation vanishes as the value of n increases with a rate proportional to $(2^{ps}psn)^{-(1-1/S)}$. Using a conjugacy argument, we can deduce from Theorem 3 the convergence results of Schoenberg and Marsden et al. [6, 14]. Specifically, these authors showed that in the limit $(n\to+\infty)$, the approximation error measured in the time domain tends to zero.

PROOF OF THEOREM 3. To establish our claim, we will split the argument into three parts making estimates on four different regions of the real line:

$$[-\tfrac{1}{2},\tfrac{1}{2}],\ (\tfrac{1}{2},\tfrac{3}{2}]\cup[-\tfrac{3}{2},-\tfrac{1}{2}),\ (\tfrac{3}{2},\tfrac{5}{2}]\cup[-\tfrac{5}{2},-\tfrac{3}{2})$$

and

$$(\tfrac{5}{2},\infty)\cup(-\infty,-\tfrac{5}{2}).$$

First, we look at the difference between the Fourier transforms of the function and its interpolation in the interval $[-\frac{1}{2},\frac{1}{2}]$, we use (3.31) and the Hölder inequality to get

$$\begin{aligned}
&\|G(f)-G_n(f)\|^p_{L_p(-1/2,1/2)} \\
&= \|G(f)-H^n(f)G_\delta(f)\|^p_{L_p(-1/2,1/2)} \\
&\leqslant \|H^n(f)-1\|^p_{L_{pr}(-1/2,1/2)}\|G(f)\|^p_{L_{ps}} \\
&\leqslant (A_{\rm R}(n,pr))^{1/r}\|G(f)\|^p_{L_{ps}}, \qquad (3.42)
\end{aligned}$$

where $1/r+1/s=1$ and where G_δ is the Fourier transform of g_δ and is characterized by the periodic repetition of G.

For the interval $(\frac{5}{2},\infty)\cup(-\infty,-\frac{5}{2})$ we use (3.13) and the Hölder inequality to obtain

$$\begin{aligned}
&2\int_{5/2}^{\infty}|H^n(f)G_\delta(f)|^p\,\mathrm{d}f \\
&\leqslant 2\times 2^{-p(n+1)}\|G(f)\|^p_{L_p}\sum_{i=1}^{i=\infty}(3/2+i)^{-p(n+1)} \\
&\leqslant \frac{(3)^{-p(n+1)+1}}{(np+p-1)}\|G(f)\|^p_{L_p} \\
&\leqslant \frac{(3)^{-p(n+1)+1}}{(np+p-1)}\|G(f)\|^p_{L_{ps}}. \qquad (3.43)
\end{aligned}$$

Again, using (3.13) and the Hölder inequality, for the interval $(\frac{3}{2},\frac{5}{2}]\cup[-\frac{3}{2},-\frac{5}{2})$ we have

$$\begin{aligned}
2\int_{3/2}^{5/2}|H^n(f)G_\delta(f)|^p\,\mathrm{d}f &\leqslant 2(3)^{-np-p}\|G(f)\|^p_{L_p} \\
&\leqslant 2(3)^{-np-p}\|G(f)\|^p_{L_{ps}}. \qquad (3.44)
\end{aligned}$$

For the intervals $(\frac{1}{2},\frac{3}{2}]\cup[-\frac{3}{2},-\frac{1}{2})$, we use (3.32) and the Hölder inequality to get

$$\begin{aligned}
&2\int_{1/2}^{3/2}|H^n(f)G_\delta(f)|^p\,\mathrm{d}f \\
&\leqslant 2\|G(f)\|^p_{L_{ps}}\left(\int_{1/2}^{3/2}|H^n(f)|^{pr}\,\mathrm{d}f\right)^{1/r} \\
&\leqslant 2\|G(f)\|^p_{L_{ps}}(\Delta_{\rm H}(n,pr)/2)^{1/r} \\
&\leqslant 2\|G(f)\|^p_{L_{ps}}(A_{\rm H}(n,pr)/2)^{1/r}, \qquad (3.45)
\end{aligned}$$

where r and s are related by conjugacy $(1/r + 1/s = 1)$.

The theorem then follows by collecting the terms in the inequalities (3.42)–(3.45). □

4. Conclusion

In this paper, we interpreted B-spline interpolation as a continuous filtering process and demonstrated the stability of the discrete direct B-spline filters of all orders. We proved that the resolution error and the interpolation error of the cardinal spline filters converge to zero as the order of the spline tends to infinity and provided their rates of convergence. It is possible to use these rates to choose the spline order needed to maintain a given maximal error tolerance. For instance, we have shown that a spline of order five produces a resolution error that is no larger than 2.6% in energy and an interpolation error no larger than 2%; errors that are usually acceptable for practical application in image processing. Finally, we obtained the rates at which bandlimited signals can be recovered from their samples as the order of the spline tends to infinity.

We believe that the present results provide a theoretical as well as a practical ground for signal processing using cardinal spline interpolation. These results together with the availability of fast B-spline interpolation algorithms should give a good and practical alternative to currently accepted interpolation techniques in image and signal processing.

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