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OBLIQUE MULTIWAVELET BASES: EXAMPLES

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ABSTRACT

Orthogonal, semiorthogonal and biorthogonal wavelet bases are special cases of oblique multiwavelet bases. One of the advantage of oblique multiwavelets is the flexibility they provide for constructing bases with certain desired shapes and/or properties. The decomposition of a signal in terms of oblique wavelet bases is still a perfect reconstruction filter bank. In this paper, we present several examples that show the similarity and differences between the oblique and other types of wavelet bases. We start with the Haar multiresolution to illustrate several examples of oblique wavelet bases, and then use the Cohen-Daubechies-Plonka multiscaling function to construct several oblique multiwavelets.

Keywords: multiwavelet, multiscaling function, oblique wavelet bases, biorthogonal wavelet, semiorthogonal wavelet, perfect reconstruction filter bank, vector filter bank

1 INTRODUCTION

The goal of this paper is to show several examples of the oblique wavelet and multiwavelet bases described in.[1,2] These wavelet bases contain, as special cases, the orthogonal, semiorthogonal, and biorthogonal theory of multiwavelets.[7,10,13,14,23] The main advantage of oblique wavelets is that they give more flexibility in choosing wavelet bases, without compromising the fast filter bank implementation algorithms.

1.1 An oblique wavelet based on the Haar multiresolution

This first example introduces the concept of oblique wavelets. Our starting point is a piecewise constant function $f_0(x)$ which belongs to the Haar multiresolution $V_0(\chi_{[0,1]}) = \{\sum c_0(k)\chi_{[0,1]}(x-k);\ c_0 \in l_2\}$ (see Fig. 1), where $\chi_{[a,b]}$ is the characteristic function on the interval $[a,b]$ (i.e., $\chi_{[a,b]}(x) = 1, \forall x \in [a,b]$, and $\chi_{[a,b]}(x) = 0, \forall x \notin [a,b]$). We wish to approximate the function

$$f_0(x) = \sum_{k\in\mathcal{Z}} c_0(k)\chi_{[0,1]}(x-k), \tag{1}$$

in V_0 by a coarser function $f_{V_1}(x)$ in V_1. One way to do this is to simply downsample $f_0(x)$, and then to hold

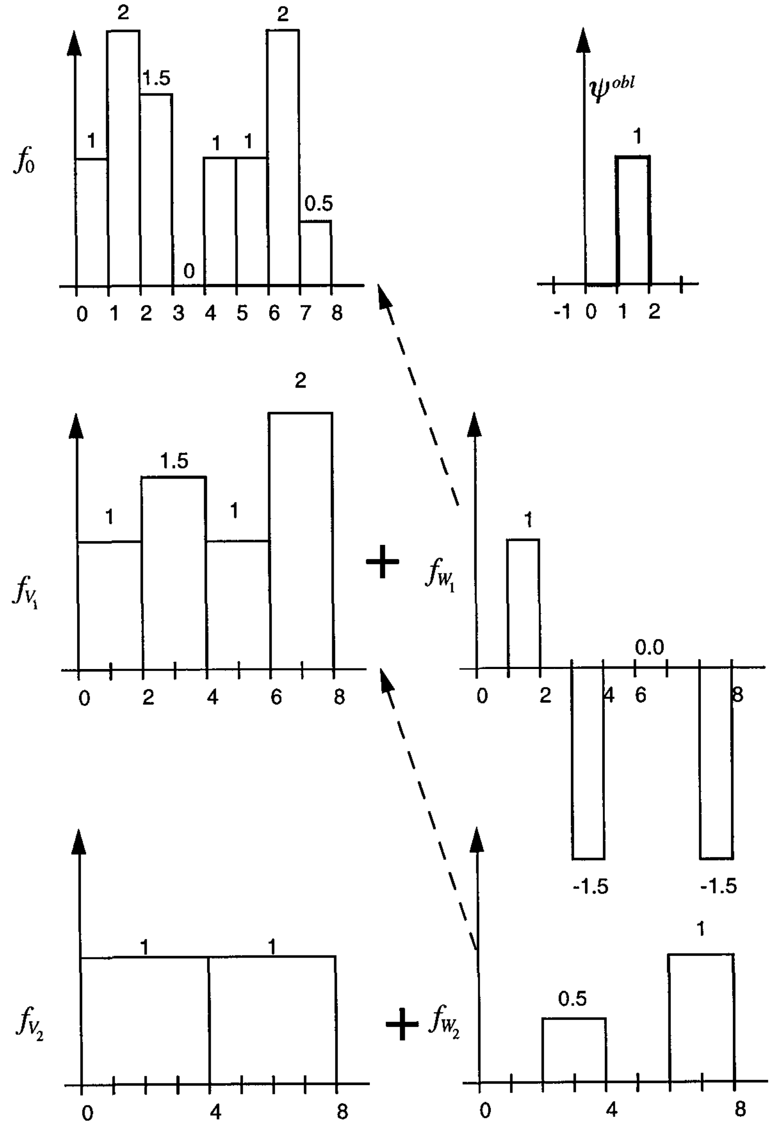

Figure 1: To obtain the approximation f_{V_1}, The function f_0 is ignored in the intervals $[2k-1, 2k]$, while its values on the intervals $[2k, 2k+1]$ are extended. The error $f_{W_1} = f_0 - f_{V_1}$ is clearly a linear combination of the rectangular pulse ψ^{obl} and its shifts on the grid of even integers $2\mathcal{Z}$.

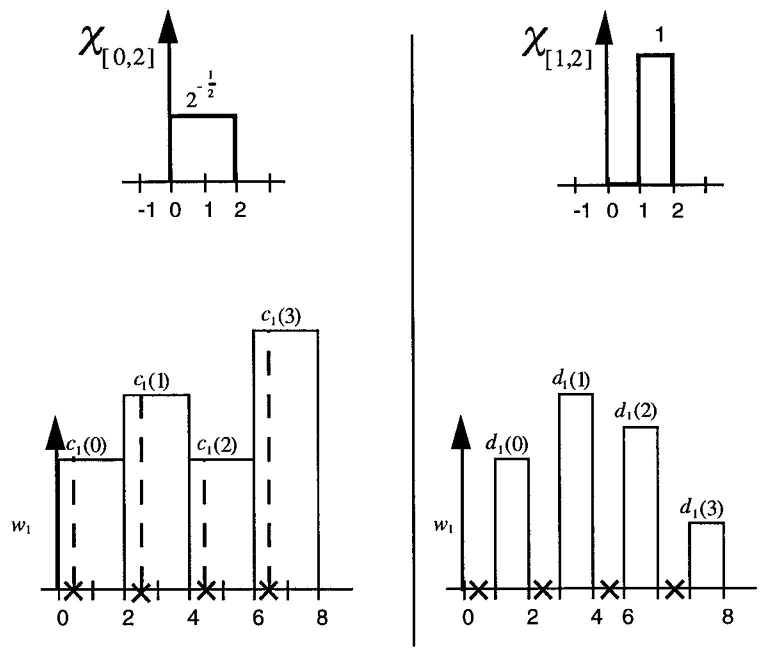

Figure 2: A function w_1 that belongs to V_1 must be a linear combination of $\{\chi_{[0,2]}(x-2k)\}$ (left panel), while a function w_1 that belongs W_1 must be a linear combination of $\{\chi_{[1,2]}(x-2k)\}$ (right panel). Thus, the only possible function that belongs to $V_1 \cap W_1$ is the zero function $w_1 = 0$.

the sample values for interval lengths of size two as shown in Fig. 1:

$$f_{V_1}(x) = \sum c_0(2k)\chi_{[0,2]}(x-2k). \tag{2}$$

The difference between the function f_{V_1} and the starting function $f_0(x)$ is given by

$$f_{W_1}(x) = \sum \left(c_0(2k+1) - c_0(2k)\right)\chi_{[1,2]}(x-2k), \tag{3}$$

where the error function f_{W_1} belongs to the space W_1, defined by

$$W_1 = \left\{ \sum_{k\in\mathcal{Z}} d_1(k)\chi_{[1,2]}(x-2k);\ d_1 \in l_2 \right\}. \tag{4}$$

It is not difficult to see that our original function $f_0(x)$ can be written as

$$\begin{aligned} f_0 &= f_{V_1} + f_{W_1} \\ &= \sum_{k\in\mathcal{Z}} c_0(2k)\chi_{[0,2]}(x-2k) + \sum_{k\in\mathcal{Z}} \left(c_0(2k+1) - c_0(2k)\right)\chi_{[1,2]}(x-2k). \end{aligned}$$

Clearly, $V_1 \subset V_0$ and $W_1 \subset V_0$. Thus, we can apply the same decomposition algorithm to f_{V_1} and f_{W_1}, now both viewed as elements of V_0. By inspection, we see that this procedure leaves f_{V_1} and f_{W_1} invariant. Moreover, we have that the intersection $V_1 \cap W_1 = \{0\}$. To see this (see Figure 2), we simply note that if $w_1 \in W_1$, then $w_1(2k+1/2) = 0$ for any $k \in \mathcal{Z}$. On the other hand, if $w_1 \in V_1$, then $w_1(2k+1/2) = c_1(k)$. Thus, $c_1(k) = 0$ for all $k \in \mathcal{Z}$, and therefore $w_1(x) = 0$. Therefore, we conclude that $f_{V_1} = P_{V_1//W_1} f_0$ is the oblique projection of f_0 in V_1 in a direction parallel to W_1, and $f_{W_1} = P_{W_1//V_1} f_0$ is the oblique projection of f_0 in W_1 in a direction parallel to V_1 (see Figure 3). Moreover we have

$$f_0 = P_{V_1//W_1} f_0 + P_{W_1//V_1} f_0. \tag{5}$$

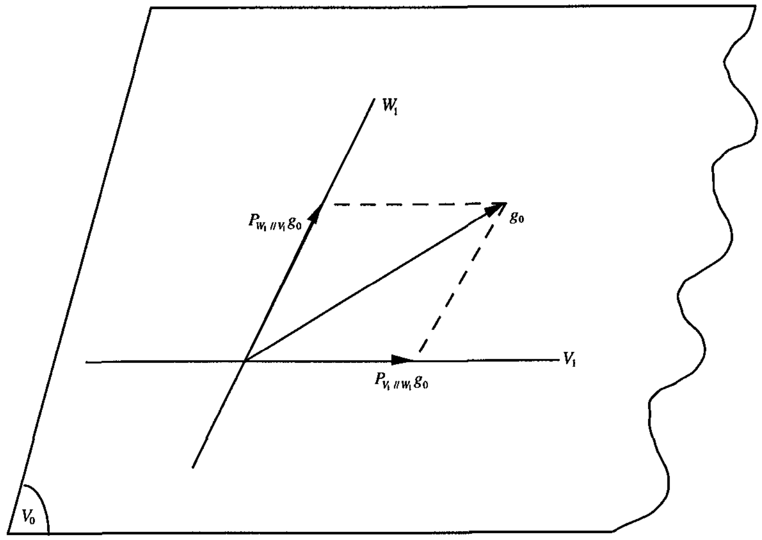

Figure 3: Schematic representation of the oblique projection of g_0 on V_1 in a direction parallel to W_1.

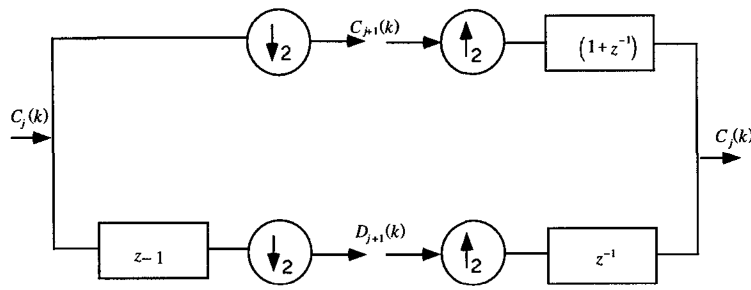

Figure 4: Perfect reconstruction filter-bank for computing a Haar oblique wavelet transform of order zero. The transfer functions of the filters are obtained by setting $z = e^{-i2\pi f}$.

It is easy to see that the set $\{\phi_{1,k}(x) = 2^{-1/2}\chi_{[0,2]}(x-2k);\ k \in \mathcal{Z}\}$ is an orthogonal basis of V_1, and that $\{\psi_{1,k}(x) = \chi_{[1,2]}(x-2k);\ k \in \mathcal{Z}\}$ is an orthogonal basis of W_1. Moreover, it can be shown that the set

$$\{\phi_{1,k}(x),\ \psi_{1,k}(x);\ k \in \mathcal{Z}\}$$

forms a Riesz basis for the space of piecewise constants with integer knot-points V_0.[1]

If we now repeat the decomposition procedure, we obtain a series of coarser approximation of $f_0(t)$ and a series of error functions as shown in Fig. 1. The decomposition algorithm consists of the repetitive application of the perfect reconstruction filter algorithm shown in Figure 4. Therefore, for any function $f \in L_2$, we can choose a sufficiently small value J_2 so that the orthogonal projection $f_{J_2} = P_{V_{J_2}} f$ of f into the space of piecewise constants V_{J_2} is arbitrarily close to f. Then, we can repeatedly apply the decomposition algorithm in (5) to obtain

$$f_{J_2} = P_{V_{J_1}//W_{J_1}} f_{J_2} + \sum_{j=J_2+1}^{J_1} P_{W_j//V_j} f_{J_2}. \tag{6}$$

It follows that the set $\left\{\psi_{j,k}(x) = 2^{-\frac{j}{2}}\chi_{[1,2]}(2^{-j}x - 2k);\ (j,k) \in \mathcal{Z}^2\right\}$ is a basis of L_2. However, this set is not a

Riesz basis of L_2, but simply a countable basis of L_2. This means that any finite number of members in this set are linearly independent, and that finite linear combinations of this set are dense in L_2. What is interesting is that we still have a fast and stable computational algorithm for calculating the expansion in terms of this countable basis.

We call the set $\left\{\psi_{j,k} = 2^{-\frac{j}{2}}\psi(2^j x - k);\ (j,k) \in \mathcal{Z}^2\right\}$ an *oblique wavelet basis* of L_2, and we call the spaces W_j that are generated by the Riesz bases

$$\left\{\psi_{j,k} = 2^{-\frac{j}{2}}\psi(2^j x - k);\ k \in \mathcal{Z}\right\}$$

oblique wavelet spaces. Unlike other wavelets, the function $\psi(x) = 2^{\frac{1}{2}}\chi_{[1/2,1]}$ does not have an average of zero. Thus, it is not an orthogonal, semiorthogonal, or biorthogonal wavelet.[4,19] In contrast with biorthogonal wavelets, this Haar oblique wavelet has neither an associated dual wavelet spaces, nor a dual wavelet basis. What is remarkable is that this transform is faster than the classical Haar transform. Moreover, the support of the wavelet (in this case the function $\chi_{[\frac{1}{2},1]}$) is half the support of the orthogonal Haar wavelet.

2 MULTISCALING FUNCTIONS AND MULTIWAVELET BASES

2.1 Multiscaling functions

If the multiresolution spaces $\{V_j\}_{j\in\mathcal{Z}}$

$$\cdots V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset \cdots$$

are generated by the translations and dilations of $r > 1$ functions $\phi^1(x), \ldots, \phi^r(x)$, then the vector function $\Phi = (\phi^1(x), \ldots, \phi^r(x))^T$ is called a *multiscaling function* (Here, $()^T$ denote the matrix transpose operator). Thus, we have that

$$\begin{aligned} V_j &= \left\{\sum_{i=1}^{r}\sum_{k\in\mathcal{Z}} c_j^i(k) 2^{-j/2}\phi^i\left(\frac{x}{2^j} - k\right); c_j^i \in l_2,\ i = 1, \cdots, r\right\} \\ &= \left\{\sum_{k\in\mathcal{Z}} \mathbf{C}_j^T(k)\Phi_j(x - 2^j k); \mathbf{C}_j(k) \in l_2^r\right\}, \end{aligned}$$

where $\mathbf{C}_j(k)$ is the vector $\mathbf{C}_j = (c_j^1(k), c_j^2(k), \ldots, c_j^r(k))^T$, $l_2^r = l_2 \times \cdots \times l_2$, and the vector $\Phi_j(x)$ is defined to be

$$\begin{aligned} \Phi_j(x) &= \left(\phi_j^1(x), \phi_j^2(x), \cdots, \phi_j^r(x)\right)^T \\ &= 2^{-j/2}\left(\phi^1\left(\frac{x}{2^j}\right), \phi^2\left(\frac{x}{2^j}\right), \cdots, \phi^r\left(\frac{x}{2^j}\right)\right)^T. \end{aligned}$$

In particular, a function $f_0 \in V_0$ is given by

$$\begin{aligned} f_0 &= \sum_{k\in\mathcal{Z}} c_0^1(k)\phi^1(x-k) + \ldots + c_0^r(k)\phi^r(x-k) \\ &= \sum_{k\in\mathcal{Z}} \mathbf{C}_0^T\Phi(x-k). \end{aligned}$$

2.2 Multiwavelets

If the spaces $\{W_j\}_{j\in\mathcal{Z}}$ that complement the multiresolution spaces $\{V_j\}_{j\in\mathcal{Z}}$ (i.e., $V_{j+1}+W_{j+1} = V_j$) are generated by a set of r functions $\psi^1(x), \psi^2(x), \cdots, \psi^r(x)$, then the vector $\Psi(x) = \left(\psi^1(x), \psi^2(x), \cdots, \psi^r(x)\right)^T$ is called a *multiwavelet*, as long as the set

$$\left\{\psi^i_{j,k}(x) = 2^{-\frac{j}{2}}\psi^1_j(x-2^jk), \cdots, 2^{-\frac{j}{2}}\psi^r_j(x-2^jk);\ k\in\mathcal{Z}\right\}$$

forms a Riesz basis of

$$\begin{aligned} W_j &= \left\{\sum_{i=1}^{r}\sum_{k\in\mathcal{Z}} d^i_j(k)2^{-j/2}\psi^i\left(\frac{x}{2^j}-k\right); d^i_j\in l_2,\ i=1,\cdots,r.\right\} \\ &= \left\{\sum_{k\in\mathcal{Z}}\mathbf{D}^T_j(k)\Psi_j(x-2^jk); \mathbf{D}_j(k)\in l^r_2\right\}, \end{aligned}$$

where $\Psi_j(x)$ is defined to be

$$\Psi_j(x) := 2^{-j/2}\Psi\left(\frac{x}{2^j}\right). \tag{7}$$

Note that we do not require orthogonality between the spaces V_j and W_j. Therefore, the spaces W_j are not necessarily orthogonal to each other.

2.3 Classification of wavelet bases

According to the angles between the spaces $\{W_j\}_{j\in\mathcal{Z}}$ and the choice of bases for these spaces, wavelets and multiwavelets can be classified into several categories:

- *Orthogonal* wavelet and multiwavelet bases[8,10,16,17,21]: For this case, there are two conditions that must be satisfied:
  1. The wavelet spaces must be orthogonal to each other:
  $$W_j \perp W_l,\ \forall j\neq l.$$
  2. The set $\left\{\psi^i_{j,k};\ k\in\mathcal{Z},\ i=1,\cdots,r.\right\}$ must be an orthogonal basis of W_j.
- *Semiorthogonal* wavelet and multiwavelet bases,[3,5,9,20,22]: For these bases, the only requirement is that $W_j\perp W_l,\ \forall j\neq l$. In this case, $\left\{\psi^i_{j,k};\ k\in\mathcal{Z},\ i=1,\cdots,r.\right\}$ is not necessarily an orthogonal basis of W_j.
- *biorthogonal* wavelet bases[6]: This case consists of a pair of wavelet bases $\left\{\psi^i_{j,k};\ k\in\mathcal{Z},\ i=1,\cdots,r.\right\}$ and $\left\{\tilde{\psi}^i_{j,k};\ k\in\mathcal{Z},\ i=1,\cdots,r.\right\}$ generating a pair of wavelet spaces W_j and $\tilde{W}_j$, and satisfying the biorthogonality condition
  $$\left\langle\psi^i_{j,k}, \tilde{\psi}^l_{m,n}\right\rangle_{L_2} = \delta_0(j-m)\delta_0(k-n)\delta_0(i-l). \tag{8}$$
- *Oblique* wavelet and multiwavelet bases[2]: In this case, we require only that the set $\left\{\psi^i_{j,k};\ k\in\mathcal{Z},\ i=1,\cdots,r.\right\}$ be a Riesz basis of W_j. We do not require orthogonality between the wavelet spaces, as in the orthogonal and semiorthogonal cases; and unlike the biorthogonal case, we do not require the associated wavelet spaces $\tilde{W}_j$ or the existence of a biorthogonal wavelet.

The first orthogonal multiwavelet bases were introduced by Donovan, Geronimo, Hardin, and Massopust.[11] These bases have the remarkable properties of being orthogonal, regular, compactly supported, and symmetrical. It was not possible to construct wavelet bases ($r = 1$) satisfying these four properties before.[8] Other orthogonal multiwavelet bases have also been found.[21] Semiorthogonal constructions of multiwavelets were first introduced in.[13,12] The first biorthogonal multiwavelet constructions of which we are aware has been introduced in.[18] These multiwavelets are special cases of oblique multiwavelets.[2] In fact, the theory underlying oblique multiwavelet bases encompasses the theories of orthogonal, semiorthogonal, and biorthogonal multiwavelets.

2.4 Two-scale equations

Since $V_1 \subset V_0$ and $W_1 \subset V_0$, both $\Phi\left(\frac{x}{2}\right)$ and $\Psi\left(\frac{x}{2}\right)$ must be linear combination of the basis of V_0. Therefore, we must have the two-scale relations

$$\Phi\left(\frac{x}{2}\right) = 2\sum_{k\in\mathcal{Z}} \mathbf{H}_1(k)\Phi(x-k), \tag{9}$$

where $\mathbf{H}_1(k)$ is an $r \times r$ matrix-sequence called the *generating sequence*, and

$$\Psi\left(\frac{x}{2}\right) = 2\sum_{k\in\mathcal{Z}} (\delta_1 * \mathbf{G}_1)(k)\Phi(x-k), \tag{10}$$

where $\mathbf{G}_1(k)$ is a matrix-sequence; $\delta_i(k)$ is the unit impulse sequence located at $k = i$, and the *generalized convolution* "$*$" is an operator that acts on sequences of matrices. Specifically, the convolution $\mathbf{C}(k) = (\mathbf{A} * \mathbf{B})(k)$ between the $m \times r$ matrix-sequence $\{\mathbf{A}(k)\}_{k\in\mathcal{Z}}$ and the $r \times n$ matrix-sequence $\{\mathbf{B}(k)\}_{k\in\mathcal{Z}}$ is the $m \times n$ matrix-sequence, defined in terms of the convolution between the entries of $\mathbf{A}$ and $\mathbf{B}$ as

$$\mathbf{C}_{i,j}(k) := \sum_{l=1}^{l=r}\sum_{h\in\mathcal{Z}} \mathbf{A}_{i,l}(h)\mathbf{B}_{l,j}(k-h). \tag{11}$$

From this definition, it follows that the unit impulse $\delta_1(k)$ in (10) is used to shift the sequence $\mathbf{G}_1(k)$, i.e., $(\delta_1 * \mathbf{G}_1)(k) = \mathbf{G}_1(k-1)$.

As an example of two-scale relation, the Cohen-Daubechies-Plonka multiscaling function[7] consists of a pair of functions $\Phi(x) = (\phi^1(x), \phi^2(x))^T$ that are characterized by the Fourier transform of their 2×2 two-scale sequence $\hat{\mathbf{H}}(z)\Big|_{z=e^{-i2\pi f}}$:

$$\hat{\mathbf{H}}_1(f) = \frac{1}{4}\begin{bmatrix} z^1 + 2z^2 + z^3 & (z^1 - 2z^3 + z^5)/2 \\ 1/32 & z \end{bmatrix}. \tag{12}$$

3 CONSTRUCTION OF OBLIQUE WAVELET AND MUTLIWAVELET BASES

The main result for finding the oblique multiwavelets relies on a construction in the Fourier domain. Specifically, we use the $r \times r$ matrix-function $\hat{\mathbf{H}}_1(f)$ (which is usually known) to construct a $2r \times 2r$ invertible matrix function

$$\hat{\mathbf{S}}_1(f) = \begin{bmatrix} \hat{\mathbf{H}}_1^T(f) & \hat{\mathbf{G}}_1^T(f) \\ \hat{\mathbf{H}}_1^T(f-\frac{1}{2}) & -\hat{\mathbf{G}}_1^T(f-\frac{1}{2}) \end{bmatrix}. \tag{13}$$

The main goal is to choose the $r \times r$ matrix-function $\hat{\mathbf{G}}_1^T(f)$ appropriately so that $(\hat{\mathbf{S}}_1)(f))^{-1}$ exits. Then, the inverse Fourier transform $\mathbf{G}_1(k)$ of $\hat{\mathbf{G}}_1^T(f)$ gives rise to an oblique wavelet by the formula (10).[1,2] The main result can be stated as follows[1]:

THEOREM 3.1. *If $\hat{\mathbf{S}}_1^{-1}(f)$ exists for almost all f, and if the matrix-norms of $\hat{\mathbf{S}}_1(f)$ and $\hat{\mathbf{S}}_1^{-1}(f)$ are uniformly bounded, i.e., if there exist two constants $m > 0$ and $M > 0$ independent of f such that $\left\|\hat{\mathbf{S}}_1(f)\right\| \leq M$ and $\left\|\hat{\mathbf{S}}_1^{-1}(f)\right\| \leq m$, then*

$$\Psi_1(x) = 2^{-\frac{1}{2}}\Psi\left(\frac{x}{2}\right) = 2^{\frac{1}{2}} \sum_{k\in\mathcal{Z}} (\delta_1 * \mathbf{G}_1)(k)\Phi(x-k) \tag{14}$$

is an oblique multiwavelet. It is associated with the multiresolution generated by the multiscaling function $\Phi(x)$, whose two-scale sequence is $\mathbf{H}_1(k)$. The set

$$\{\psi_{j,k}^i(x);\ k \in \mathcal{Z},\ i = 1, \cdots, r.\}$$

is a Riesz basis of W_j; the set

$$\{\psi_{j,k}^i(x);\ (j,k) \in \mathcal{Z}^2,\ i = 1, \cdots, r.\}$$

is a basis of L_2; and for any $J_2 \in \mathcal{Z}$, the set

$$\left\{\phi_{j,k}^{J_1}(x), \psi_{j,k}^i(x);\ k \in \mathcal{Z},\ J_2 + 1 \leq j \leq J_1,\ i = 1, \cdots, r.\right\}$$

is a Riesz basis of V_{J_2}. Moreover, for any $g \in V_{J_2}$, we have the decomposition

$$g = P_{V_{J_1}//W_{J_1}} g + \sum_{j=J_2+1}^{J_1} P_{W_j//V_j} g. \tag{15}$$

4 EXAMPLES OF OBLIQUE WAVELET AND MULTIWAVELET BASES

4.1 The Haar oblique wavelets

We start by reviewing our example in Section 1.1 using the results of Theorem 3.1. For this case, r=1, and $H_1(k) = 2^{-1}(\delta_0(k) + \delta_1(k))$. For $G_1(k)$, we first choose $G_1(k) = 2^{-\frac{1}{2}}\delta_0(k)$. The 2×2 matrix function $\hat{\mathbf{S}}_1(f)$ is given by

$$\hat{\mathbf{S}}_1(f) = \begin{bmatrix} \frac{1+z^{-1}}{2} & 2^{-\frac{1}{2}} \\ \frac{1-z^{-1}}{2} & -2^{-\frac{1}{2}} \end{bmatrix}, \tag{16}$$

where $z = e^{i2\pi f}$. We have that $\det(\hat{\mathbf{S}}_1(f)) = -2^{-\frac{1}{2}}$ for all f; thus, $\hat{\mathbf{S}}_1(f)$ has an inverse. Moreover, it is easy to see that the conditions of Theorem 3.1 on the norms of $\hat{\mathbf{S}}_1(f)$ and its inverse are satisfied. Therefore, Equation (14) gives us the oblique wavelet $\psi_1(x/2) = \chi_{[1,2]}(x)$. The filter bank implementing the decomposition and reconstruction algorithms is depicted in Figure 4.

If, instead of the previous choice for G_1, we set $G_1(k) = 2^{-1}(\delta_{-1}(k) - \delta_0(k))$, we obtain the well-known orthogonal Haar wavelet. It is easy to check that, in this case, $\hat{\mathbf{S}}_1(f)$ is given by

$$\hat{\mathbf{S}}_1(f) = 2^{-1}\begin{bmatrix} 1 + z^{-1} & z - 1 \\ 1 - z^{-1} & z + 1 \end{bmatrix} \tag{17}$$

and that $\det(\hat{\mathbf{S}}_1(f)) = 1$ for all f. As expected, the wavelet function given by (14) is precisely the orthogonal Haar wavelet.

Another choice for $\hat{G}_1(f)$ is to take the trigonometric polynomial given by $\hat{G}_1(f) = (z-1)^n$. For $n = 0, 1$, we get the previous two cases. For $n = 2$ we get that $\det(\hat{\mathbf{S}}_1(f)) = -3 - e^{-i2\pi f}$, which is nonzero for all f. Hence, we again obtain oblique wavelets, but now with two vanishing moments.

4.2 Multiwavelets based on the Cohen-Daubechies-Plonka multiscaling function

The Cohen-Daubechies-Plonka multiscaling function described in[7] has the 2×2 generating sequence given by (12) We select sequences $\mathbf{G}_1^n$ of the form $\hat{\mathbf{G}}_1^n(f) = (z-1)^n \mathbf{I}_2$, where $\mathbf{I}_2$ is the 2×2 identity matrix. For $n = 0$, we easily check that the determinant of $\hat{\mathbf{S}}_1^0(f)$ is zero for all f. Therefore, $\hat{\mathbf{S}}_1^0(f)$ is not invertible and we cannot find a multiwavelet for this choice. For $n = 1$, the determinant of $\hat{\mathbf{S}}_1^1(z)$ is given by

$$\det\left(\hat{\mathbf{S}}_1^1(z)\right) = \frac{z^2}{256}\left(63z^4 + 194z^2 - 1\right), \tag{18}$$

which is nonzero for $z = e^{-i2\pi f}$. We conclude that the inverse $\hat{\mathbf{S}}_1^2(f)$ exists, and its entries are ratios of trigonometric polynomials. It follows that the conditions of Theorem 3.1 are satisfied. Therefore, the choice $\hat{\mathbf{G}}_1^1(f) = (z-1)\mathbf{I}_2$ gives rise to an oblique multiwavelet. It should be noted that this multiwavelet has its moments of order 1 equal to zero. It can be shown that $n = 3, 5, 7$ also gives rise to multiwavelets.

For $n = 2$, and in fact for any even integer $n = 2m$, a simple calculation shows that $\hat{\mathbf{S}}_1^{2m}(\pm\frac{1}{4})$ is singular. Since $\hat{\mathbf{S}}_1^{2m}(f)$ is continuous in f, the matrix-norm of the inverse of $\hat{\mathbf{S}}_1^{2m}(f)$ cannot be bounded for almost all f. Thus, it is not possible to construct oblique multiwavelets using sequences of the form $\hat{\mathbf{G}}_1^{2m}(f) = (z-1)^{2m}\mathbf{I}_2$.

5 FAST FILTER-BANK ALGORITHMS

Oblique wavelet and multiwavelet transforms can be implemented using perfect reconstruction filter banks.[2,1] Any function $g \in \mathbf{L}_2(\mathcal{R})$ can be decomposed into a low resolution approximation $g_J \in V_J$ and the sum of the error terms in the spaces $\{W_j\}_{j \geq J}$

$$g = \uparrow_{2^J}[\mathbf{C}_J] * \Phi_J^T + \sum_{j=J}^{-\infty} \uparrow_{2^j}[\mathbf{D}_j] * \Psi_j^T. \tag{19}$$

In practice, $g(x)$ belongs to a multiresolution space, e.g., $g = \mathbf{C}_0 * \Phi^T$. In this case, the procedure for finding the coefficients $\mathbf{C}_J(k)$ and $\{\mathbf{D}_j(k)\}_{j=J,\ldots,1}$ from the coefficients $\mathbf{C}_0(k)$ can be obtained by a fast vector-filter-bank algorithm depicted in the left part of Figure 5.

Similarly, the procedure for finding the coefficients $\mathbf{C}_0(k)$ from the knowledge of $\mathbf{C}_J(k)$ and $\{\mathbf{D}_j(k)\}_{j=J,\ldots,1}$ can be obtained by a vector-filter-bank algorithm depicted in the right part of Figure 5. The filters $\mathbf{H}_1$, $\mathbf{H}_2$, $\mathbf{G}_1$, and $\mathbf{G}_2$ of Figure 5 are the filters defined in the previous section and are associated with the oblique wavelets. The decomposition and reconstruction filter-banks constitute together the perfect reconstruction filter-bank structure.

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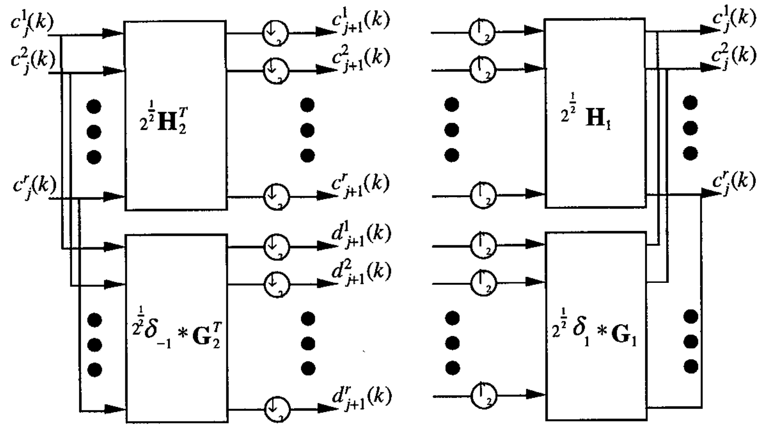

Figure 5: Perfect reconstruction filter-bank for computing the oblique multiwavelet transform. The coefficients $\mathbf{C}_0$ are decomposed into two sequences $\mathbf{C}_1$ and $\mathbf{D}_1$ (left filter-bank pair). The two sequences $\mathbf{C}_1$ and $\mathbf{D}_1$ can then be combined by the right pair of filters to reconstruct $\mathbf{C}_0$.

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