Families of Wavelet Transforms in Connection with Shannon's Sampling Theory and the Gabor Transform

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Abstract. In this chapter, we look at the algebraic structure of nonorthogonal scaling functions, the multiresolutions they generate, and the wavelets associated with them. By taking advantage of this algebraic structure, it is possible to create families of multiresolution representations and wavelet transforms with increasing regularity that satisfy some desired properties. In particular, we concentrate on two important aspects. First, we show how to generate sequences of scaling functions that tend to the ideal low-pass filter and for which the corresponding wavelets converge to the ideal bandpass filter. We give the conditions under which this convergence occurs and provide the link between Mallat's theory of multiresolution approximations and the classical Shannon Sampling Theory. This offers a framework for generating generalized sampling theories. Second, we construct families of nonorthogonal wavelets that converge to Gabor functions (modulated Gaussians). These latter functions are optimally concentrated in both time and frequency and are therefore of great interest for signal and image processing. We obtain the conditions under which this convergence occurs; thus allowing us to create whole classes of wavelets with asymptotically optimal time-frequency localization. We illustrate the theory using polynomial splines.

§1. Introduction

A recurrent question in signal processing is how to best represent signals. This issue arises naturally in applications such as detection, data storage, data compression, signal analysis, signal transmission, etc. When the signals are continuous, it is usually desired to represent them by discrete sequences. One approach is simply to sample the functions on uniform grids. However, there are infinitely many continuous functions having the same sample values. For this reason, a signal is first preprocessed and sampled so that, after the whole process, the sample values are the "best" description of the original signal.

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This procedure is contained in the well known Shannon Sampling Theorem ([15] and [21]). The theory as well as the class of functions that are completely characterized by their sample values will be reviewed in Section 6.

Another widely used approach is to discretize a signal by describing it as a weighted sum of elementary signal packets (in fact Shannon's sampling theory can also be described in this way). The Gabor representation uses modulated Gaussian functions as the elementary packets [14]. This representation has been praised because modulated Gaussian functions are optimal in terms of their time-frequency localization. This means that the expansion coefficients describe, in an optimal sense, the frequency components at their occurrence in time (or space).

Recently, the concepts of multiresolutions and wavelet transforms have offered an infinite variety of discrete representations to choose from ([13], [17], and [19]). Moreover, the wavelet representations have the desirable property of being localized in both the time (space) and the frequency domains. However, no wavelets can achieve the optimal time-frequency localization of Gabor functions. Still, we will show how to construct families of wavelets that tend to Gabor functions.

In this chapter, we present a unified view of these various signal representation theories. To do this, we identify their essential properties. We use these properties to extend some results from multiresolution analysis and to generalize Shannon's Sampling Theory. We then introduce the new concept of wavelet sequences and apply it to create "Shannon's-type" and "Gabor's-type" scaling and wavelets sequences.

The presentation is organized as follows: in Section 3, we briefly review Stephane Mallat's theory of multiresolution analysis and orthogonal wavelet transforms. In Section 4, we show how to extend some of his results to nonorthogonal scaling functions. We define equivalent classes of scaling and wavelet functions and construct basis functions with certain special properties. In Section 5, we define related sequences of scaling and wavelet functions with an increasing regularity index n. We concentrate on three particular sequences: the basic, the cardinal, and the orthogonal sequences. The connection with Shannon's Theory is provided in Section 6. In particular, we show that the cardinal and orthogonal sequences tends to ideal lowpass and bandpass filters as the regularity index n tends to infinity. Finally, in Section 7, we discuss the Gabor transform which is optimal in terms of its time-frequency localization. We then show that the basic wavelet sequence constructed in Section 5 tends to Gabor functions. This implies that we can obtain wavelets that are nearly time-frequency optimal. The various aspects of the theory are illustrated using polynomial splines. Many results in this chapter are new; their proofs can be found in [1] and will be published elsewhere.

§2. Definitions and notations

The signals considered here are defined on $\mathbb{R}$ and belong to the space of measurable, square-integrable functions: L_2. The space of square summable sequences (discrete functions) is denoted by l_2.

The symbol "$*$" will be used for three slightly different binary operations that are defined below: the convolution, the mixed convolution, and the discrete convolution . The ambiguity should be easily resolved from the context.

For two functions f and g defined on $\mathbb{R}$, $*$ denotes the usual convolution:

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(\xi)g(x-\xi)d\xi, \quad x \in \mathbb{R}. \tag{1}$$

The mixed convolution between a sequence $b(k), k \in \mathbb{Z}$ and a function f on $\mathbb{R}$ is the function $b * f$ on $\mathbb{R}$ given by:

$$(b * f)(x) = \sum_{k=-\infty} b(k)f(x-k), \quad x \in \mathbb{R}. \tag{2}$$

The discrete convolution between two sequences a and b is the sequence $a * b$:

$$(a * b)(k) = \sum_{i=-\infty}^{i=+\infty} a(i)b(k-i), \quad k \in \mathbb{Z}. \tag{3}$$

Whenever it exists, the convolution inverse $(b)^{-1}$ of a sequence b is defined to be:

$$\left((b)^{-1} * b\right)(k) = \delta_0(k), \tag{4}$$

where δ_0 is the unit impulse; *i.e.*, $\delta_0(0) = 1$ and $\delta_0(k) = 0$ for $k \neq 0$.

The reflection of a function f (resp., a sequence b) is the function f' (resp., the sequence b') given by:

$$f'(x) = f(-x), \quad \forall\, x \in \mathbb{R}, \tag{5}$$

$$b'(k) = b(-k), \quad \forall\, k \in \mathbb{Z}. \tag{6}$$

The modulation $\tilde{b}(k)$ of a sequence b is obtained by changing the signs of the odd components of b:

$$\tilde{b}(k) = (-1)^k b(k). \tag{7}$$

The decimation (or down-sampling) operator by a factor of two $[.]_{\downarrow 2}$ assigns to a sequence b the sequence $[b]_{\downarrow 2}$ which consists of the even components of b only:

$$[b]_{\downarrow 2}(k) = b(2k), \quad \forall\, k \in \mathbb{Z}. \tag{8}$$

A pseudo-inverse to the decimation operator is the up-sampling operator $[.]_{\uparrow 2}$. This operator expands a sequence b by inserting zeros between its components:

$$[b]_{\uparrow 2}(k) = \begin{cases} b(k/2), & k \text{ even} \\ 0, & k \text{ odd} \end{cases}. \tag{9}$$

§3. Multiresolution approximations

As defined by Stephane Mallat, a multiresolution structure of L_2 is a sequence of spaces $V_{(j)}$ satisfying:

(i) $\text{clos}\left(V_{(0)}\right) = V_{(0)} \subset L_2$;

(ii) $V_{(j)} \subset V_{(j-1)}$;

(iii) $\text{clos}\left(\bigcup_{j=-\infty}^{j=+\infty} V_{(j)}\right) = L_2$;

(iv) $\bigcap_{j=-\infty}^{j=+\infty} V_{(j)} = \{0\}$;

(v) $f(x) \in V_{(j)} \Leftrightarrow f(2x) \in V_{(j-1)}$;

(vi) there exists an isomorphism I *from* $V_{(0)}$ onto l_2 which commutes with the action of $\mathbb{Z}$.

Relative to such a structure, a multiresolution approximation $\{\ldots, s_{(-1)}, s_{(0)}, s_{(1)}, \ldots\}$ of a signal s consists of the L_2 least square approximations $s_{(j)}$ of s on the spaces $V_{(j)}$. Because of property (ii), if $i < j$ then $s_{(i)}$ can be viewed as a finer approximation than $s_{(j)}$. In signal processing, $\{\ldots, s_{(-1)}, s_{(0)}, s_{(1)}, \ldots\}$ is called a fine-to-coarse pyramid representation. It is used for finding efficient algorithms in applications such as coding, edge detection, texture discrimination, fractal analysis, etc. ([6], [17], [23], and [29]).

Properties (i), (ii), and (iii) allow us to approximate a function to any desired accuracy by taking a sufficiently large approximation space $V_{(j)}$ while property (iv) states that coarser and coarser approximations will eventually contain no information about the original function. The similarity property (v) and property (vi) are essential for numerical computations. An important result due to S. Mallat is that, in a multiresolution structure, each subspace $V_{(j)}$ can be induced by an orthonormal basis $\left\{2^{-j/2}\phi(2^{-j}x - k)\right\}_{k\in\mathbb{Z}}$. The basis vectors are generated by dilating and translating a unique function ϕ called the orthogonal scaling function. Formally, this can be stated as follows:

$$V_{(j)}(\phi) = \left\{ v : v(x) = \sum_{k=-\infty}^{\infty} c_{(j)}(k)\phi_{2^j}(x - 2^j k), \quad c_{(j)} \in l_2 \right\}, \tag{10}$$

where

$$\phi_{2^j}(x) = 2^{-j/2}\phi(2^{-j}x). \tag{11}$$

Associated with the space $V_{(j)}$, is the wavelet space $W_{(j)}$ which is defined to be the orthogonal complement of $V_{(j)}$ relative to $V_{(j-1)}$:

$$W_{(j)} \oplus V_{(j)} = V_{(j-1)}. \tag{12}$$

Similarly to $V_{(j)}$, the wavelet space $W_{(j)}$ can also be generated by an orthonormal basis $\left\{2^{-j/2}\psi(2^{-j}x - k)\right\}_{k\in\mathbb{Z}}$ induced by the orthogonal wavelet function

ψ ([13], [17], and [19]):

$$W_{(j)}(\psi) = \left\{ w : w(x) = \sum_{k=-\infty}^{\infty} d_{(j)}(k)\psi_{2^j}(x - 2^j k), \quad d_{(j)} \in l_2 \right\}, \tag{13}$$

where

$$\psi_{2^j}(x) = 2^{-j/2}\psi(2^{-j}x). \tag{14}$$

Equation (12) implies that the difference between the two approximations $s_{(j)}$ and $s_{(j-1)}$ of a function s is equal to the orthogonal projection of s onto $W_{(j)}$. For this reason, the spaces $W_{(j)}$ are also called the detail spaces, or sometimes the residual spaces. They constitute a direct sum of the square integrable function space L_2:

$$L_2 = \bigoplus_j W_{(j)} := \ldots \oplus W_{(1)} \oplus W_{(0)} \oplus W_{(-1)} \oplus \ldots. \tag{15}$$

Thus, a function s can be represented by the coefficients $\{\ldots\mathbf{d}_{(-1)}, \mathbf{d}_{(0)}, \mathbf{d}_{(1)}, \ldots\}$ (in Equation (13)) of the orthogonal projection of s onto $W_{(j)}$. This representation is what is called the discrete wavelet transform.

The simplest example of orthogonal scaling function is the rectangular or indicator function $\chi_{[0,1)}$ which takes the value 1 in the interval $[0, 1)$ and the value 0 elsewhere. The corresponding wavelet is the Haar function which takes the value 1 in $[0, \frac{1}{2})$ and -1 in the interval $[\frac{1}{2}, 1)$. The associated multiresolution spaces $V_{(j)}$ are the piecewise constant functions or splines of order zero. Other examples are the orthogonal spline scaling functions and their corresponding orthogonal spline wavelets which have been found independently by Lemarié and Battle ([5] and [16]).

Another important scaling function is the sinc function (*i.e.*, the impulse response of the ideal lowpass filter) used in the Shannon's Sampling Theory. The associated wavelet is a modulated sinc (the ideal bandpass filter) ([15], [21]). The link between Shannon Sampling Theory and polynomial splines approximations can be found in some of our previous work ([2] and [27]). The theory for discrete signals has also been considered by Aldroubi *et al.* ([3] and [4]).

§4. Nonorthogonal scaling functions and wavelets

A useful and constructive method to build a multiresolution structure is to start with a scaling function ϕ and use Equations (10) and (11) to define the subspaces $V_{(j)}$. Clearly, the scaling function cannot be chosen arbitrarily. In fact, because $V_{(1)} \subset V_{(0)}$, there must be a sequence $u(k)$ that relates the dilated function ϕ_2 and the scaling function ϕ:

$$\phi_2(x) = (u * \phi)(x), \tag{16}$$

where the $*$ is the mixed convolution defined by Equation (2). The sequence $u(k)$ in the relation will be called the generating sequence and the relation is usually referred to as the two scale relation.

Sufficient conditions on $u(k)$ and its Fourier transform $U(f)$ as well as a method of constructing the orthogonal scaling function can be found in [17]. Fast algorithms for computing the projections onto the corresponding spaces $V_{(j)}$ and $W_{(j)}$ are described in [18]. The construction method as well as the computational algorithms are closely related to the multirate filter banks developed in signal processing ([12] and [30]).

If we drop the orthogonality constraint and require only that $\{2^{-j/2}\phi(2^{-j}x-k) : k \in \mathbb{Z}\}$ be an unconditional basis of $V_{(j)}$ then we can start from a sequence $u(k)$ with Fourier transform $U(f)$ and, similarly to [17], construct a scaling function λ with Fourier transform $L(f)$ as described in the following proposition:

Proposition 1. *Let* $U(f) = \sum_{k=-\infty}^{\infty} u(k)e^{-i2\pi kf}$ *be such that:*

$$2^{-1/2}\,|U(0)| = 1, \tag{17}$$

$$2^{-1/2}\,|U(f)| \leq 1, \tag{18}$$

$$U(f) \neq 0, \quad \forall\, f \in [-1/4, 1/4], \tag{19}$$

$$\prod_{i=1}^{n} 2^{-1/2}U\left(\frac{f}{2^i}\right) \xrightarrow{L_2} L(f), \tag{20}$$

$$L(f) = O(|f|^{-1-\epsilon}), \tag{21}$$

then $L(f)$ *is the Fourier transform of a scaling function* λ *which generates a multiresolution* $V_{(j)}(\lambda)$ *as in Equations (10) and (11).*

The conditions in Proposition 1 are the same as the one given by Mallat in [17] except that we have droped the Quadrature Mirror Filter requirement and added Equation (21). Except for the technical condition (19), all the others can be inferred by taking the Fourier transform of Equation (16) and solving for $L(f)$. Condition (21) is a regularity requirement which implies the continuity of the function λ.

The set $\{2^{-j/2}\lambda(2^{-j}x-k) : k \in \mathbb{Z}\}$ generated from a scaling function obtained by the procedure of Proposition 1 are not necessarily orthogonal. If $U(f)$ is chosen to be real and symmetric then λ will also be real and symmetric. Moreover, if $u(k)$ has finitely many nonzero values then λ will have compact support; *i.e.*, $\lambda(x)$ will be zero outside a closed bounded set [13]. Nonorthogonal scaling functions and wavelets using splines were proposed by Chui and Wang ([9], [10], and [11]) and, in the context of signal processing, by Unser *et al.* ([24], [27], and [28]).

4.1. Law of composition of scaling functions

It should be noted that, if $U_1(f)$ and $U_2(f)$ are the Fourier transforms of two sequences satisfying the conditions of Proposition 1 then so does their weighted product :

$$U = U_1 \bullet U_2 = 2^{-1/2} U_1 U_2. \tag{22}$$

This remark depicts an obvious algebraic structure which we state in the following proposition.

Proposition 2. *The product (22) defines an internal law of composition for the set G which consists of all sequences satisfying the conditions of Proposition 1.*

Using the property that the Fourier transform converts a convolution product into a multiplication product, this last proposition allows us to use convolution to generate new scaling functions. Thus, the convolution $\lambda * \zeta$ of two scaling functions λ and ζ generated by the sequences $u_1(k)$ and $u_2(k)$ is also a scaling function generated by the discrete convolution $u_1 * u_2$. An interesting property is that the new function $\lambda * \zeta$ is more regular than its individual constitutive atoms λ and ζ. In fact, if λ is r_1 regular and ζ is r_2 regular, then $\lambda * \zeta$ has $r_1 r_2$ regularity. Another interesting property is that if $U_1(f)$ has a zero of order p_1 at $f = 1/2$ and $U_2(f)$ has a zero of order p_2 at $f = 1/2$ then $U_1(f)U_2(f)$ has zero of order $p_1 p_2$ at $f = 1/2$. This implies that the degree of the polynomials that can be exactly represented in terms of $\lambda * \zeta$ is $p_1 p_2$ instead of p_1 if the representation λ is used or p_2 if ζ is used [22].

Also, there are useful properties that are invariant under convolution. For instance, symmetry is preserved; *i.e.*, if λ and ζ are symmetric then $\lambda * \zeta$ is also symmetric. Moreover, if λ and ζ have compact support, then $\lambda * \zeta$ have compact support as well.

4.2. Equivalent scaling functions

It should be noted that two non-identical scaling functions may generate the same multiresolution $V_{(j)}$. In fact, given a scaling function λ that generates the multiresolution $V_{(j)}(\lambda)$, we can use the mixed convolution to construct another function φ generating the same multiresolution $V_{(j)}(\lambda) = V_{(j)}(\varphi)$. This is done by convolving λ with an invertible convolution operator $p(k)$ on l_2 (mixed convolution as in Equation (2)):

$$\varphi(x) = (p * \lambda)(x). \tag{23}$$

Thus, the set of scaling functions can be partitioned into equivalence classes by the relation that associates two scaling functions whenever they generate the same multiresolution. Moreover, two scaling functions belonging to the same equivalence class are always related by Equation (23).

As an example, we can choose p to be the inverse of the operator-square-root of the autocorrelation function $a(k)$ of λ:

$$p(k) = (a)^{-1/2}(k), \tag{24}$$

$$a(k) = (\lambda * \lambda')(k), \quad k \in \mathbb{Z}, \tag{25}$$

where the operator-square-root $a^{1/2}$ is such that $a(k) = (a^{1/2} * a^{1/2})(k)$. For this choice of p we obtain the orthogonal scaling function ϕ of Stéphane Mallat. To verify this claim, a simple calculation will show that:

$$(\phi * \phi')(k) = \begin{cases} 1, & k = 0 \\ 0, & k = \pm 1, \pm 2, ... \end{cases} \tag{26}$$

which is equivalent to the orthogonality condition.

4.3. Nonorthogonal wavelets

Similarly to the case of scaling functions, there are infinitely many ways to choose a wavelet ω so that $\{2^{-j/2}\omega(2^{-j}x - k) : k \in \mathbb{Z}\}$ is a basis of $W_{(j)}$. However, although $\{2^{-j/2}\omega(2^{-j}x - k) : k \in \mathbb{Z}\}$ are not necessarily orthogonal at a given level j, they retain the orthogonality between two wavelet spaces at different resolutions. The reason for this is that $W_{(j)}$ is still the orthogonal complement of $V_{(j)}$ relative to $V_{(j-1)}$.

Since $W_{(j)}$ is included in the space $V_{(j-1)}$, any wavelet function ω generating $W_{(j)}$ can be expressed in terms of the scaling function λ generating $V_{(j)}$. For our purpose, we begin by choosing the basic wavelet μ defined by:

$$2^{-1/2}\mu(x/2) = \mu_2(x) = \sum_{k=-\infty}^{\infty} (\delta_1 * \tilde{u}' * \tilde{a})(k)\lambda(x - k), \tag{27}$$

where δ_1 is the unit pulse centered at 1 (*i.e.*, $\delta_1(1) = 1$ and $\delta_1(k) = 0$ for $k \neq 1$), where $a(k)$ is the sampled autocorrelation of λ as in (25), and where $u(k)$ is its generating sequence. An interesting property of the construction (27) is that if the generating sequence $u(k)$ has finitely many nonzero values, then the wavelet μ has compact support.

To obtain other wavelet functions ω generating the same spaces $W_{(j)}$, we use an invertible convolution operator $q(k)$ on l_2 as in (23) to get:

$$\omega(x) = (q * \mu)(x). \tag{28}$$

For example, we can choose q to obtain the orthonormal wavelet function ψ of Mallat:

$$q(k) = \left(a * [\tilde{a} * a]_{\downarrow 2}\right)^{-1/2}(k), \tag{29}$$

where a is the sampled autocorrelation function defined by (25) where $\tilde{a}$ is the modulation of a as defined by Equation (7) and where the decimation operator $[.]_{\downarrow 2}$ is defined by Equation (8).

§5. Families of scaling and wavelet functions

As we have mentioned in Subsection 4.1, we can generate new multiresolution structures simply by convolution of known scaling functions. We can even start with a single function and generate, by repeated convolution, an infinite sequence of scaling functions. Well known cases that can be obtained in this fashion are the B-spline functions $\beta^n(x)$ of order n, which are used as basis for generating certain polynomial spline function spaces. In fact, it is the structure and properties of B-spline functions that motivated part of this work. In this sense, the constructions that follow can be viewed as a generalization of polynomial splines.

5.1. Basic scaling functions and wavelets

We start with a scaling function λ and generate, by convolution, a sequence of increasingly regular scaling functions λ^n as follows:

$$\lambda^n(x) = \lambda * \lambda * \lambda * \ldots * \lambda. \quad (n-1 \text{ convolution}). \tag{30}$$

These functions will be called the basic scaling functions of order n. For each λ^n, there is a corresponding basic wavelet function μ^n obtained from the construction (27):

$$2^{-1/2}\mu^n(x/2) = \mu_2^n(x) = \sum_{k=-\infty}^{\infty} (\delta_1 * \tilde{u}^{n\prime} * \tilde{a}^n)(k)\lambda^n(x-k), \tag{31}$$

$$a^n(k) = (\lambda^n * \lambda^{n\prime})(k), \quad \forall\, k \in \mathbb{Z}, \tag{32}$$

$$u^n = 2^{-(n-1)/2} u * u * \ldots * u, \quad (n-1 \text{ discrete convolutions}). \tag{33}$$

The remarks of Subsection of 4.1 imply that the regularity of the scaling function λ^n and the wavelet μ^n increases with increasing values of the order n. Also, if the starting generating sequence u is finite then the basic scaling and wavelet functions λ^n and μ^n have compact support. Moreover, λ^n and μ^n will be "linear phase" (have a vertical axe of symmetry) if u has a vertical axe of symmetry. This property is relevant in signal processing for obtaining representations that have no phase distortions.

5.2. Cardinal families of scaling functions and wavelets

The basic scaling functions λ^n and the corresponding wavelets μ^n generate the multiresolution spaces $V_{(j)}^n$ and the wavelet spaces $W_{(j)}^n$. We can use Equations (23) and (28) to create other scaling functions and wavelets associated with the same multiresolution spaces $V_{(j)}^n$. A particular scaling function of interest to us is the cardinal scaling function η^n which has zeros at all the integers except at the origin where its value is 1:

$$\eta^n(k) = \delta_0(k) = \begin{cases} 1 & k = 0 \\ 0 & k = \pm 1, \pm 2, \ldots \end{cases} \tag{34}$$

The expression of the cardinal scaling function in terms of λ^n is given by:

$$\eta^n(x) = \left((b^n)^{-1} * \lambda^n\right)(x), \tag{35}$$

$$b^n(k) = \lambda^n(k), \quad \forall\, k \in \mathbb{Z}. \tag{36}$$

An associated wavelet which has the same property (34) as the cardinal scaling function is the cardinal wavelet given by:

$$2^{-1/2}\nu^n(x/2) = \nu_2^n(x) = \left([c^n]_{\uparrow 2} * \mu_2^n\right)(x), \tag{37}$$

$$c^n(k) = 2^{-1/2}\left([\delta_1 * b^n * \tilde{u}^{n\prime} * \tilde{a}^n]_{\downarrow 2}\right)^{-1}(k) \quad \forall\, k \in \mathbb{Z}, \tag{38}$$

where μ_2^n is the basic wavelet in Equation (31).

An important property of a cardinal representation of a signal s is that the expansion coefficients in the representation are precisely equal to its sampled values. This implies that the expansion coefficients can themselves be viewed as a faithful representation of the sampled signal $s(k), k \in \mathbb{Z}$. This property will be further discussed in Section 6.

5.3. Orthogonal families of scaling functions and wavelets

Another family of interest is the orthogonal scaling sequence ϕ^n. It is obtained from the basic sequence λ^n using Equations (23)-(25) as follows:

$$\phi^n(x) = \left((a^n)^{-1/2} * \lambda^n\right)(x), \tag{39}$$

$$a^n(k) = (\lambda^n * \lambda^{n\prime})(k), \quad \forall\, k \in \mathbb{Z}. \tag{40}$$

The corresponding orthogonal wavelet ψ^n is obtained from the basic wavelet μ^n using Equations (28) and (29):

$$\psi_2^n(x) = \left([o^n]_{\uparrow 2} * \mu_2^n\right)(x), \tag{41}$$

where

$$o^n(k) = \left(a^n * [\tilde{a}^n * a^n]_{\downarrow 2}\right)^{-1/2}(k), \quad \forall\, k \in \mathbb{Z}. \tag{42}$$

5.4. Examples using polynomial splines

Polynomial spline functions of order m are C^{m-1} functions that are formed by patching together polynomials of degree m at grid points called the knot points. Here, we will only consider uniformly spaced knot points. In this case, Schoenberg in a remarkable paper shows that any polynomial spline function $s(x)$ of order m can be represented in terms of a unique bell shaped function $\beta^m(x)$; the B-spline function of order m [20]. As an example, when the grid is $\mathbb{Z}$, $s(x)$ can be written as :

$$s(x) = \sum_{k=-\infty}^{\infty} c(k)\beta^m(x-k), \tag{43}$$

where the symmetric B-spline $\beta^m(x)$ is given by:

$$\beta^m(x) = (\beta^0 * \beta^0 * \cdots * \beta^0)(x), \quad (m \text{ convolution}), \tag{44}$$

and where $\beta^0(x)$ is the rectangular function rect(x) defined by Equation (47).

The fact that spline functions are constructed from pieces of polynomials and that they can be easily represented by compactly supported functions $\beta^m(x)$ makes them computationally, as well as theoretically, very attractive. Moreover, it can be easily shown that, for m odd, $\beta^m(x)$ are scaling functions. For m even, the shifted functions $\beta^m(x-1/2)$ are also scaling functions. In fact, there are numerous examples of spline scaling functions and spline wavelets. The simplest of all, the Haar basis function used in the Haar transform, is a spline wavelet of order zero. The Lemarié and Battle orthogonal scaling and wavelet functions are splines of order m ([5] and [16]). For $m = 2n+1$, they can be obtained from the tent function $\tau = \beta^1$ by substituting τ for λ in Equations (39)-(42). The generating sequence u for the tent function τ (see Equation (16)) is given by:

$$u(k) = \begin{cases} 2^{-1/2}, & k = 0 \\ 2^{-3/2}, & k = \pm 1 \\ 0, & \text{otherwise} \end{cases} . \tag{45}$$

Nonorthogonal spline scaling and wavelet functions with various desired properties have also been proposed ([7-11], [24-25], and [27]). For example the cardinal spline functions and wavelets are obtained by replacing λ^n in Equations (35)-(38) by τ^n. The basic spline wavelets proposed by Chui and Wang [11] and by Unser *et al.* [24] are obtained from Equation (31) by replacing λ^n by τ^n.

§6. Multiresolution families and Shannon's Sampling Theory

Shannon's Sampling Theory describes how to process a function before a uniform sampling. The goal is to "best" reconstruct the original function from the sample values. Clearly, there are many different functions having the same sample values. Thus, it is important to describe the class of functions that are fully characterized by a uniform sampling. This class is the set of bandlimited functions (*i.e.*, functions that have a compactly supported Fourier transforms). In particular, a function $s(x)$ with Fourier transform $S(f)$ supported in $[-\frac{1}{2}, \frac{1}{2}]$ (*i.e.*, $S(f) = 0$, $\forall f \notin [-\frac{1}{2}, \frac{1}{2}]$) can be recovered from its samples $s(k)$, $k \in \mathbb{Z}$, using the sinc-interpolation:

$$s(x) = \sum_{k=-\infty}^{\infty} s(k)\mathrm{sinc}(x-k). \tag{46}$$

where $\mathrm{sinc}(x) = \sin(\pi x)/\pi x$. Equivalently, $S(f)$ can be recovered by multiplying the Fourier transform of the sampled sequence with the ideal lowpass filter. The ideal lowpass filter is the rect function :

$$\mathrm{rect}(f) = \begin{cases} 1, & |f| < 1/2 \\ 0, & \text{elsewhere.} \end{cases} \tag{47}$$

In general, however, the functions to be represented by their samples are not band-limited. To address this problem, the functions are first preprocessed. This is done by first applying an ideal lowpass filter, thus forcing them to be bandlimited before the sampling. The whole process, which is depicted in Figure 1, is what is known as Shannon's Sampling Theory.

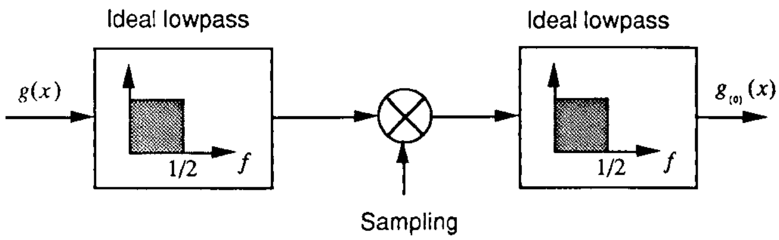

Figure 1. Block diagram for Shannon's sampling procedure. The function g is prefilter by the ideal lowpass filter before sampling. The reconstruction $g_{(0)}$ is obtained from the samples by a second ideal lowpass filtering.

6.1. Multiresolution interpretation of Shannon' Sampling Theory

If we consider the subspace of L_2 generated by the span of all the shifted sinc functions:

$$V_{(0)} = \left\{ v : v(x) = \sum_{k=-\infty}^{\infty} c(k)\mathrm{sinc}(x-k), \quad c \in l_2 \right\}, \tag{48}$$

then, Shannon's Sampling Theory is equivalent to the least square approximation of L_2 in $V_{(0)}$. Taking this point of view, the whole procedure in Figure 1 applied to a function g is expressed as:

$$g_{(0)}(x) = \sum_{k=-\infty}^{\infty} (g * \mathrm{sinc})(k)\mathrm{sinc}(x). \tag{49}$$

In fact, this interpretation was suggested by Shannon in his original paper [21].

It is easy to show that the sinc function is a scaling function. Thus, by defining $\phi(x) := \mathrm{sinc}(x)$ in Equations (10) and (11), we generate a multiresolution structure $V_{(j)}$ of L_2. Since the sinc function vanishes at all integers except at the origin, the coefficients $(g * \mathrm{sinc})(k)$ in Equation (49) are exactly equal to the samples of the approximation $g_{(0)}(k)$. Because of this property, the sinc function is a cardinal scaling function and the resolution associated with it is a cardinal multiresolution. Interestingly enough, sinc is also an orthogonal scaling function.

6.2. Generalized sampling theory

By adopting the previous interpretations, we can generalize Shannon's Sampling Theory. This is achieved simply by using an arbitrary scaling function φ. The whole sampling procedure applied to a function g is reduced to finding the L_2 projection of g on the space $V_{(0)}(\varphi)$:

$$V_{(0)}(\varphi) = \left\{ v : v(x) = \sum_{k=-\infty}^{\infty} c(k)\varphi(x-k), \quad c \in l_2 \right\}. \tag{50}$$

By computing the projection of g and by making the appropriate identification we get:

$$g_{(0)}(x) = \sum_{k=-\infty}^{\infty} (g * \overset{o}{\varphi})(k)\varphi(x), \tag{51}$$

where $\overset{o}{\varphi}$ is also called the dual of φ and is given by:

$$\overset{o}{\varphi}(x) = ((a)^{-1} * \varphi')(x), \tag{52}$$

$$a(k) = (\varphi * \varphi')(k), \quad \forall\, k \in \mathbb{Z}. \tag{53}$$

In this context, the function g is first preprocessed using the prefilter with impulse response $\overset{o}{\varphi}$ and then sampled. The sample values $(g * \overset{o}{\varphi})(k)$ can then be used to reconstruct the "best" (least square) approximation $g_{(0)}$ of g given by Equation (51).

By analogy with Shannon's Sampling Theory, we can say that the functions that are fully characterized by their sample values are the φ-limited functions (*i.e.*, the class that is described by Equation (50)).

Finally, the assumption that φ be a scaling function is in no way crucial to the generalization. The only important requirement is that the space $V_0(\varphi)$ be a closed subspace of L_2.

6.3. Sampling with cardinal and orthogonal functions

It should be noted that, in general, the samples $(g * \overset{o}{\varphi})(k)$ are not equal to the sampled approximation $g_{(0)}(k)$. If we require equality, then we need to replace φ in Equations (50)-(53) by the cardinal scaling function η. The function η vanishes at all integers except at 0 where it takes the value 1. It can be obtained from φ as described by Equations (35) and (36):

$$\eta(x) = \left((b)^{-1} * \varphi\right)(x), \tag{54}$$

$$b(k) = \varphi(k), \quad \forall\, k \in \mathbb{Z}. \tag{55}$$

More generally, starting from a scaling function λ, we obtain a sequence of cardinal scaling functions η^n by using Equations (30), (35), and (36). If we replace φ in Equations (50)-(53) by η^n, we obtain a cardinal sampling procedure for each member of the family. In fact, under the appropriate conditions on λ, the sequence of cardinal sampling procedures tends to the scheme of Shannon as n goes to infinity. This result, which further emphasizes the link with Shannon's Sampling Theory is stated in the following theorem.

Theorem 3. *Let $\lambda(x)$ be a scaling function and $L(f)$ its Fourier transform. Let η^n be the sequence of cardinal scaling functions generated by Equations (35) and (36) and let $\overset{o}{\eta}{}^n$ be the corresponding duals. If the Fourier transform $L(f)$ of $\lambda(x)$ satisfies:*

$$|L(f)| > |L(f-i)|, \quad \forall\, f \in (-1/2, 1/2), \forall\, i \in \mathbb{Z} \quad i \neq 0, \tag{56}$$

$$\sum_i \frac{|L(f-i)|}{|L(f)|} < \infty, \quad f \in (-1/2, 1/2), \tag{57}$$

then the filters $H^n(f)$ and $\overset{o}{H}{}^n(f)$ corresponding to η^n and $\overset{o}{\eta}{}^n$ converge to the ideal lowpass filter as the order n tends to infinity:

$$\lim_{n\to\infty} H^n(f) = rect(f), \tag{58}$$

$$\lim_{n\to\infty} \overset{o}{H}{}^n(f) = rect(f). \tag{59}$$

With a few more conditions on the scaling function, L_p convergence can also be obtained as in the case of polynomial splines ([2] and [28]). Conditions (56) and (57) essentially mean that $L(f)$ is a non-ideal lowpass filter in the frequency band $(-\frac{1}{2}, \frac{1}{2})$. They can be satisfied by appropriately choosing $U(f)$ (*e.g.*, $U(f)$ smooth and $U(f)$ monotonically decreasing). The theorem can be viewed as stating that the ideal lowpass filter can be approximated as closely as needed by the sequences η^n and $\overset{o}{\eta}{}^n$. These are obtained by repeated convolutions and a simple correction of a single non-ideal lowpass filter. Moreover, by the remarks in Subsection 4.1, η^n and $\overset{o}{\eta}{}^n$ increase their regularity (smoothness) with increasing values of n.

Instead of cardinal scaling functions η^n, we can select orthogonal scaling functions ϕ^n (*i.e.*, $\{2^{-j/2}\phi(2^{-j}x-k)\}_{k\in\mathbb{Z}}$ is an orthogonal set). They are obtained from λ using Equations (39) and (40). As previously noted, this orthogonality property also holds for the sinc scaling function. An interesting property of orthogonal functions is that they are equal to the reflection of their duals. This means that the preprocessing and the reconstruction filters in the sampling procedure are the same. Similarly to the cardinal scaling family, the Fourier transforms of the sequence ϕ^n also tend to the ideal lowpass filter as n goes to infinity. It can also be shown that the Fourier transforms of the cardinal and orthogonal wavelets (37) and (41) tend to the ideal bandpass filter as the order n goes to infinity.

In Figure 2, spline scaling functions are used to illustrate the extension to Shannon's theory and the above theorem. This figure shows the totality of the sampling and representation procedure; it consists of a prefiltering followed by a sampling and finally a reconstruction using a postfilter. As can be seen from the graphs, the cardinal and orthogonal spline filters of order 3 are already a good approximations to the ideal filter. The graphs also show that the orthogonal filters and their duals are equal.

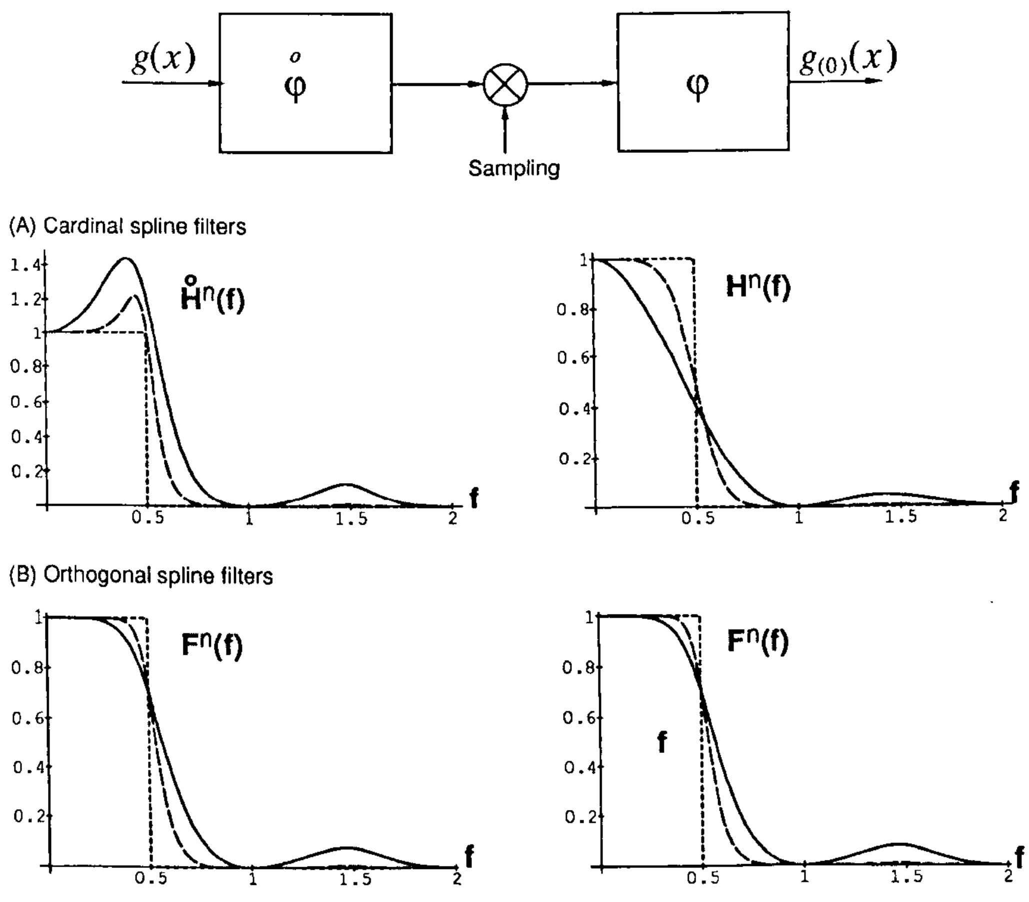

Figure 2. Block diagram for the general sampling theory.

Graph (A) corresponds to the cardinal spline filters and Graph (B) corresponds to the Orthogonal spline filters. The filters of order $n = 1$ are drawn with a continuous lines, the ones of order $n = 3$ are in dashed lines (- - - -), and the ideal lowpass filter is in dotted line $(\cdots)$.

§7. Multiresolution families in connection with Gabor transform

An important property of wavelets is that they are localized in both space (or time) and in frequency. This property is useful because a decomposition of a signal in terms of a wavelet ω determines the signal frequency content at "each" location in space (or time). A measure of the time and frequency localization of a function is given by the product of the standard deviations of its squared modulus and the squared modulus of its Fourier transform. This measure is bounded below by the constant $(4\pi)^{-2}$ [14]. The lower bound cannot be achieved by any wavelet. The only time-frequency optimal functions that achieve the lower bound are the Gabor functions:

$$g(x) = \exp\left(i\Omega(x - x_0) - i\theta\right) \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x - x_0)^2}{2\sigma^2}\right). \qquad (60)$$

These are modulated Gaussian functions with four parameters: the offset x_0, the standard deviation σ, the modulation frequency Ω, and the phase shift θ. All the parameters can be chosen arbitrarily.

It can be shown that there are no values of the parameters that can force the corresponding function to form a wavelet basis of L_2. Moreover, this also holds for the real and complex parts of Gabor functions, even though it is still possible to create a sequence of wavelets that approaches the real part of Gabor functions as close as we choose. One way of doing this is by choosing the nonorthogonal family of wavelets given by Equation (31). This family is easy to construct since the only operations that are involved are: convolutions, discrete modulations and a single reflection. The way in which this family tends to the real part of a Gabor function as the order n tends to infinity is stated in the following theorem.

Theorem 4. *Let λ be a scaling function, $L(f)$ its Fourier transform and $u(k)$ the corresponding generating function. Let μ_2^n be the sequence of nonorthogonal scaling function generated by Equations (31), (32), and (33). Let $U(f)$ be symmetric and $U(-1/2) = 0$. Furthermore assume that :*

$$|L(f)| > |L(f - i)|, \quad \forall\, f \in (-1/2, 1/2), \forall\, i \in \mathbb{Z} \quad i \neq 0, \qquad (61)$$

$$\sum_i |L(f - i - 1/2)| < \infty, \qquad (62)$$

then the Fourier transforms $G_2^n = e^{i2\pi f} M_2^n(F)$ of $\delta_{-1} * \mu_2^n$ have the convergence property given by

$$\lim_{n\to+\infty}\left\{\frac{1}{(Z(f_0))^n}\left(G_2^n\left(\frac{f}{2\pi\sigma_0\sqrt{n}}\pm f_0\right)\right)\right\}=\exp\left(-f^2/2\right), \tag{63}$$

where $Z(f)$, $f_0\in(-\frac{1}{2},\frac{1}{2})$, *and* σ_0 *are given by*

$$Z(f)=2^{-1/2}U(f-1/2)L(f)|L|^2(f-1/2), \tag{64}$$

$$Z(f_0)>|Z(f)|,\quad \forall\, f\in R, f\neq f_0, \tag{65}$$

$$\left.\frac{d^2Z}{df^2}\right|_{f=f_0}=-(2\pi\sigma_0)^2Z(f_0). \tag{66}$$

The conditions that $U(f)$ be symmetric and $U(1/2)=0$ are not restrictive. The first is simply the requirement that λ be real symmetric. The second is a standard condition that is usually desirable as mentioned in Subsection 4.1. Conditions (61) and (62) constitute the requirement that $L(f)$ essentially be a lowpass filter in the frequency band $(-\frac{1}{2},\frac{1}{2})$. They can be met by choosing $U(f)$ appropriately (*e.g.*, $U(f)$ smooth and decreasing). Roughly speaking, the theorem states that the basic wavelet of order n is essentially the real part of a Gabor function. It is centered at the origin, its standard deviation is $n^{1/2}\sigma_0$ and its modulation frequency is $\Omega=2\pi f_0\sigma n^{1/2}$. If (63) also holds in L_2 then Parseval identity gives us that

$$\frac{\sigma_0\sqrt{n}}{(Z(f_0))^n}\delta_{-1}*\mu_2^n\left((\sigma_0\sqrt{n})x\right)\overset{L_2}{\to}\frac{\cos(2\pi f_0\sigma_0\sqrt{n}x)}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right). \tag{67}$$

It should be noted that the sequence of wavelets satisfying the conditions of the theorem above is symmetric. They also have linear phase, and their regularity increases with increasing values of the order n. Moreover, if the generating sequence $u(k)$ is finite, then the wavelets have compact support (see Subsection 4.1).

As a particular case, the B-spline wavelets proposed in ([11] and [26]) can be obtained by replacing λ^n in Equations (30)-(33) by the function τ^n defined in Subsection 5.4. They satisfy all the properties of Theorem 4 and converge to the real part of the Gabor function with frequency $f_0\cong 0.41$. Moreover, the convergence (63) is in all L_p norms and (67) holds in L_q for $q\in[2,\infty)$ [26].

Figure 3 illustrates the basic B-spline wavelets and their relation to the real part of Gabor functions. Even though the theorem is only an asymptotic result, the graph already shows a very good agreement between the B-spline wavelet of order 3 and its Gabor limit. In fact, the time-frequency localization measure defined at the beginning of this section is found to be within 2% of the optimal number $(2\pi)^{-2}$.

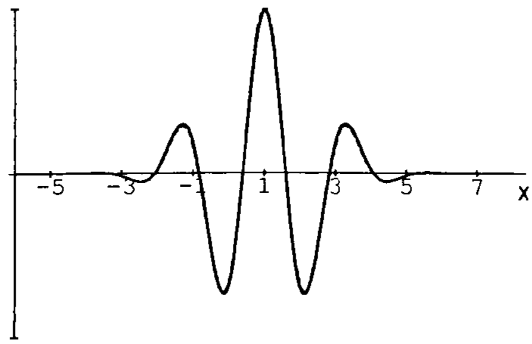

Figure 3. Gabor function (dashed lines) and the Basic spline function of order $n = 3$ (continuous line).

§8. Concluding remarks

Multiresolutions and the related wavelet representations offer an infinite variety of signal representations. Which one to choose depends on the application. As we have shown here, we have an easy procedure to construct scaling and wavelet functions that have certain desired properties. We only need to start from a generating sequence or a scaling function. Then, by using simple convolutions and appropriate corrections, we obtain multiresolution and wavelet transforms with some special chosen features. For example, we can construct arbitrarily regular cardinal and orthogonal functions that are linear phase. Also, we can construct smooth wavelets that tend to a Gabor functions and cardinal sequences that tend to the ideal filters of Shannon. The techniques that we have developed here are general. They can be easily adapted to obtain scaling and wavelets functions with some special features other than the ones presented in this chapter.

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