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OBLIQUE PROJECTIONS IN DISCRETE SIGNAL SUBSPACES OF l_2 AND THE WAVELET TRANSFORM

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ABSTRACT

We study the general problem of oblique projections in discrete shift-invariant spaces of l_2 and we give error bounds on the approximation. We define the concept of discrete multiresolutions and wavelet spaces and show that the oblique projections on certain subclasses of discrete multiresolutions and their associated wavelet spaces can be obtained using perfect reconstruction filter banks. Therefore we obtain a discrete analog of the Cohen-Daubechies-Feauveau results on biorthogonal wavelets.

Keywords: shift-invariant spaces, angle, oblique projection, discrete signals, sequence spaces, multiresolution, wavelet, perfect reconstruction filter bank

1. INTRODUCTION

The multiresolution theory of the wavelet transform in L_2 and the equivalence with perfect reconstruction filter banks in now well understood. These types of decompositions are usually classified in three main categories: orthogonal[6], semi-orthogonal[1], and biorthogonal[4,10], where these properties are understood with respect to the continuous L_2-inner product. The more general class of biorthogonal wavelet bases includes all the other ones. Cohen, Daubechies , and Feauveau[4] provide an elegant interpretation of these decompositions in terms of oblique projections that involves the interplay of two dual multiresolution ladders of subspaces in L_2. However, there are many applications such as digital signal processing and coding, in which such a continuous interpretation is not particularly relevant, and where one would prefer to adopt a purely discrete point of view. In the orthogonal case, switching to a discrete interpretation is particularly easy because of the perfect equivalence of the underlying L_2 and l_2-norms[11]. Other concepts such as semi-orthogonality can also be carried over to the discrete domain but the underlying filter structures are usually not homogeneous; i.e., they vary from one scale to the other. Rioul[7] has investigated the general discrete biorthogonal case and has shown that such decompositions could be understood in term of non-orthogonal projections. However, he did not explicitly characterize the underlying projection operators and their corresponding approximation spaces. Thus, he did not fully bring out the parallel with the continuous case.

The purpose of this paper is to investigate discrete multiresolution and wavelet decompositions from the perspective of oblique projections, which has been neglected so far. In particular, we generalize the concept of multiresolutions of L_2 to the discrete sequence space l_2. It turns out that there is also a similar link between perfect reconstruction filter banks and the discrete multiresolutions of l_2 and their associated wavelets. In both the discrete and analog cases, the link comes from the interplay between shift-invariant spaces and oblique projections. For this reason, we first study the general theory of oblique projections in discrete shift-invariant subspaces of l_2. This is done in section 3. We then define the concepts of discrete multiresolutions and some of their subclasses in section 4. In section 5, we show that the decomposition/reconstruction algorithm for certain biorthogonal pairs of homogeneous discrete multiresolutions $\{S_{(j)}, V_{(j)}\}_{j\in\mathbb{N}}$ and their associated pair of discrete wavelet spaces $\{T_{(j)}, W_{(j)}\}_{j\in\mathbb{N}^+}$ can be obtained by a perfect reconstruction filter bank.

2. DEFINITIONS AND NOTATION

The Fourier transform of a sequence $s(k)$ denoted by $\hat{s}(f)$ is defined to be

(1) $$\hat{s}(f) = \sum_{k\in Z} s(k)e^{-i2\pi fk}$$

The convolution between two sequences a and b is denoted by $a * b$:

(2) $$(a*b)(l) = \sum_{k=-\infty}^{k=+\infty} a(k)b(l-k), \quad l \in \mathbb{Z}$$

Whenever it exits, the convolution inverse $(b)^{-1}$ of a sequence b is defined by

(3) $$\left((b)^{-1} * b\right)(k) = \delta_0(k)$$

where $\delta(k)$ is the unit impulse; *i.e.*, $\delta(0) = 1$ and $\delta(k) = 0$ for $k \neq 0$.

The reflection of a sequence b is the function $b^{\vee}$, given by

(4) $$b^{\vee}(k) = b(-k), \quad \forall k \in \mathbb{Z}$$

The modulation $\tilde{b}(k)$ of a sequence b is obtained by changing the signs of the odd components of b:

(5) $$\tilde{b}(k) = (-1)^k b(k)$$

The operator $\downarrow_m$ of down-sampling by the integer factor m assigns to a sequence b the sequence $\downarrow_m [b]$, given by

(6) $$(\downarrow_m [b])(k) = b(mk), \quad \forall k \in \mathbb{Z}$$

The operator $\uparrow_m$ of up-sampling by the integer factor m takes a discrete signal b and expands it by adding $m-1$ zeros between consecutive samples:

(7) $$(\uparrow_m [b])(k) = \begin{cases} b(k'), & k = mk' \\ 0, & \text{elsewhere} \end{cases}$$

3. OBLIQUE PROJECTIONS IN SEQUENCE SPACES

In this section we consider the general problem of projecting signals on a space S in a direction orthogonal to a space V, possibly different from S. When $S \neq V$ we get the ***oblique projection*** $P_{S \perp V}$. We will restrict our attention to the shift invariant subspaces of l_2, although some of the result are more general.

3.1. Shift-invariant sequence spaces

We define the *m-shift-invariant sequence space* (or *m-shift-invariant discrete signal space*) to be a subspace $S(u,m) \subset l_2$ that is generated by the translation of a single sequence $u(k)$ (for notation, see previous section):

$$(8) \qquad S(u,m) = \left\{ s(k) := \sum_{i \in \mathbb{Z}} c(i) u(k - mi), = \uparrow_m [c] * u \quad c \in l_2 \right\}.$$

When it is clear from the context, we will write S or $S(u)$ for $S(u,m)$, and we will say shift-invariant space instead of m-shift-invariant space. We will require $S(u)$ to be closed and to have $\{u(k-mi)\}_{i \in \mathbb{Z}}$ as its Riesz basis. The following theorem gives the necessary and sufficient condition for the above requirements to be satisfied:

Theorem 1. If there exists two positive constants $\alpha > 0$ and $\beta > 0$ such that the sequence $u \in l_2$ satisfies

$$(9) \qquad \alpha \leq \Lambda(f) := \sum_{i=0}^{m-1} |\hat{u}((f-i)/m)|^2 \leq \beta,$$

then the space $S(u)$ is a closed subspace of l_2, and $\{u(k-mi)\}_{i \in \mathbb{Z}}$ is its Riesz basis.

Proof. From Parseval identity we get that

$$(10) \qquad \|\uparrow_m [c] * u\|_{l_2}^2 = \int_0^1 |\hat{u}(f)|^2 \, |\hat{c}(mf)|^2 \, df$$

Using the change of variable $\xi = mf$ and using the fact that $\hat{c}(f+1) = \hat{c}(f)$, we rewrite the right hand side of (10) to get

$$(11) \qquad \int_0^1 |\hat{u}(f)|^2 \, |\hat{c}(mf)|^2 \, df = m^{-1} \int_0^1 \sum_{i=1}^{m-1} |\hat{u}((\xi - i)/m)|^2 \, |\hat{c}(\xi)|^2 d\xi$$

The theorem then follows from the equation above and the definition of Riesz bases. ■

Remark 3.1. It should be noted that the converse is also true, i.e., if $S(u)$ is a closed subspace of l_2 and $\{u(k-mi)\}_{i \in \mathbb{Z}}$ is its Riesz basis, then condition (9) holds. A proof of this last assertion can be obtained by an argument similar to the one in theorem 2 of[2].

3.2. Oblique projection on S in the direction orthogonal to V

For the remainder of this paper, we will require the shift-invariant spaces to be closed and generated by a Riesz basis that satisfies condition (9) of theorem 1. The oblique projection $P_{S\perp V}g$ of the sequence $g \in l_2$ on the space $S(u,m)$ in a direction orthogonal to the space $V(h,m)$ must satisfy

$$\langle (g - P_{S\perp V}g)(k), h(k-ml)\rangle_{l_2} = 0 \quad \forall l \in \mathbb{Z} \tag{12}$$

By letting $P_{S\perp V}g = \uparrow_m [c] * u$, we rewrite the above equation as

$$\downarrow_m [\chi] * c = \downarrow_m [g * h^\vee] \tag{13}$$

where χ is the cross-correlation between u and h (i.e., $\chi = u * h^\vee$). If $\downarrow_m [\chi]$ is invertible, then the oblique projection is well defined. It is given by

$$c = \downarrow_m \left[\mathring{h} * g\right] \tag{14}$$

where

$$\mathring{h} = \uparrow_m \left[(\downarrow_m [\chi])^{-1}\right] * h^\vee \tag{15}$$

In fact, the converse is also true and we have the following theorem:

Theorem 2. The oblique projection $P_{S\perp V}$ is well defined if and only if $\downarrow_m [\chi]$ has an inverse in l_2. The oblique projection is given by $P_{S\perp V}g = \uparrow_m [c] * u$ where c is given by (14).

Proof. If the oblique projection is well defined (i.e., existence and uniqueness of the projection), then uniqueness implies that the operator $\downarrow_m [\chi] * \bullet$ is injective. Moreover, if we let $g = \uparrow_m [d] * h$ in (13), then we can choose the right hand side to be any element of l_2 since by (9), $\downarrow_m [h * h^\vee]$ is invertible. Thus, by the existence of a projection for each vector in l_2, we conclude that $\downarrow_m [\chi] * \bullet$ is surjective. The closed graph theorem then implies the boundedness of $(\downarrow_m [\chi])^{-1}$. ∎

The whole procedure of finding the oblique projection can be interpreted in terms of filtering as shown in Fig. 1.

3.3 Angles between spaces and error bounds

The oblique projection $P_{S\perp V}g$ of a function $g \in l_2$ on a space S in a direction orthogonal to V can be viewed as an approximation of g in the space S. The error $e = g - P_{S\perp V}g$ is orthogonal to V. The relation between this approximation and the least squares solution depends on the angle between the two spaces S and V. The *angle* between two closed subspaces S and V of a Hilbert space H is defined to be[5,9]

$$\cos(\vartheta(S,V)) = \operatorname*{ess\,inf}_{s\in S, \|s\|=1} (\|P_V s\|) \tag{16}$$

where P_V is the orthogonal projection on V. It should be noted that $\vartheta(S,V)$ is not equal to $\vartheta(V,S)$ in general. Moreover, for the oblique projection $P_{S\perp V}$ to make sense, it is necessary that $\cos(\vartheta(S,V)) < 1$. Otherwise, the space $V \cap S^\perp$ would contain a non-zero vector and the oblique projection would not be well defined. It should also be noted that if $\vartheta(S,V)=0$ then $S\subset V$. Thus if $\vartheta(S,V)=\vartheta(V,S)=0$, then $S=V$ and the oblique projection is equal to the orthogonal projection. If S and V are discrete shift invariant spaces, then we can prove the following theorem:

Theorem 3. If $S(u,m)$ and $V(h,m)$ are two discrete shift invariant spaces satisfying the condition (9) of theorem 1, then

$$\cos(\vartheta(S,V))=\cos(\vartheta(V,S))=\rho_{S,V} \tag{17}$$

where

$$\rho_{S,V}=\operatorname*{ess\,inf}_{f\in[-\frac{1}{2},\frac{1}{2}]}\left(\frac{\left|\sum_{i=0}^{m-1}\overline{\hat{u}((f+i)/m)}\hat{h}((f+i)/m)\right|}{\left(\sum_{i=0}^{m-1}\left|\hat{h}((f+i)/m)\right|^2\right)^{1/2}\left(\sum_{i=0}^{m-1}\left|\hat{u}((f+i)/m)\right|^2\right)^{1/2}}\right) \tag{18}$$

and where $\hat{u},\hat{h}$ are the Fourier transforms of u,h respectively.

We have similar results for the continuous case[9], and similarly, we will call $\rho_{S,V}$ the *spectral coherence*.

It should be noted that if $\rho_{S,V}>0$, then the oblique projection of a signal g is well defined. Moreover, our result in[9] implies that the error $e=P_{S\perp V}g-g$ is of the same order of magnitude as the smallest error that can be obtained when approximating g by an element in V. We have :

Theorem 4. If $\rho_{S,V}>0$ then the oblique projection $P_{s\perp V}$ is well defined. Moreover, we have the error bound

$$\|g-P_{S\perp V}g\|\leq(\rho_{S,V})^{-1}\|g-P_Vg\| \tag{19}$$

4. DISCRETE MULTIRESOLUTIONS OF l_2

4.1. Multiresolutions of l_2

Definition: A discrete multiresolution $\{S_j\}_{j\in\mathbb{N}}$ is a set of subspaces of l_2 satisfying

i: $\ldots\subset S_2\subset S_1\subset S_0=l_2$;

ii: S_j closed $\forall j\in\mathbb{N}$.

Our definition of multiresolution is very broad. It does not endow any structure to the spaces S_j, $j\in\mathbb{N}$. One possible structure is to require shift-invariance as defined below:

Definition : A shift invariant multiresolution $\{S_j\}_{j\in\mathbb{N}}$ is a multiresolution in which each space S_j is generated by a Riesz basis of the form $\{u_{m_j}(k-m_jl)\}_{l\in\mathbb{Z}}$:

$$S_j=\left\{s(k):=\sum_{i\in Z}c(i)u_{m_j}(k-m_ji),=\uparrow_{m_j}[c]*u_{m_j}\quad c\in l_2\right\} \tag{20}$$

Since the spaces S_j must be nested, we must have the relation

$$u_{m_j}=\uparrow_{m_{j-1}}\left[b_{m_j}\right]*u_{m_{j-1}} \tag{21}$$

where b_{m_j} is a sequence in l_2.

If we choose $m_j = p^j$, we obtain the shift-invariant multiresolution $\{S_{(j)} = S_{p^j}\}_{j\in\mathbb{N}}$ generated by $u_{(j)} = u_{p^j}$ (note the new notation), and we can prove the following theorem:

Theorem 5. The set of subspaces $\{S_{(j)}\}_{j\in\mathbb{N}}$ is a discrete shift-invariant multiresolution if and only if

$$u_{(j)} = \uparrow_{p^{j-1}} [b_{(j)}] * u_{(j-1)} \quad \forall j = +1, +2, \ldots \tag{22}$$

and there exists positive constants $\alpha_j, \beta_j > 0$ such that

$$\alpha_j \leq \sum_{i=0}^{p-1} \left|\hat{b}_{(j)}\left((f+i)/p\right)\right|^2 \leq \beta_j \tag{23}$$

An example of such multiresolution that uses spline functions for its definition can be found in[3].

An important special case of discrete shift-invariant multiresolutions is what we will call *homogeneous discrete multiresolution* defined below:

Definition: A homogeneous discrete multiresolution is a discrete shift-invariant multiresolution in which $u_{(0)}(k) = \delta(k)$ and $b_{(j)}(k) = u \;\; \forall j \in \mathbb{N}^+$.

For homogeneous discrete multiresolutions we have the following corollary of theorem 5:

Corollary 6. $\{S_{(j)}\}_{j\in\mathbb{N}}$ is a discrete homogeneous multiresolution if and only if there exists two positive constants $\alpha, \beta > 0$ such that

$$\alpha \leq \sum_{i=0}^{p-1} \left|\hat{u}\left((f+i)/p\right)\right|^2 \leq \beta \tag{24}$$

Discrete multiresolutions do not include the concept of self-similarity between the spaces $\{S_j\}_{j\in\mathbb{N}}$ as is the case for multiresolutions of L_2. We can introduce this concept as follows:

Definition: A multiresolution is self-similar if

(i): $\forall s \in S_j \Rightarrow \downarrow_{m_j} [s] \in S_{j-1}$;

(ii): $\forall x \in S_{j-1}$ there exists a unique element $s \in S_j$ such that $\downarrow_{m_j} [s] = x$.

For homogeneous multiresolutions $\{S_{(j)}(u) = S_{p^j}(u)\}_{j\in\mathbb{N}}$, we can prove that self-similarity can be characterized by discrete interpolating filters. Discrete interpolating filters have been described in[8]. We have the following result:

Proposition 7. A homogeneous multiresolution $\{S_{(j)}\}_{j\in\mathbb{N}}$ is self-similar if and only if

$$u_{(1)}(0) = 1 \tag{25}$$

and

$$u_{(1)}(2k) = 0 \quad \forall k \in \mathbb{Z} \tag{26}$$

5. OBLIQUE PROJECTIONS IN HOMOGENEOUS MULTIRESOLUTION SPACES OF l_2 AND THE WAVELET TRANSFORM

Let $\{S_{(j)}(u) = S_{p^j}(u)\}_{j \in \mathbb{N}}$ and $\{V_{(j)}(h) = V_{p^j}(h)\}_{j \in \mathbb{N}}$ be two homogeneous multiresolutions, then using theorem 1 we obtain the following result for the oblique projection on $S_{(j)}(u)$ orthogonal to $V_{(j)}(h)$:

Theorem 8. The oblique projection of $g \in l_2$ on $S_{(i)}(u)$ in a direction perpendicular to $V_{(i)}(h)$ exists if and only if the sequence

$$\downarrow_{p^i} [\chi_i] = \downarrow_p [\chi * \downarrow_p [\chi * \downarrow_p [\chi * ...]]] \, (\chi \text{ repeated } i \text{ times}) \tag{27}$$

has an l_2 inverse $(\downarrow_{p^i} [\chi_i])^{-1}; (\chi = u * h^\vee)$. The projection is then given by

$$P_{S_{(i)} \perp V_{(i)}} g = \uparrow_{p^i} \left[\downarrow_{p^i} \left[\mathring{h}_{(i)} * g \right] \right] * u_{(i)} \tag{28}$$

where

$$\mathring{h}_{(i)} = \uparrow_{p^i} \left[\left(\downarrow_{p^i} [\chi_i] \right)^{-1} \right] * h^\vee_{(i)} \tag{29}$$

Because of the nested property of multiresolutions, we have the important property that the projection $P_{S_{(i)} \perp V_{(i)}} g$ can be obtained by the oblique projection of the vector $P_{S_{(i-1)} \perp V_{(i-1)}} g$ into $S_{(i)}$. Thus, the projection of a vector g on $S_{(i)}$ can be obtained by projecting on $S_{(i)}$ any finer approximation $P_{S_{(j)} \perp V_{(j)}} g$, $i > j$. We have

Theorem 9.

$$P_{S_{(i)} \perp V_{(i)}} = P_{S_{(i)} \perp V_{(i)}} P_{S_{(i-1)} \perp V_{(i-1)}} \tag{30}$$

Proof. we can decompose g uniquely as

$$g = P_{S_{(i-1)} \perp V_{(i-1)}} g + (I - P_{S_{(i-1)} \perp V_{(i-1)}}) g$$

with $P_{S_{(i-1)} \perp V_{(i-1)}} g$ in $S_{(i-1)}$, and $(I - P_{S_{(i-1)} \perp V_{(i-1)}}) g$ in $V^\perp_{(i-1)}$. By decomposing $(I - P_{S_{(i-1)} \perp V_{(i-1)}}) g$ in $S_{(i)}$ and $V^\perp_{(i)}$, we can rewrite the above equation as:

$$g = P_{S_{(i)} \perp V_{(i)}} P_{S_{(i-1)} \perp V_{(i-1)}} g + (I - P_{S_{(i)} \perp V_{(i)}}) P_{S_{(i-1)} \perp V_{(i-1)}} g + (I - P_{S_{(i-1)} \perp V_{(i-1)}}) g$$

Since $V_{(i)} \subset V_{(i-1)}$, we have that $V^\perp_{(i-1)} \subset V^\perp_{(i)}$. Thus the sum of the last two terms in the previous equation belong to $V^\perp_{(i)}$, while the term $P_{S_{(i)} \perp V_{(i)}} P_{S_{(i-1)} \perp V_{(i-1)}} g$ belongs to $S_{(i)}$. Therefore, the uniqueness of the decomposition by projection implies that $P_{S_{(i)} \perp V_{(i)}} P_{S_{(i-1)} \perp V_{(i-1)}} g = P_{S_{(i)} \perp V_{(i)}} g$. Since g was arbitrary, the proof is complete. ■

Theorem 9 implies that the coefficients c_i of the oblique projection $P_{S_{(i)} \perp V_{(i)}} g = \uparrow_{p^i} [c_i] * u_{(i)}$ can be obtained from the coefficients c_{i-1} of the oblique projection $P_{S_{(i-1)} \perp V_{(i-1)}} g = \uparrow_{p^{i-1}} [c_{i-1}] * u_{(i-1)}$. If we iterate this procedure, we obtain coarser and coarser approximation of the signal g and we have the following algorithm:

The decomposition algorithm

$$(31)\qquad \begin{cases} c_j = \downarrow_p \left[\mathring{h}_{(1)} * y_{j-1}\right] \\ y_{j-1} = K_{j-1} * c_{j-1} \quad j \geq 1 \\ c_0 = g. \end{cases}$$

where the operator K_{j-1} is given by

$$(32)\qquad K_{j-1} = \uparrow_p \left[\left(\downarrow_{p^j}[\chi_j]\right)^{-1} * \downarrow_p [\chi]\right] * \downarrow_{p^{j-1}}[\chi_{j-1}]$$

Since the operator K_{j-1} depends on j, the algorithm (31) depends on the level index j in general. If however, the algorithm is independent of the index j, then it consists of the repetitive application of a single procedure and we say that the algorithm is *pyramidal*. Clearly, if $K_{j-1}(k) = \delta(k)$, $\forall j \geq 1$, then the algorithm is pyramidal. We have the following theorem:

Theorem 10. The following statements are equivalent

(i): $K_{j-1} = \delta(k) \quad \forall j \geq 1$.
(ii): $\downarrow_p [\chi](k) = \delta(k)$.

Proof. (ii) implies (i): This follows from the expression (27) of $\downarrow_p [\chi_j](k)$ and equation (32)
(i) implies (ii): From the expression (32) of K_{j-1} we deduce that

$$(33)\qquad \downarrow_{p^{j-1}}[\chi_{j-1}] = \uparrow_p [\gamma] \quad \forall j > 2$$

for some $\gamma \in l_2$. From this equation and the expression (27), we can use a recurrence argument to show that for any $n \in \mathbb{N}$, there exists a sequence γ_n such that

$$(34)\qquad \downarrow_p [\chi] = \uparrow_{p^n} [\gamma_n]$$

Thus, $\downarrow_p [\chi] = \delta(k)$. ∎

If the conditions of theorem 10 above hold, then it is straightforward to see that we have the recurrence relation

$$\mathring{h}_{(j)} = \uparrow_{p^{j-1}} \left[\mathring{h}_{(1)}\right] * \mathring{h}_{(j-1)} = h^{\vee}_{(j)} \quad \forall j > 1$$

5.1. Biorthogonal discrete wavelet spaces and the wavelet transform

If in the previous section we let $p = 2$, we get two sets of homogeneous multiresolutions $\{V_{(j)} = V_{2^j}\}_{j \in \mathbb{N}}$ and $\{S_{(j)} = S_{2^j}\}_{j \in \mathbb{N}}$. Under the condition of theorem 10, the approximation error $e_{j+1} = s_j - P_{S_{(j+1)} \perp V_{(j+1)}}$ for a sequence $s_j \in S_{(j)}$ can be obtained by an oblique projection. To see this we first define the two spaces $T_{(j)}$ and $W_{(j)}$, $j \in \mathbb{N}^+$ by

$$(35)\, W_{(j)} = \left\{ w(k) := \sum_{n \in \mathbb{Z}} d(n) o_{(j)}(k - 2^j n), = \uparrow_{2^j} [d] * o_{(j)} \quad d \in l_2 \right\} \quad \forall j \geq 1$$

where

$$o_{(j+1)} = \uparrow_{2^j} \left[\delta_1 * \tilde{u}^{\vee}_{(1)}\right] * h_{(j)} \tag{36}$$

$$T_{(j)} = \left\{ t(k) := \sum_{n \in \mathbb{Z}} d(n) x_{(j)}(k - 2^j n), = \uparrow_{2^j} [d] * x_{(j)} \quad d \in l_2 \right\} \tag{37}$$

where

$$x_{(j+1)} = \uparrow_{2^j} \left[\delta_1 * \tilde{h}^{\vee}_{(1)}\right] * u_{(j)} \tag{38}$$

Here $\tilde{u}, \tilde{h}$ are the modulation of u, h respectively as defined by (5) in section 2.

It is not difficult to show that $T_{(j)}$ belongs to $S_{(j-1)}$ and is orthogonal to $V_{(j)}$: $T_{(j)} \subset S_{(j-1)} \cap V^{\perp}_{(j)}$. Similarly $W_{(j)} \subset V_{(j-1)} \cap S^{\perp}_{(j)}$. We also have the following theorem which shows that the error $e_{i+1} = s_i - P_{S_{(i+1)} \perp V_{(i+1)}} s_i$ is given by an oblique projection on the error space (or wavelet space) $T_{(j)}$ orthogonaly to $W_{(j)}$:

Theorem 11.

$$e_{j+1} = P_{T_{(j)} \perp W_{(j)}} s_j = \uparrow_{2^j} [d_j] * x_{(j)} \tag{39}$$

Moreover, the coefficients $d_{j+1}(k)$ can be obtained from the coefficients c_j of $s_j = \uparrow_{2^j} [c_j] * u_{(j)}$ by the simple filtering algorithm

$$d_{j+1} = \downarrow_2 [\delta_{-1} * \tilde{u} * c_j] \tag{40}$$

There is a perfect analogy between our result and the biorthogonal formulation of Cohen-Daubecies-Feauveau for analog signals[4]. From theorem 11 and the decomposition algorithm (31), we have a biorthogonal wavelet decomposition on the two homogeneous multiresolutions $V_{(j)}$ and $S_{(j)}$ and the two wavelet spaces $W_{(j)}$ and $T_{(j)}$ associated with them, as long as the cross correlation $\downarrow_2 [(u * h^{\vee})(k)] = \delta(k)$. In fact we have that any function $g \in l_2$ can be decomposed as follows:

$$g = P_{S_{(J)} \perp V_{(J)}} g + \sum_{n=J}^{1} P_{T_{(n)} \perp W_{(n)}} g \tag{41}$$

The decomposition/reconstruction algorithms which determine the coefficients of the projections at a given level from the knowledge of the coefficients at adjacent levels is depicted in Fig. 2. It should be noted that this is the biorthogonal perfect reconstruction filter bank used in the context of the wavelet transform for analog signals. Rioul has also studied a discrete multiresolution theory for discrete signals[7] . However, his theory does not rely on the explicit formulation of the oblique projection on specific discrete multiresolution and wavelet spaces. Instead, he uses down-scaling and up-scaling operators that are tied to filter banks.

6. CONCLUSION

Our results on oblique projections in certain subspaces of l_2 (homogeneous multiresolutions, etc.), are equivalent to Perfect Reconstruction Filter Banks (PRFB). Given any PRFB, we can explicitly define the pair of multiresolutions and the associated oblique projection that can be implemented by the chosen PRFB. Thus, we have the discrete analog of the Cohen-Deaubechies-Feauveau results for the continuous biorthogonal wavelets.

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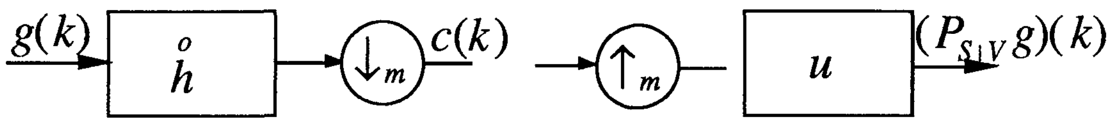

Figure 1: Linear filtering process corresponding to the oblique projection of the discrete signal $g(k)$ on the space $S(u,m)$ in a direction orthogonal to the space $V(h,m)$. The whole projection procedure consists of a pre-filtering and down-sampling by a factor m , then an up-sampling by a factor m followed by a post-filtering.

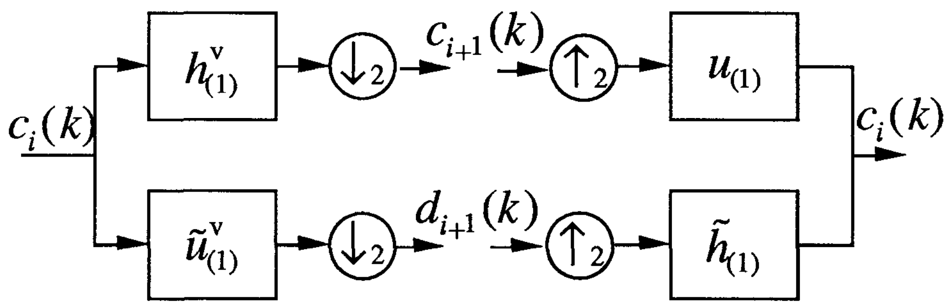

Figure 2: The perfect reconstruction filter bank associated with the pair of homogeneous multiresolutions $\{S_{(j)}, V_{(j)}\}_{j\in N}$ and corresponding wavelet spaces $\{T_{(j)}, W_{(j)}\}_{j\in N}$, generated by $u_{(1)}, h_{(1)}$.