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Complex-order scale-invariant operators and self-similar processes

Arash Amini [a,*], Julien Fageot [b], Michael Unser [c]

[a] Advanced Communication Research Institute (ACRI), EE department, Sharif University of Technology, Tehran, Iran
[b] AudioVisual Communications Laboratory (LCAV), École polytechnique fédérale de Lausanne (EPFL), Lausanne, Switzerland
[c] Biomedical Imaging Group (BIG), École polytechnique fédérale de Lausanne (EPFL), Lausanne, Switzerland

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ABSTRACT

In this paper, we perform the joint study of scale-invariant operators and self-similar processes of complex order. More precisely, we introduce general families of scale-invariant complex-order fractional-derivation and integration operators by constructing them in the Fourier domain. We analyze these operators in detail, with special emphasis on the decay properties of their output. We further use them to introduce a family of complex-valued stable processes that are self-similar with complex-valued Hurst exponents. These random processes are expressed via their characteristic functionals over the Schwartz space of functions. They are therefore defined as generalized random processes in the sense of Gel'fand. Beside their self-similarity and stationarity, we study the Sobolev regularity of the proposed random processes. Our work illustrates the strong connection between scale-invariant operators and self-similar processes, with the construction of adequate complex-order scale-invariant integration operators being preparatory to the construction of the random processes.

1. Introduction

1.1. Scale-invariant differential operators and self-similar random processes

Differential operators are of great interest to model physical phenomena [1]. They are tightly linked with mathematical notions such as splines [2] and stochastic processes [3]. In this paper, we focus on complex-order differential operators and their use in the specification of complex-valued one-dimensional random processes $S = (S(x))_{x\in\mathbb{R}}$.

An operator L is called scale-invariant if it commutes with time rescaling up to some proportional factor. Formally, for a function f and a time-rescaling factor $T > 0$, the operator L is scale-invariant if $\mathrm{L}\{f(T\cdot)\} = c_T \mathrm{L}\{f\}(T\cdot)$, where $c_T > 0$ is a multiplicative scalar that depends on $T > 0$ and not on f. It is known that $c_T = T^\gamma$ for some fixed γ that depends on L (see [4]), or that

$$\mathrm{L}\{f(T\cdot)\} = T^\gamma \mathrm{L}\{f\}(T\cdot) \tag{1}$$

for every f and T. For instance, when $\mathrm{L} = \mathrm{D}$, we have that $\gamma = 1$. All D^k operators ($k \in \mathbb{N}$) are scale-invariant with $\gamma = k$. More generally, the Riemann-Liouville fractional-derivatives D^γ with $\gamma \in \mathbb{R}^+$ are also scale-invariant. However, D^γ is not local when γ is not an integer.

* Corresponding author.
E-mail address: aamini@sharif.edu (A. Amini).

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A real-valued random process $S = (S(x))_{x\in\mathbb{R}}$ is called *self-similar* if, for each $T > 0$, there exists a constant $d_T > 0$ such that $S(T\cdot) \stackrel{d}{=} d_T S$. This means that the random processes $S(T\cdot) = (S(Tx))_{x\in\mathbb{R}}$ and $d_T S = (d_T S(x))_{x\in\mathbb{R}}$ have the same finite-dimensional marginal laws for every $T > 0$. It is known that $d_T = T^H$ for some $H \in \mathbb{R}$ called the Hurst exponent of S [5]. This implies that

$$S(T\cdot) \stackrel{d}{=} T^H S \tag{2}$$

for any T. Fractional Brownian-motion (fBm) processes are prototypical examples of Gaussian self-similar random processes, covering the whole range of $0 < H < 1$, with $H = \frac{1}{2}$ representing the standard Brownian motion. The fBm processes are the only self-similar Gaussian processes with stationary increments. There also exist non-Gaussian self-similar random processes, such as stable ones [6], which could admit even complex-valued Hurst exponents $H \in \mathbb{C}$. As we show in this paper, scale-invariant differential operators and stable self-similar processes are deeply connected. In fact, for many self-similar processes, we can identify a scale-invariant operator (called whitening operator) that will decouple the random process by transforming it into a stable white noise. Conversely, one can start from the white noise and generate a self-similar process by applying the inverse of a scale-invariant operator, which then coincides with the classical definition via a stochastic integral. For instance, the fBm process S with Hurst exponent $0 < H < 1$ can be whitened (*i.e.*, linearly transformed into a white noise) via the differential operator D, in the sense that

$$\mathrm{D}^{\gamma} S = W \tag{3}$$

is a Gaussian white noise W for $\gamma = H + \frac{1}{2}$.

An important specificity of our work is to define self-similar random processes in the framework of *generalized random processes* [7] (see Section 3). This allows us to include stable white noises and other singular random processes (*i.e.*, generalized random processes with no point-wise interpretation) and to rely on their characteristic functional (the infinite-dimensional generalization of the characteristic function of random variables). This approach necessitates one to carefully study complex-order fractional-derivative operators together with their (right-)inverse integral operators.

1.2. Related works

Fractional-differential operators Differential operators are of great interest to model physical phenomena [1]. They are tightly linked with mathematical notions such as splines [2], wavelets [8], and stochastic processes [3]. The derivative operator $\mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}x}$ is the simplest differential operator with a non-trivial null-space. Beside being linear, this operator is also shift- and scale-invariant. Differential operators have been extended to fractional [9,10] and complex-order ones [11,12]. The connection between fractional calculus and self-similar processes is well established [13,14] and is a cornerstone of our work.

Self-similar random processes The specification of fBm by Mandelbrot and van Ness[1] in [17] led to a substantial increase in the popularity of fractals and fractional-derivatives. Mandelbrot and van Ness define fBM as

$$B_H(x) = \frac{1}{\Gamma(H+\frac{1}{2})} \int_{\mathbb{R}} \left((x-\tau)_+^{H-\frac{1}{2}} - (-\tau)_+^{H-\frac{1}{2}} \right) \mathrm{d}B(\tau), \tag{4}$$

where the random process $B(x)$ is the classical Brownian motion and the notation $(x)_\pm^r$ refers to $|x|^r \mathbb{1}_{\pm x>0}$ with $\pm$ representing either $+$ or $-$, but consistent with $(x)_\pm^r$. Due to their self-similarity and long-range-dependence properties, fBm found applications in various fields such as traffic modeling [18], image processing [19–22], modeling scatterings from rough surfaces [23], and finance [24]. The self-similarity property implies that these random processes have the same structure at various scales. As wavelets provide a multiresolution representation of signals, the study of the wavelet representation of fBm processes has been the center of extensive research [8,25–32].

The extension of fBm has been considered in a number of aspects. While early extensions generalized the random processes to two [33] and higher dimensions [34], the introduction of non-Gaussian distributions were another generalization. Indeed, Gaussian distributions belong to the larger family of α-stable distributions ($0 < \alpha \le 2$). As shown in [6], the random processes

$$\begin{aligned} S_{\alpha,H}(a,b;x) = \int_{\mathbb{R}} \Big(& a\big((x-\tau)_+^{H-\frac{1}{\alpha}} - (-\tau)_+^{H-\frac{1}{\alpha}}\big) \\ & + b\big((x-\tau)_-^{H-\frac{1}{\alpha}} - (-\tau)_-^{H-\frac{1}{\alpha}}\big) \Big) \mathrm{d}\mu_\alpha(\tau), \end{aligned} \tag{5}$$

are also self-similar with α-stable marginal distributions, where a, b are arbitrary reals, H is any real in the interval $]0,1[$ other than $\frac{1}{\alpha}$, and μ_α is a suitable α-stable motion. The case of $H = \frac{1}{\alpha}$ could be defined as the conventional stable motion process, while $H = 1$ represents a degenerate case (lines with random slopes). In addition, $S_{2,H}\big(\frac{1}{\Gamma(H+\frac{1}{2})}, 0; x\big)$ associated with $\alpha = 2$ is equivalent to (4). This wider class of distributions allows for a wider range of applications [35,36] and with better modeling capabilities [37]. For $\alpha \neq 2$, the stable distributions have heavy tails with infinite variance. Self-similar stable processes have been extensively studied [6,38–40].

[1] The fBm processes were previously studied by Kolmogorov [15] and Lévy [16]; however, they were made popular by Mandelbrot and van Ness.

The notion of self-similarity itself is extended to operator-self-similarity in [41] and later [42], where invariance of the distribution with respect to scaling by means of a positive-definite self-adjoint linear operator is considered. The operator-self-similar stable processes are investigated in [43–45] while [46] focuses on operator-self-similar random fields with Gaussian laws. The extension to multivariate operator-self-similar stable random fields is provided in [47] and the complex-valued processes are introduced in [48]. Operator self-similarity not only generalizes self-similarity, but also allows for processes with complex-valued Hurst exponents. Our work can therefore be seen as a particular case of this rich literature, for which we provide new results.

Generalized random processes Self-similar processes are usually defined as classical random processes, typically via stochastic integrals [6]. Our work relies on a different framework and specifies the considered random processes as random elements in the Schwartz space of generalized functions [7,49]. This approach was previously adopted by some authors for the construction and analysis of self-similar processes [40,50–53]. One closely related work is [54], where fractional α-stable motions (and sheets) are defined for $\alpha \in]1,2[$ and $\gamma \in]1, 2-\frac{1}{\alpha}[$. These processes are shown to be self-similar with Hurst exponent $H = \gamma + \frac{1}{\alpha} - 1 \in]\frac{1}{\alpha}, 1[$ [53, Proposition 4.2]. Our construction in this paper is more general, as we cover the whole range $0 < \alpha \leq 2$ and consider much more general smoothness parameters γ.

One of the advantages of generalized random processes is the inclusion of continuous-domain white-noise processes (which do not admit point-wise definitions) and their linear transformations (*e.g.* filtered white-noise). Besides, ordinary random processes could also be considered as special cases of generalized random processes; therefore, generalized random processes provide us with a richer framework.

1.3. Contributions and outline

The connection between self-similar processes, operators, and splines in [2] and, particularly, complex-order B-splines in [55] and complex-order exponential splines in [56], forms the inspiration for this work. Specifically, our goal is to construct self-similar random processes with *complex-valued* Hurst exponent H with the help of scale-invariant operators with *complex-valued* order that can be whitened. Our contributions can be summarized as follows.

- *Systematic Characterization of Scale-Invariant Operators with Complex Order.* We introduce an extended family of scale-invariant derivatives and integration operators. Their respective construction is made in the Fourier domain and allows for complex-valued γ in (1). We provide an in-depth analysis of these fractional operators, with special emphasis on the decay properties of their output. We believe these results to be interesting by themselves, but they are also preparatory to the construction of self-similar processes.
- *Complex-order Fractional Generalized Random Processes.* We rely on the framework of generalized random processes and construct the self-similar random processes via their characteristic functionals over the (complex-valued) Schwartz space $\mathcal{S}$ of rapidly decaying and smooth test functions. This allows us to include singular self-similar processes that do not admit a point-wise interpretation.
- *Construction of Self-Similar Stable Processes with Complex-Valued Hurst Exponent.* We introduce an extended family of fractional processes that are self-similar, stable, and that have a *complex-valued* Hurst exponent H. Those random processes are solutions of stochastic differential equations involving a fractional-differential operator.
- *Invariance and Regularity.* We study the invariance (self-similarity and stationarity) and regularity (in terms of Sobolev spaces) of our self-similar stable processes.
- *Whitening.* We point out the tight interplay between scale-invariant operators and self-similar processes. First, we use scale-invariant integration operators to construct stable self-similar processes as filtered versions of the stable white noise. Second, we show that scale-invariant derivatives can be applied to many self-similar processes to whiten them, in other words, to transform them into a stable white noise.

Our work relies on the powerful Fourier machinery in two ways: (i) scale-invariant derivatives and integration operators are defined in the Fourier domain, and (ii) random processes are specified via their characteristic functional, which is the infinite-dimensional Fourier transform of the underlying probability law.

The paper is organized as follows. Complex-order fractional-derivative and integral operators are introduced and analyzed in Section 2. The family of self-similar stable processes is defined and studied in Section 3. Section 4 collects useful results for the proofs of the main results, which are provided in Section 5. Finally, we conclude in Section 6.

2. Complex fractional operators and their properties

We study complex-order fractional-derivative operators and their associated complex-order fractional-integral operators. Special attention is given to identify the right-inverse operators to the complex-order derivatives together with their adjoint operators, which is preparatory to the construction of complex-order self-similar random processes in Section 3. We introduce complex-order fractional-derivative and fractional-integral operators in Section 2.1. The properties of derivative operators are studied in Section 2.2, while integral operators are considered in Section 2.3.

2.1. Construction of complex-order fractional operators

The Fourier and inverse Fourier transforms, $\mathcal{F}\{\cdot\}$ and $\mathcal{F}^{-1}\{\cdot\}$, respectively, are understood as

$$\hat{f}(\omega) = \mathcal{F}\{f\}(w) = \int_{\mathbb{R}} f(x)\mathrm{e}^{-\mathrm{i}\omega x}\mathrm{d}x, \tag{6}$$

$$f(x) = \mathcal{F}^{-1}\{\hat{f}\}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\omega)\mathrm{e}^{\mathrm{i}\omega x}\mathrm{d}\omega, \tag{7}$$

for any $x, \omega \in \mathbb{R}$, provided that f and/or $\hat{f}$ are integrable. The circumflex accent $(\hat{\cdot})$ always denotes the Fourier transform of a function with the symbol ω used for the frequency variable. It is well-known that the (inverse) Fourier transforms of Schwartz functions are themselves Schwartz functions. Further, all tempered generalized functions also have valid (inverse) Fourier transforms in the distributional sense [57]. The Fourier transforms of $f(Tx)$ and of $\frac{\mathrm{d}}{\mathrm{d}x}f(x)$ are given by $\frac{1}{|T|}\hat{f}(\frac{\omega}{T})$ and $\mathrm{i}\omega\hat{f}(\omega)$, respectively. Accordingly, all integer-order derivative operators D^k can be redefined as

$$\big(\mathrm{D}^k\varphi\big)(x) = \mathcal{F}^{-1}\big\{(\mathrm{i}\cdot)^k\hat{\varphi}(\cdot)\big\}(x). \tag{8}$$

To extend the framework to fractional exponents, we define

$$h^{\gamma}_{a,b}(\omega) = \begin{cases} a\,(\omega)^{\gamma}_{+} + b\,(\omega)^{\gamma}_{-}, & \omega \neq 0, \\ 0, & \omega = 0, \end{cases} \tag{9}$$

where $a, b \in \mathbb{C}\backslash\{0\}$ and $\gamma \in \mathbb{C}$. It is easy to check that $h^{\gamma}_{a,b}(T\omega) = T^{\gamma}h^{\gamma}_{a,b}(\omega)$ for all $T > 0$ and $\omega \in \mathbb{R}$. In other words, the dilation of $h^{\gamma}_{a,b}$ translates into an amplitude scaling. The generalized functions that satisfy this property are called homogeneous. In fact, except for singularities at $\omega = 0$ (Dirac's impulse and its derivatives), (9) describes the complete family of homogeneous generalized functions [4]. It follows that any well-behaved[2] linear and shift-invariant operator L (acting on $\mathcal{S}$) that is also scale-invariant corresponds to a Fourier multiplier of the form $h^{\gamma}_{a,b}$. By comparing these operators with (8), we observe that such scale-invariant operators can be interpreted as γ-order derivatives.

Definition 2.1 *(Complex-order derivatives).* The γ-order derivative operator $\mathrm{D}^{\gamma}_{a,b}$ with parameters $\gamma \in \mathbb{C}$ with $\mathrm{Re}(\gamma) > -1$ and $a, b \in \mathbb{C}\backslash\{0\}$ is defined as

$$\big(\mathrm{D}^{\gamma}_{a,b}\varphi\big)(x) = \mathcal{F}^{-1}\Big\{\hat{\varphi}(\cdot)h^{\gamma}_{a,b}(\cdot)\Big\}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\varphi}(\omega)h^{\gamma}_{a,b}(\omega)\mathrm{e}^{\mathrm{i}\omega x}\mathrm{d}\omega \tag{10}$$

for any $x \in \mathbb{R}$ and $\varphi \in \mathcal{S}$.

Definition 2.1 is valid because the function $h^{\gamma}_{a,b}$ specifies a tempered generalized function: it is integrable at the origin for $\mathrm{Re}(\gamma) > -1$ and asymptotically grows like a polynomial. It is therefore the frequency response; *i.e.* the Fourier transform of the impulse response of a convolution operator $\mathrm{D}^{\gamma}_{a,b} : \mathcal{S} \mapsto \mathcal{S}'$ [57].

Real fractional operators are covered by Definition 2.1 when restricting to $\gamma \in \mathbb{R}$. The fractional-Laplacian $(-\Delta)^{\gamma/2}$, whose Fourier multiplier is $\omega \mapsto |\omega|^{\gamma}$, corresponds to $a = b = 1$ in (9). The fractional-derivative D^{γ} is obtained with $a = \mathrm{i}^{\gamma}$ and $b = (-\mathrm{i})^{\gamma}$.

For $\mathrm{Re}(\gamma) > 0$, the Fourier multiplier $h^{\gamma}_{a,b}(\omega)$ associated with fractional-derivatives vanishes at $\omega = 0$. Hence, the operator might not be invertible (similar to the integer-order case). However, we can introduce integration operators as possible inverses of the fractional-derivatives as follows. We reserve $\mathbb{N}$ for the set of integers equal or larger than 1.

Definition 2.2 *(Complex-order integrators).* The γ-order integration operator $\mathrm{I}^{\gamma;k}_{a,b}$ with parameters $\gamma \in \mathbb{C}$ with $\mathrm{Re}(\gamma) > 0$, $a, b \in \mathbb{C}\backslash\{0\}$, and $k \in \mathbb{N}$ such that $k \geq \lfloor \mathrm{Re}(\gamma) \rfloor$ is defined as

$$\big(\mathrm{I}^{\gamma;k}_{a,b}\varphi\big)(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\varphi}(\omega)\frac{\mathrm{e}^{\mathrm{i}\omega x} - \sum_{j=0}^{k-1}\frac{(\mathrm{i}x)^j}{j!}\omega^j}{h^{\gamma}_{a,b}(\omega)}\mathrm{d}\omega, \tag{11}$$

for every $x \in \mathbb{R}$ and $\varphi \in \mathcal{S}$.

Again, one can verify that Definition 2.2 is valid since the function $\omega \mapsto \frac{\mathrm{e}^{\mathrm{i}\omega x} - \sum_{j=0}^{k-1}\frac{(\mathrm{i}x)^j}{j!}\omega^j}{h^{\gamma}_{a,b}(\omega)}$ is locally integrable (in particular around the origin, since $k \geq \lfloor \mathrm{Re}(\gamma) \rfloor$) and bounded by some polynomial. Hence $\big(\mathrm{I}^{\gamma;k}_{a,b}\varphi\big)$ specifies a tempered generalized function.

[2] As before, we are excluding point singularities such as $\delta(\omega)$ and its derivatives.

More precisely, we shall show that $\mathrm{I}_{a,b}^{\gamma;k}$ forms a proper right-inverse for $\mathrm{D}_{a,b}^{\gamma}$ if $k = \lfloor \mathrm{Re}(\gamma) \rfloor$ or $k = \lceil \mathrm{Re}(\gamma) \rceil$ (see Proposition 2.5). Moreover, the adjoint operators $\mathrm{I}_{a,b}^{(\gamma;k)*}$ for such integration operators are characterized by the relation $\langle \mathrm{I}_{a,b}^{(\gamma;k)*}\varphi, \psi\rangle = \langle \varphi, \mathrm{I}_{a,b}^{\gamma;k}\psi\rangle$ for any $\varphi, \psi \in \mathcal{S}$ and are given by

$$\big(\mathrm{I}_{a,b}^{(\gamma;k)*}\varphi\big)(x) = \mathcal{F}^{-1}\left\{\frac{\widehat{\varphi}(\cdot) - \sum_{j=0}^{k-1}\frac{\widehat{\varphi}^{(j)}(0)}{j!}(\cdot)^j}{h_{a,b}^{\gamma}(-\cdot)}\right\}(x), \tag{12}$$

as we shall see in Proposition 2.5. Next, we characterize the key properties of the fractional-derivative and integration operators $\mathrm{D}_{a,b}^{\gamma}$, $\mathrm{I}_{a,b}^{\gamma;k}$ and $\mathrm{I}_{a,b}^{(\gamma;k)*}$.

2.2. Properties of complex-order fractional-derivative operators

We characterize the smoothness and decay properties of complex-order fractional-derivatives $\mathrm{D}_{a,b}^{\gamma}\varphi$ for a given test function φ.

Theorem 2.3. *Let a, b, γ be arbitrary complex numbers with $\mathrm{Re}(\gamma) > -1$, and let $h_{a,b}^{\gamma}$ be the homogeneous generalized function of degree γ defined in (9) corresponding to the γ-order derivative operator $\mathrm{D}_{a,b}^{\gamma}$ (defined in (10)). Then, $\big(\mathrm{D}_{a,b}^{\gamma}\varphi\big)(x)$ is infinitely differentiable for $\varphi \in \mathcal{S}$. Further,*

(i) *if $\mathrm{Re}(\gamma) \notin \mathbb{N} \cup \{0\}$, then, $\mathrm{D}_{a,b}^{\gamma}$ is a continuous mapping from $\mathcal{S}$ into L_p for $p > \frac{1}{\mathrm{Re}(\gamma)+1}$, and there exists a constant $c > 0$ such that*

$$\forall \varphi \in \mathcal{S},\ x \in \mathbb{R}: \quad \left|\big(\mathrm{D}_{a,b}^{\gamma}\varphi\big)(x)\right| \le \frac{c}{1 + |x|^{\mathrm{Re}(\gamma)+1}}. \tag{13}$$

(ii) *if $\gamma \in \mathbb{N}$ and $h_{a,b}^{\gamma}(\omega) \equiv d\omega^{\gamma}$ for some $d \in \mathbb{C}$, then, $\mathrm{D}_{a,b}^{\gamma}$ is a continuous mapping from $\mathcal{S}$ into $\mathcal{S}$.*

(iii) *if $\mathrm{Re}(\gamma) \in \mathbb{N}$ but $h_{a,b}^{\gamma}(\omega) \not\equiv c\omega^{\gamma}$, then, $\mathrm{D}_{a,b}^{\gamma}$ is a continuous mapping from $\mathcal{S}$ into L_p for $p > \frac{1}{\mathrm{Re}(\gamma)+1}$, and there exists a constant $c > 0$ such that*

$$\forall \varphi \in \mathcal{S},\ x \in \mathbb{R}: \quad \left|\big(\mathrm{D}_{a,b}^{\gamma}\varphi\big)(x)\right| \le c\,\frac{1 + \log(1 + |x|)}{1 + |x|^{\mathrm{Re}(\gamma)+1}}. \tag{14}$$

Remark 2.4. Based on Lemma 4.1, the above decay estimates extend to larger classes of functions subject to the Sobolev-like constraint that $\widehat{\varphi}$ be $n_\gamma = \lceil \mathrm{Re}(\gamma) \rceil + 1$ times differentiable such that $\left|\widehat{\varphi}^{(j)}(\omega)\right|\big(1 + |\omega|^{j+r}\big)$ is bounded for all $0 \le j \le n_\gamma$ (r is a real strictly larger than $-\{-\mathrm{Re}(\gamma)\}$).

2.3. Properties of complex-order fractional-integration operators

We now study the class of integration operators $\mathrm{I}_{a,b}^{\gamma;k}$ introduced in Definition 2.2. We show that these operators are scale-invariant right-inverses of the complex-order fractional-derivative operators in Proposition 2.5. Special attention is given to the adjoint operators $\mathrm{I}_{a,b}^{(\gamma;k)*}$ and to the smoothness and decay properties of $\mathrm{I}_{a,b}^{(\gamma;k)*}\varphi$, for some test function φ.

Proposition 2.5. *Let $\gamma \in \mathbb{C}$ with $\mathrm{Re}(\gamma) > 0$, and a, b be arbitrary non-zero complex numbers. As in (11), define the integration operator $\mathrm{I}_{a,b}^{\gamma;k}$ of order γ associated with the homogeneous generalized function $h_{a,b}^{\gamma}$, where $k \ge \max\big(1, \lfloor \mathrm{Re}(\gamma) \rfloor\big)$. Then,*

(i) *$\mathrm{I}_{a,b}^{\gamma;k}$ defines a scale-invariant operator of degree $(-\gamma)$ over $\mathcal{S}$,*

(ii) *$\mathrm{I}_{a,b}^{\gamma;k}$ is a right-inverse of $\mathrm{D}_{a,b}^{\gamma}$ if $k = \lfloor \mathrm{Re}(\gamma) \rfloor$ or $k = \lceil \mathrm{Re}(\gamma) \rceil$,*

(iii) *the adjoint operator of $\mathrm{I}_{a,b}^{\gamma;k}$ denoted by $\mathrm{I}_{a,b}^{(\gamma;k)*}$ is given in (12).*

Remark 2.6. Since there are two admissible values of k in (ii) for $\mathrm{Re}(\gamma) \ge 1$ and $\mathrm{Re}(\gamma) \notin \mathbb{N}$, Proposition 2.5 introduces two right-inverse operators for each linear shift-invariant (LSI) derivative operator $\mathrm{D}_{a,b}^{\gamma}$. For $\mathrm{Re}(\gamma) \in \mathbb{N}$ the two integration operators coincide. The case $0 < \mathrm{Re}(\gamma) < 1$ is more elaborate. Let $h_{a,b}^{\gamma}(\omega)$ denote the homogeneous Fourier multiplier (frequency response) of the derivative operator $\mathrm{D}_{a,b}^{\gamma}$ with $0 < \mathrm{Re}(\gamma) < 1$. Proposition 2.5 introduces only the integration operator $\mathrm{I}_{a,b}^{\gamma;1}$ such that

$$\big(\mathrm{I}_{a,b}^{\gamma;1}\varphi\big)(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\varphi}(\omega)\frac{\mathrm{e}^{\mathrm{i}\omega x}-1}{h_{a,b}^{\gamma}(\omega)}\mathrm{d}\omega, \tag{15}$$

which is not shift-invariant (similar to all the other operators introduced in Proposition 2.5). We should, nevertheless, mention that Theorem 2.3 covers the range $\mathrm{Re}(\gamma) > -1$. Particularly, for $-1 < \mathrm{Re}(\gamma) < 0$, we obtain a shift-invariant integration operator. Therefore, the second right-inverse operator for $\mathrm{D}_{a,b}^{\gamma}$ when $0 < \mathrm{Re}(\gamma) < 1$ is the shift-invariant operator $\mathrm{D}_{a',b'}^{-\gamma}$:

$$\varphi(x) \overset{\mathrm{D}^{-\gamma}_{a',b'}}{\longmapsto} \big(\mathrm{D}^{-\gamma}_{a',b'}\,\varphi\big)(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\varphi}(\omega) h^{-\gamma}_{a',b'}(\omega) \mathrm{e}^{\mathrm{i}\omega x}\mathrm{d}\omega = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\widehat{\varphi}(\omega)\mathrm{e}^{\mathrm{i}\omega x}}{h^{\gamma}_{a,b}(\omega)}\mathrm{d}\omega, \tag{16}$$

where $a' = \frac{1}{a}$ and $b' = \frac{1}{b}$.

Theorem 2.7. *Let $\mathrm{I}^{(\gamma;k)*}_{a,b}$ be the scale-invariant operator associated with the homogeneous generalized function $h^{\gamma}_{a,b}(\omega)$ of degree $\gamma \in \mathbb{C}$ ($\mathrm{Re}(\gamma) > 0$) as defined in (12), such that $k \geq \max(1\,,\,\lfloor \mathrm{Re}(\gamma)\rfloor)$. Then, $\big(\mathrm{I}^{(\gamma;k)*}_{a,b}\,\varphi\big)(x)$ is a well-defined and continuous function at $x \neq 0$ for all $\varphi \in \mathcal{S}$. Moreover,*

(i) *if $\mathrm{Re}(\gamma) \notin \mathbb{N}$ and $k = \lfloor \mathrm{Re}(\gamma)\rfloor$, then, $\big(\mathrm{I}^{(\gamma;k)*}_{a,b}\,\varphi\big)(x)$ is bounded around $x = 0$ and the function belongs to L_p for $p > \frac{1}{k+1-\mathrm{Re}(\gamma)}$. More precisely, there exists a constant $c \in \mathbb{R}^+$ such that*

$$\forall\, x \in \mathbb{R},\ 1 \leq |x|:\quad \Big|\big(\mathrm{I}^{(\gamma;k)*}_{a,b}\,\varphi\big)(x)\Big| \leq \frac{c}{|x|^{k+1-\mathrm{Re}(\gamma)}}. \tag{17}$$

(ii) *if $\gamma \in \mathbb{N}$, $h^{\gamma}_{a,b}(\omega) \equiv d\,\omega^{\gamma}$ for some $d \in \mathbb{C}$ and $k = \lfloor \mathrm{Re}(\gamma)\rfloor$, then, $\mathrm{I}^{(\gamma;k)*}_{a,b}\,\varphi$ is bounded around $x = 0$ and belongs to L_p for all $p > 0$.*

(iii) *if $\mathrm{Re}(\gamma) \in \mathbb{N}$ but $h^{\gamma}_{a,b}(\omega) \not\equiv c\,\omega^{\gamma}$ and $k = \lfloor \mathrm{Re}(\gamma)\rfloor$, then, $\big(\mathrm{I}^{(\gamma;k)*}_{a,b}\,\varphi\big)(x)$ is singular at $x = 0$ but $\frac{|(\mathrm{I}^{(\gamma;k)*}_{a,b}\,\varphi)(x)|}{\log|x|}$ is bounded around $x = 0$. Further, $I^{(\gamma;k)*}_{a,b}\,\varphi$ belongs to L_p for $p > 1$, and there exists a constant $c \in \mathbb{R}^+$ such that*

$$\forall\, x \in \mathbb{R},\ 1 \leq |x|:\quad \Big|\big(\mathrm{I}^{(\gamma;k)*}_{a,b}\,\varphi\big)(x)\Big| \leq c\,\frac{1+\log|x|}{|x|}. \tag{18}$$

(iv) *if $\mathrm{Re}(\gamma) \notin \mathbb{N}$ and $k > \lfloor \mathrm{Re}(\gamma)\rfloor$, then, $|x|^{k-\mathrm{Re}(\gamma)}|(\mathrm{I}^{(\gamma;k)*}_{a,b}\,\varphi)(x)|$ is bounded around $x = 0$ (possible singularity of $\big(\mathrm{I}^{(\gamma;k)*}_{a,b}\,\varphi\big)(x)$ at $x = 0$). In addition, $\mathrm{I}^{(\gamma;k)*}_{a,b}\,\varphi$ belongs to L_p for $\frac{1}{k+1-\mathrm{Re}(\gamma)} < p < \frac{1}{k-\mathrm{Re}(\gamma)}$, and there exists a constant $c \in \mathbb{R}^+$ such that*

$$\forall\, x \in \mathbb{R},\ 1 \leq |x|:\quad \Big|\big(\mathrm{I}^{(\gamma;k)*}_{a,b}\,\varphi\big)(x)\Big| \leq \frac{c}{|x|^{k+1-\mathrm{Re}(\gamma)}}. \tag{19}$$

(v) *if $\mathrm{Re}(\gamma) \in \mathbb{N}$ and $k > \lfloor \mathrm{Re}(\gamma)\rfloor$, then, $|x|^{k-\mathrm{Re}(\gamma)}|(\mathrm{I}^{(\gamma;k)*}_{a,b}\,\varphi)(x)|$ is bounded around $x = 0$ (possible singularity of $\big(\mathrm{I}^{(\gamma;k)*}_{a,b}\,\varphi\big)(x)$ at $x = 0$). In addition, $\mathrm{I}^{(\gamma;k)*}_{a,b}\,\varphi$ belongs to L_p for $\frac{1}{k+1-\mathrm{Re}(\gamma)} < p < \frac{1}{k-\mathrm{Re}(\gamma)}$, and there exists a constant $c \in \mathbb{R}^+$ such that*

$$\forall\, x \in \mathbb{R},\ 1 \leq |x|:\quad \Big|\big(\mathrm{I}^{(\gamma;k)*}_{a,b}\,\varphi\big)(x)\Big| \leq c\,\frac{1+\log|x|}{|x|^{k+1-\mathrm{Re}(\gamma)}}. \tag{20}$$

Remark 2.8. Theorem 2.7 (in parts (i) and (iv)), especially shows how the integer parameter k controls the space L_p to which $\big(\mathrm{I}^{(\gamma;k)*}_{a,b}\,\varphi\big)(x)$ belongs. It therefore indicates that, for $\gamma \in \mathbb{C}$ with $\mathrm{Re}(\gamma) \geq 1$, it is possible to set k such that $\mathrm{I}^{(\gamma;k)*}_{a,b} : \mathcal{S} \to L_p$ for any given $p > 0$, except when $\frac{1}{p} + \mathrm{Re}(\gamma) \in \mathbb{Z}$. For $\mathrm{Re}(\gamma) < 1$ (even negative $\mathrm{Re}(\gamma)$), as $k \geq 1$ is required in Theorem 2.7, the p values above $\frac{1}{1-\mathrm{Re}(\gamma)}$ are excluded. In some sense, $k = 0$ is required to cover the (almost) full range of $p > 0$ values. Indeed, $k = 0$ can be interpreted as

$$\begin{aligned}\big(\mathrm{I}^{\gamma;0}_{a,b}\,\varphi\big)(x) &= \mathcal{F}^{-1}\left\{\frac{\widehat{\varphi}(\omega)}{h^{\gamma}_{a,b}(\omega)}\right\}(x),\\ \big(\mathrm{I}^{(\gamma;0)*}_{a,b}\,\varphi\big)(x) &= \mathcal{F}^{-1}\left\{\frac{\widehat{\varphi}(\omega)}{h^{\gamma}_{a,b}(-\omega)}\right\}(x).\end{aligned} \tag{21}$$

This suggests that $\mathrm{I}^{\gamma;0}_{a,b} = \mathrm{D}^{-\gamma}_{a',b'}$ and $\mathrm{I}^{(\gamma;0)*}_{a,b} = \mathrm{D}^{-\gamma}_{b',a'}$, where $\mathrm{Re}(-\gamma) > -1$, and $a' = \frac{1}{a}$, $b' = \frac{1}{b}$. Theorem 2.3 shows that $\mathrm{I}^{(\gamma;0)*}_{a,b}\,\varphi$ is in L_p for $p < \frac{1}{1-\mathrm{Re}(\gamma)}$. Thus, by extending the family of γ-order integration operators and adjoints to include $k = 0$ as above (for $\mathrm{Re}(\gamma) < 1$), we are always able to choose a k such that $\mathrm{I}^{(\gamma;k)*}_{a,b} : \mathcal{S} \to L_p$, except when $\frac{1}{p} + \mathrm{Re}(\gamma) \in \mathbb{Z}$.

2.4. Impulse responses of fractional-derivative and integration operators

To evaluate the output of complex-order operators, we start by simple inputs such as Dirac's delta function and its shifted versions. This then yields the Schwartz kernel of these operators.

Theorem 2.9. *Let $\mathrm{D}^{\gamma}_{a,b}$, $\mathrm{I}^{\gamma;k}_{a,b}$ and $\mathrm{I}^{(\gamma;k)*}_{a,b}$ be the scale-invariant operators as defined in (10), (11) and (12), respectively, where $k \geq \max(1\,,\,\lfloor \mathrm{Re}(\gamma)\rfloor)$ and $\gamma \notin \mathbb{Z}$. Then, the response of $\mathrm{D}^{\gamma}_{a,b}$, $\mathrm{I}^{\gamma;k}_{a,b}$ and $\mathrm{I}^{(\gamma;k)*}_{a,b}$ to the shifted Dirac impulse $\delta(\cdot - \tau)$ is given by*

$$
\begin{aligned}
\mathrm{D}_{a,b}^{\gamma}\{\delta(\cdot-\tau)\}(x) &= \frac{\Gamma(\gamma+1)}{2\pi}\left(\frac{a}{(\mathrm{i}\tau-\mathrm{i}x)^{\gamma+1}}+\frac{b}{(\mathrm{i}x-\mathrm{i}\tau)^{\gamma+1}}\right),\\
\mathrm{I}_{a,b}^{\gamma;k}\{\delta(\cdot-\tau)\}(x) &= \frac{\Gamma(1-\gamma)}{2\pi}\left(\frac{(\mathrm{i}\tau-\mathrm{i}x)^{\gamma-1}}{a}+\frac{(\mathrm{i}x-\mathrm{i}\tau)^{\gamma-1}}{b}\right)\\
&\quad -\frac{\Gamma(1-\gamma)}{2\pi}\left(\frac{(\mathrm{i}\tau)^{\gamma-1}}{a}+\frac{(-\mathrm{i}\tau)^{\gamma-1}}{b}\right)\sum_{j=0}^{k-1}\binom{j-\gamma}{j}\left(\tfrac{x}{\tau}\right)^j,\\
\mathrm{I}_{a,b}^{(\gamma;k)*}\{\delta(\cdot-\tau)\}(x) &= \frac{\Gamma(1-\gamma)}{2\pi}\left(\frac{(\mathrm{i}x-\mathrm{i}\tau)^{\gamma-1}}{a}+\frac{(\mathrm{i}\tau-\mathrm{i}x)^{\gamma-1}}{b}\right)\\
&\quad -\frac{\Gamma(1-\gamma)}{2\pi}\left(\frac{(\mathrm{i}x)^{\gamma-1}}{a}+\frac{(-\mathrm{i}x)^{\gamma-1}}{b}\right)\sum_{j=0}^{k-1}\binom{j-\gamma}{j}\left(\tfrac{\tau}{x}\right)^j,
\end{aligned}
\tag{22}
$$

where $\binom{x}{0}=1$ *and* $\binom{x}{j}=\frac{\Gamma(x+1)}{j!\Gamma(x+1-j)}=\frac{x(x-1)\dots(x-j+1)}{j!}$ *extends the standard definition of* $\binom{n}{j}$.

Remark 2.10. Theorem 2.9 enables us to represent the operators in terms of integrals. Indeed, for $\varphi\in\mathcal{S}$, we know that

$$\varphi(x)=\int_{\mathbb{R}}\varphi(\tau)\,\delta(x-\tau)\mathrm{d}\tau.$$

According to Schwartz' kernel theorem [58,59], the action of $\mathrm{L}:\mathcal{S}\to\mathcal{S}'$ can be represented as

$$\big(\mathrm{L}\varphi\big)(x)=\int_{\mathbb{R}}\varphi(\tau)\,\big(\mathrm{L}\delta(\cdot-\tau)\big)(x)\mathrm{d}\tau,$$

where L stands for any of $\mathrm{D}_{a,b}^{\gamma}$, $\mathrm{I}_{a,b}^{\gamma;k}$ and $\mathrm{I}_{a,b}^{(\gamma;k)*}$. The integral forms are particularly useful for numerical evaluation of $\big(\mathrm{L}\varphi\big)(x)$.

Remark 2.11. With the particular choice of $a=\mathrm{i}^{\gamma}$ and $b=(-\mathrm{i})^{\gamma}$ for $\gamma\notin\mathbb{Z}$, the kernels in Theorem 2.9 simplify to more familiar forms of

$$
\begin{aligned}
\big(\mathrm{D}_{a,b}^{\gamma}\delta(\cdot-\tau)\big)(x) &= -\frac{\Gamma(1+\gamma)\sin(\pi\gamma)}{\pi}(x-\tau)_+^{-1-\gamma},\\
\big(\mathrm{I}_{a,b}^{\gamma;k}\delta(\cdot-\tau)\big)(x) &= \frac{\Gamma(1-\gamma)\sin(\pi\gamma)}{\pi}\left((x-\tau)_+^{\gamma-1}-(-\tau)_+^{\gamma-1}\sum_{j=0}^{k-1}\binom{j-\gamma}{j}\left(\tfrac{x}{\tau}\right)^j\right),\\
\big(\mathrm{I}_{a,b}^{(\gamma;k)*}\delta(\cdot-\tau)\big)(x) &= \frac{\Gamma(1-\gamma)\sin(\pi\gamma)}{\pi}\left((\tau-x)_+^{\gamma-1}-(-x)_+^{\gamma-1}\sum_{j=0}^{k-1}\binom{j-\gamma}{j}\left(\tfrac{\tau}{x}\right)^j\right).
\end{aligned}
\tag{23}
$$

3. Self-similar stable processes with complex-valued Hurst exponent

As seen in Section 1.2, self-similar random processes are more traditionally specified as *classical random processes*, which are random processes $S=\big(S(x)\big)_{x\in\mathbb{R}}$ whose sample paths are classical functions. In this section, we define a class of self-similar stable processes with complex-valued Hurst exponent. We construct and study this class in the framework of *generalized random processes* introduced by Gel'fand and Vilenkin [7], which involves random elements in the Schwartz space $\mathcal{S}'$ of tempered generalized functions [49]. The construction of generalized random processes relies on their characteristic functional, which is the infinite-dimensional generalization of the characteristic function of random vectors. Meanwhile, our construction of self-similar stable processes relies on the Bochner-Minlos theorem and benefits from the study of complex-order operators.

Generalized random processes and characteristic functional are introduced in Section 3.1. We then use the integration operators defined in Section 2 to construct our family of self-similar stable processes in Section 3.2. The invariance and regularity properties of self-similar stable processes are studied in Sections 3.3 and 3.4 respectively. Finally, we provide simulations of realizations of the considered class of random processes in Section 3.5.

3.1. Generalized random processes and their characteristic functional

The theory of generalized random processes has been formalized by Gelfand and Itô [7,49]. It is the probabilistic counterpart of the theory of generalized functions of Schwartz [57] and allows for the construction of broad classes of random processes, including the ones that do not admit point-wise representations such as the stable white noises [7, Chapter 3]. We briefly recap this formalism, with special emphasis on the characteristic functional.

A tempered generalized random process S (or simply a generalized random process) is a collection of random variables $\langle S,\varphi\rangle$, where φ is a test function in the Schwartz space $\mathcal{S}$, such that

- *Linearity:* for any $\varphi_1,\varphi_2\in\mathcal{S}$ and $\lambda\in\mathbb{R}$, $\langle S,\varphi_1+\lambda\varphi_2\rangle=\langle S,\varphi_1\rangle+\lambda\langle S,\varphi_2\rangle$ almost surely;

- *Continuity:* for any converging sequence $\varphi_n \to \varphi$ in $\mathcal{S}$, the random variables $\langle S, \varphi_n \rangle$ converge in probability to $\langle S, \varphi \rangle$.

The space $\mathcal{S}'$ of tempered generalized functions is endowed with the weak* topology [59]. This topology defines a Borel σ-field on $\mathcal{S}'$ and a tempered generalized random process S can then be seen as a random element of the space of tempered generalized functions. The probability law of a generalized random process S is the probability measure $\mathscr{P}_S$ over $\mathcal{S}'$ such that, for any Borel set $B \subset \mathcal{S}'$, we have $\mathscr{P}_S(B) = \mathscr{P}(S \in B)$. We refer the interested reader to [60] for a comprehensive introduction to these notions in the framework of tempered generalized functions and for additional references.

We are interested in complex-valued random processes; hence, we shall adapt the usual concepts to this case. Thereafter, $\mathcal{S}$ and $\mathcal{S}'$ denote the complex-valued Schwartz space and space of tempered generalized functions, respectively. The characteristic function of a complex random variable Z is given by $\widehat{\mathscr{P}}_Z(\xi) = \mathbb{E}[\mathrm{e}^{\mathrm{i}\,\mathrm{Re}(Z\bar{\xi})}]$ where $\bar{\xi}$ is the complex conjugate of $\xi \in \mathbb{C}$. The extension to complex-valued generalized random processes is as follows.

Definition 3.1 *(Characteristic functional).* The characteristic functional of a complex valued tempered generalized random process S is defined as

$$\widehat{\mathscr{P}}_S(\varphi) = \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\,\mathrm{Re}(\langle S,\bar{\varphi}\rangle)}\right] = \int_{\mathcal{S}'} \mathrm{e}^{\mathrm{i}\,\mathrm{Re}(\langle u,\bar{\varphi}\rangle)} \mathrm{d}\mathscr{P}_S(u) \tag{24}$$

for any $\varphi \in \mathcal{S}$.

The characteristic functional has been introduced by Kolmogorov [61] and popularized by Gelfand and Vilenkin [7]. The probability law of a generalized random process is characterized by its characteristic functional, which is its (infinite-dimensional) Fourier transform. In particular, two random processes with identical characteristic functionals have identical finite-dimensional marginals.

Theorem 3.2 *(Bochner-Minlos theorem [7]). The characteristic functional $\widehat{\mathscr{P}}_S$ of a complex-valued generalized random process S is continuous and positive-definite over $\mathcal{S}$ and satisfies $\widehat{\mathscr{P}}_S(0) = 1$. Conversely, any continuous and positive-definite functional $\widehat{\mathscr{P}}$ over $\mathcal{S}$ with $\widehat{\mathscr{P}}(0) = 1$ is the characteristic functional $\widehat{\mathscr{P}} = \widehat{\mathscr{P}}_S$ of some generalized random process S.*

The Bochner-Minlos theorem provides a way for one to construct tempered generalized processes via the specification of their characteristic functionals. For instance, for any $\alpha \in]0,2]$, it is known that the functional

$$\widehat{\mathscr{P}}(\varphi) = \exp\left(-\int_{\mathbb{R}} |\varphi(x)|^{\alpha} \mathrm{d}x\right) = \exp\left(-\|\varphi\|_{\alpha}^{\alpha}\right) \tag{25}$$

is continuous and positive-definite over *real-valued* Schwartz functions and satisfies $\widehat{\mathscr{P}}(0) = 1$ [62]; it is therefore the characteristic functional of the generalized random process described in Definition 3.3.

Definition 3.3. The generalized random process W_α such that $\widehat{\mathscr{P}}_{W_\alpha}(\varphi) = \exp\left(-\|\varphi\|_{\alpha}^{\alpha}\right)$ is called a *symmetric-α-stable (SαS) white noise*.

To extend the domain of $\widehat{\mathscr{P}}_{W_\alpha}$ to complex-valued test functions $\varphi \in \mathcal{S}$, according to (24), we have that

$$\widehat{\mathscr{P}}_{W_\alpha}(\varphi) = \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\,\mathrm{Re}(\langle W_\alpha,\bar{\varphi}\rangle)}\right] = \exp(-\|\mathrm{Re}(\varphi)\|_{\alpha}^{\alpha}), \tag{26}$$

which essentially defines a complex-valued generalized random process acting on complex-valued test functions. The family of stable white noises W_α will play an important role in this paper.

3.2. Construction of complex-valued fractional stable processes

We fix $\alpha \in]0,2]$ and specify fractional α-stable processes as generalized random processes via their characteristic functional (see Definition 3.1). The specification takes advantage of the fractional-integral operators studied in Section 2.1.

Construction principle Our construction relies on the Bochner-Minlos theorem (Theorem 3.2) and the specification of adequate characteristic functionals. More precisely, we define the self-similar random processes studied in this paper as filtered versions of the SαS white noise described in Definition 3.3.

Assume that U is a linear and continuous operator from $\mathcal{S}$ to L_α, where both spaces are assumed to contain complex-valued (generalized) functions.[3] The properties of both W_α and U ensure that $\varphi \mapsto \exp\left(-\|\mathrm{Re}(\mathrm{U}\{\varphi\})\|_{\alpha}^{\alpha}\right)$ is continuous, positive-definite over $\mathcal{S}$ and normalized. Hence, due to the Bochner-Minlos theorem, there does exist a generalized random process S with characteristic

[3] The space $L_\alpha = L_\alpha(\mathbb{R}, \mathbb{C})$ is the space of measurable functions $f : \mathbb{R} \to \mathbb{C}$ whose real and imaginary parts are in $L_\alpha(\mathbb{R}, \mathbb{R})$.

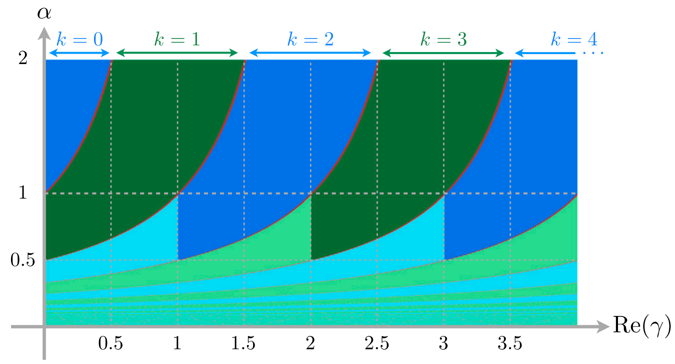

Fig. 1. The fractional stable process $S_{a,b}^{\gamma,\alpha}$ is well-defined by applying the operator $\mathrm{I}_{a,b}^{\gamma,k}$ to the symmetric-α-stable white noise with $k = \lfloor \frac{1}{\alpha} + \mathrm{Re}(\gamma) \rfloor$, whenever $\frac{1}{\alpha} + \mathrm{Re}(\gamma) \notin \mathbb{N}$ (the red curves separating the blue and green areas). For $\alpha > \frac{1}{2-\{\mathrm{Re}(\gamma)\}}$ (dark blue and dark green areas), the fractional process can be whitened by applying the fractional-derivative operator $\mathrm{D}_{a,b}^{\gamma}$. The use of green and blue colors is to better highlight the red curves separating them, and do not encode any other information. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

functional $\widehat{\mathscr{P}}_S(\varphi) = \widehat{\mathscr{P}}_{W_\alpha}(\mathrm{U}\{\varphi\}) = \exp\left(-\|\mathrm{Re}(\mathrm{U}\{\varphi\})\|_\alpha^\alpha\right)$. We use this principle to define our extended class of generalized random processes. Our construction is an alternative to the more traditional one that relies on fractional calculus and stochastic integrals, such as [14, Chapter 6].

Proposition 3.4. *Let $a, b \in \mathbb{C}\backslash\{0\}$, $\gamma \in \mathbb{C}$ with $\mathrm{Re}(\gamma) > 0$, and $\alpha \in]0,2]$. We further assume that*

$$\frac{1}{\alpha} + \mathrm{Re}(\gamma) \notin \mathbb{N} \tag{27}$$

and we set

$$k(\alpha,\gamma) = \left\lfloor \frac{1}{\alpha} + \mathrm{Re}(\gamma) \right\rfloor \in \mathbb{N}. \tag{28}$$

Then, the linear operator $\mathrm{I}_{a,b}^{(\gamma;k(\alpha,\gamma))*}$ is continuous from $\mathcal{S}$ to L_α and there exists a tempered generalized random process $S_{a,b}^{\gamma,\alpha}$ such that

$$\widehat{\mathscr{P}}_{S_{a,b}^{\gamma,\alpha}}(\varphi) = \exp\left(-\left\|\mathrm{Re}\left(\mathrm{I}_{a,b}^{(\gamma;k(\alpha,\gamma))*}\{\varphi\}\right)\right\|_\alpha^\alpha\right). \tag{29}$$

If moreover the condition

$$\alpha > \begin{cases} \frac{1}{2-\{\mathrm{Re}(\gamma)\}} & \text{if } \mathrm{Re}(\gamma) \notin \mathbb{N}, \\ 1 & \text{if } \mathrm{Re}(\gamma) \in \mathbb{N}, \end{cases} \tag{30}$$

is satisfied, then, the random process $S_{a,b}^{\gamma,\alpha}$ can be whitened in the sense that $\mathrm{D}_{a,b}^{\gamma} S_{a,b}^{\gamma,\alpha} = W_\alpha$ is a SαS white noise.

Proof. The continuity of $\mathrm{I}_{a,b}^{(\gamma;k(\alpha,\gamma))*}$ for $k = k(\alpha,\gamma) \geq 1$ is a direct consequence of Theorem 2.7, for which condition (27) is required. For the case of $k(\alpha,\gamma) = 0$, which necessitates $\mathrm{Re}(\gamma) < 1$, we recall that $\mathrm{I}_{a,b}^{(\gamma;0)*}$ is a true (left- and right-) inverse for $\mathrm{D}_{a,b}^{\gamma}$. This case is not covered by Theorem 2.7 but is discussed in Remark 2.8. Finally, the existence result simply follows from the application of the Bochner-Minlos theorem to $\varphi \mapsto \exp\left(-\|\mathrm{Re}(\mathrm{I}_{a,b}^{(\gamma;k(\alpha,\gamma))*}\varphi)\|_\alpha^\alpha\right)$.

For the claim that $S_{a,b}^{\gamma,\alpha}$ can be whitened using $\mathrm{D}_{a,b}^{\gamma}$, it is sufficient that $(\mathrm{I}_{a,b}^{\gamma;k})$ is a right-inverse of $\mathrm{D}_{a,b}^{\gamma}$. By Proposition 2.5-(ii), this property holds if $k = \lfloor \mathrm{Re}(\gamma) \rfloor$ or $k = \lceil \mathrm{Re}(\gamma) \rceil$. Since we have $k = \lfloor \frac{1}{\alpha} + \mathrm{Re}(\gamma) \rfloor$, this implies that

$$\lfloor \tfrac{1}{\alpha} + \mathrm{Re}(\gamma) \rfloor \leq \lceil \mathrm{Re}(\gamma) \rceil. \tag{31}$$

It is then not difficult to see that (31) is equivalent to (30). □

Definition 3.5. Let $\alpha \in]0,2]$, $\gamma \in \mathbb{C}$ such that $\mathrm{Re}(\gamma) > 0$ and $a, b \in C\backslash\{0\}$. We call the generalized random process $S_{a,b}^{\gamma,\alpha}$ of Proposition 3.4 a *(complex-valued) fractional stable process*.

Fig. 1 delineates the type of parameter pairs of (α,γ) for which the fractional stable random processes $S_{a,b}^{\gamma,\alpha}$ is well-defined and can be whitened. Note that $\mathrm{D}_{a,b}^{\gamma} S_{a,b}^{\gamma,\alpha}$ is not a white process when (30) is not satisfied. However, fractional random processes can always be whitened for $\alpha > 1$.

Definition 3.5 specifies self-similar stable processes as generalized random processes. We illustrate that our construction is equivalent to the classical ones, which rely on stochastic integrals for fractional Brownian motions. We recall from Remark 2.11 that, for $H \in]0,1[\setminus\{\frac{1}{2}\}$, $a = \mathrm{i}^{H+\frac{1}{2}}$ and $b = (-\mathrm{i})^{H+\frac{1}{2}}$, one has that

$$\big(\mathrm{I}_{a,b}^{H+\frac{1}{2};1}\delta(\cdot-\tau)\big)(x) = \frac{\Gamma(\frac{1}{2}-H)\cos(\pi H)}{\pi}\Big((x-\tau)_+^{H-\frac{1}{2}} - (-\tau)_+^{H-\frac{1}{2}}\Big). \tag{32}$$

Thus, if we apply the operator $\mathrm{I}_{a,b}^{H+\frac{1}{2};1}$ to the white Gaussian noise, we obtain

$$\frac{\Gamma(\frac{1}{2}-H)\cos(\pi H)}{\pi}\int_{\mathbb{R}}\Big((t-\tau)_+^{H-\frac{1}{2}} - (-\tau)_+^{H-\frac{1}{2}}\Big)\,\mathrm{d}B(\tau), \tag{33}$$

which is essentially the conventional fBm. As a result, the introduced process $S_{a,b}^{\gamma;\alpha}$ with $\alpha = 2$, $\gamma = H + \frac{1}{2}$ (real-valued H) and proper values of a, b (mentioned above), simplifies to the ordinary fBm. Although we excluded $H = \frac{1}{2}$ from (32) and (33), the case of $H = \frac{1}{2}$ (equivalently, $\gamma = 1$) accompanied with the above choice of a, b, is included in Proposition 3.4, and the resulting $S_{\mathrm{i},-\mathrm{i}}^{1,2}$ process corresponds to the standard Brownian motion. It is not difficult to show that the β-fractional α-stable processes X_α^β in [54] (that satisfy $\alpha \in]1,2[$ and $0 \le \beta < 1 - \frac{1}{\alpha}$) are the same as $S_{a,b}^{\beta,\alpha}$ with

$$a = -\frac{\Gamma(\beta)\Gamma(1-\beta)}{\pi}\sin(\pi\beta)\mathrm{e}^{\mathrm{i}\frac{\pi\beta}{2}},$$
$$b = \frac{\Gamma(\beta)\Gamma(1-\beta)}{\pi}\sin(\pi\beta)\mathrm{e}^{-\mathrm{i}\frac{\pi\beta}{2}}.$$

Remark 3.6. It is worth noting that the considered class of fractional stable processes are built over the family of *symmetric* alpha-stable SαS white noise. Consequently, they are symmetric as well ($S_{a,b}^{\gamma,\alpha}$ and $-S_{a,b}^{\gamma,\alpha}$ have the same law). It is possible to extend the family by considering non-symmetric white noises, which are fully described by four parameters [6, Definition 1.1.6]. This extension is readily achievable for $\alpha \neq 1$, but is more technical for specific cases corresponding to $\alpha = 1$, for which the characteristic exponent Ψ can include a logarithmic term. Due to the technicalities, we do not delve into the details in this paper.

3.3. *Invariance properties of complex-valued fractional stable processes*

We first adapt the notion of self-similarity to complex-valued generalized random processes.

Definition 3.7 *(Self-similar generalized random process).* A complex-valued tempered generalized random process S is self-similar with Hurst exponent $H \in \mathbb{C}$ if the probability laws of S and $T^{-H}S(T\cdot)$ are identical for any $T > 0$.

Definition 3.7 is the adaptation of the conventional self-similarity defined in (2) for the case of generalized random processes. This adaptation coincides with the conventional definition for classical random processes that have a non-negative Hurst exponent $H \ge 0$ (the case $H = 0$ corresponds to the null random process $S = 0$) [63, Theorem 1]. It is possible to construct *generalized* self-similar random processes with negative Hurst exponent. For instance, the SαS white noise W_α is self-similar and real-valued with Hurst exponent $H = (\frac{1}{\alpha} - 1) \in [-1/2, \infty)$ [39, Proposition 4.2]. Self-similar random processes with negative Hurst exponent are called *singular* since they do not admit a classical interpretation.

Proposition 3.8. *Under the conditions of Proposition 3.4, the fractional stable process $S_{a,b}^{\gamma,\alpha}$ is self-similar with Hurst exponent*

$$H = \gamma + \frac{1}{\alpha} - 1 \in \mathbb{C}. \tag{34}$$

Proof. To simplify the notations, we set $S = S_{a,b}^{\gamma,\alpha}$ and $\mathrm{I} = \mathrm{I}_{a,b}^{\gamma;k(\gamma,\alpha)}$. The self-similarity can be proved using the characteristic functional and the fact that $\widehat{\mathscr{P}}_{S_1} = \widehat{\mathscr{P}}_{S_2}$ if and only if S_1 and S_2 have the same probability law. Indeed, for $T > 0$, we have that

$$\begin{aligned}\widehat{\mathscr{P}}_{S(T\cdot)}(\varphi) &= \widehat{\mathscr{P}}_S\big(T^{-1}\varphi(\cdot/T)\big) = \mathrm{e}^{-\|T^{-1}\,\mathrm{Re}(\mathrm{I}^*\{\varphi(\cdot/T)\})\|_\alpha^\alpha}\\ &= \mathrm{e}^{-\|T^{-1-\gamma}\,\mathrm{Re}(\mathrm{I}^*\{\varphi\}(\cdot/T))\|_\alpha^\alpha} = \mathrm{e}^{-\|T^{-1-\gamma+\frac{1}{\alpha}}\,\mathrm{Re}(\mathrm{I}^*\{\varphi\})\|_\alpha^\alpha}\\ &= \widehat{\mathscr{P}}_S(T^{-1-\gamma+\frac{1}{\alpha}}\varphi) = \widehat{\mathscr{P}}_{T^{-1-\gamma+\frac{1}{\alpha}}S}(\varphi),\end{aligned} \tag{35}$$

where we used in particular the $(-\gamma)$-homogeneity of I^* for the third equality. Hence, $T^{-H}S(T\cdot) = S$ for any $T > 0$, where H is given by (34). □

Remark 3.9. It is known that nonzero classical self-similar processes have positive Hurst exponent $H > 0$ [63, Theorem 1] (the condition becomes $\mathrm{Re}(H) > 0$ for complex-valued Hurst exponents). We therefore deduce that the fractional stable process $S_{a,b}^{\gamma,\alpha}$ is singular (no classical interpretation) as long as $\mathrm{Re}(H) \leq 0$. This leads to

$$0 < \mathrm{Re}(\gamma) \leq 1 - \frac{1}{\alpha}, \tag{36}$$

which is only possible for $\alpha > 1$. This corresponds to $(\mathrm{Re}(\gamma), \alpha)$ being presented in the top-left region of Fig. 1 (blue part with $k = 0$).

We now study the invariance of fractional stable processes with respect to shift operations. Again, dealing *a priori* with processes with no point-wise interpretation, we shall adapt the usual notions to generalized random processes. For $h_0 > 0$, we define the operator $\Delta_{h_0}\{f\} = \big(f(\cdot + h_0) - f\big)$, which represents the increments of a generalized function $f \in \mathcal{S}'$. The invariance properties can all be expressed in the domain of the characteristic functional [52], which is the technique we used in our proofs.

Definition 3.10. A generalized random process S is stationary if $S(\cdot - x_0)$ and S have the same law for any $x_0 \in \mathbb{R}$. We say moreover that S has stationary increments of order $k \geq 1$ if the generalized random processes $(\Delta_{h_0})^k S$ are stationary for any $h_0 > 0$, where $(\Delta_{h_0})^k$ stands for the k-fold composition of Δ_{h_0} with itself.

Remark 3.11. Classically, we say that a random process $S = (S(x))_{x\in\mathbb{R}}$ with well-defined sampled values has stationary increments if the law of $s(x_1) - s(x_0)$ only depends on the difference $(x_1 - x_0)$ for any $x_0 < x_1$. Definition 3.10 covers and generalizes this notion. Indeed, assume that s has stationary increments of order $k = 1$ in the sense of Definition 3.10. Then, for any test function φ, $h_0 > 0$ and $t \in \mathbb{R}$, we have that $\langle \Delta_{h_0} S, \varphi\rangle$ and $\langle \Delta_{h_0} S, \varphi(\cdot - t)\rangle$ are equal in law. Picking $\varphi = \delta(\cdot - t_0)$ (which is possible for point-wise random processes), $h_0 = t_1 - t_0$, and $t = -t_0$, we deduce that

$$s(t_1) - s(t_0) \overset{(\mathcal{L})}{=} s(t_1 - t_0) - s(0). \tag{37}$$

The latter only depends on $t_1 - t_0$ and S has stationary increments in the classical sense.

Proposition 3.12. *Under the conditions of Proposition 3.4, the fractional stable process $S_{a,b}^{\gamma,\alpha}$*

- *is stationary if* $0 < \left(\mathrm{Re}(\gamma) + \frac{1}{\alpha}\right) < 1$*; and*
- *has stationary increments of order* $\lfloor \mathrm{Re}(\gamma) + \frac{1}{\alpha} \rfloor$ *if* $\left(\mathrm{Re}(\gamma) + \frac{1}{\alpha}\right) \geq 1$.

Proof. We set $S = S_{a,b}^{\gamma,\alpha}$ and $\mathrm{I} = \mathrm{I}_{a,b}^{\gamma;k(\gamma,\alpha)}$. The proof relies on the characteristic functional. For $\mathrm{Re}(\gamma) + \frac{1}{\alpha} \in]0,1[$, we have that $k(\gamma,\alpha) = 0$ (see (28)) and the operator $\mathrm{I}^* = \mathrm{I}_{a,b}^{(\gamma;0)*} = ((\mathrm{D}_{a,b}^{\gamma})^{-1})^*$ is a convolution; hence

$$\begin{aligned} \widehat{\mathscr{P}}_{S(\cdot - x_0)}(\varphi) &= \widehat{\mathscr{P}}_S(\varphi(\cdot + x_0)) = \exp(-\|\mathrm{Re}(\mathrm{I}^*\{\varphi(\cdot + x_0)\})\|_\alpha^\alpha) \\ &= \exp(-\|\mathrm{Re}(\mathrm{I}^*\{\varphi\}(\cdot + x_0))\|_\alpha^\alpha) = \exp(-\|\mathrm{Re}(\mathrm{I}^*\{\varphi\})\|_\alpha^\alpha) \\ &= \widehat{\mathscr{P}}_S(\varphi), \end{aligned} \tag{38}$$

where we used the shift-invariance of I^* in the third equality and a simple change of variables in the fourth one. Hence, S is stationary.

Assume that $\mathrm{Re}(\gamma) + \frac{1}{\alpha} \geq 1$. We treat the case $\mathrm{Re}(\gamma) + \frac{1}{\alpha} \in [1, 2[$. First, using (12) with $\Delta_{-h_0}\varphi$, we remark that $\widehat{\Delta_{-h_0}\varphi}(0) = \widehat{\varphi(\cdot - h_0)}(0) - \widehat{\varphi}(0) = 0$ and therefore

$$\mathrm{I}^*(\Delta_{-h_0}\varphi) = \mathcal{F}^{-1}\left(\frac{\widehat{\Delta_{-h_0}\varphi} - \widehat{\Delta_{-h_0}\varphi}(0)}{h_{a,b}^{\gamma}}\right) = \mathcal{F}^{-1}\left(\frac{\widehat{\Delta_{-h_0}\varphi}}{h_{a,b}^{\gamma}}\right) = (\mathrm{D}_{a,b}^{\gamma,*})^{-1}(\Delta_{-h_0}\varphi). \tag{39}$$

In particular, when restricted to functions of the form $\Delta_{-h_0}\varphi$, I^* is a convolution. As we did in (38), we prove that $\widehat{\mathscr{P}}_{(\Delta_{h_0}S)(\cdot - x_0)}(\varphi) = \widehat{\mathscr{P}}_{\Delta_{h_0}S}(\varphi)$ for any $\varphi \in \mathcal{S}$. This shows that $\Delta_{h_0} S$ is stationary, or equivalently, S has stationary increments of order $1 = \lfloor \gamma \rfloor$.

For $\gamma \geq 2$, we set $k = \lfloor \gamma \rfloor$. Then, the function $\psi = (\Delta_{-h_0})^k \varphi$ is such that $\widehat{\psi}^{(j)}(0) = 0$ for any $0 \leq j \leq k-1$. Hence, $\mathrm{I}^*\psi = (\mathrm{D}_{a,b}^{\gamma*})^{-1}\psi$ and the same argument as for $\gamma \in [1, 2[$ applies. □

Remark 3.13. Proposition 3.12 reveals that the random process $S_{a,b}^{\gamma,\alpha}$ is both stationary and self-similar for $\alpha > 1$ and $\frac{1}{\alpha} - 1 < H = \mathrm{Re}(\gamma) + \frac{1}{\alpha} - 1 < 0$ This is not possible for nonzero classical self-similar processes $S = \big(S(x)\big)_{x\in\mathbb{R}}$, for which $S(0) = 0$ [63]. We obtain a second proof that these random processes are singular (see Remark 3.11).

3.4. Regularity of fractional stable processes

We characterize the smoothness of fractional stable processes in terms of local Sobolev regularity in the space $W^{\tau}_{p,\mathrm{loc}}$ with $\tau \in \mathbb{R}$ and $1 \le p \le \infty$. Informally, for $\tau \in \mathbb{N}$, $W^{\tau}_{p,\mathrm{loc}}$ consists of functions, the derivatives of which up to order τ (including the function itself), are locally in L_p.

Definition 3.14 *(Fractional Sobolev spaces).* Let $\tau \in \mathbb{R}$ and $1 \le p \le \infty$. We say that $f \in W^{\tau}_{p}$ if

$$\mathcal{F}^{-1}\left\{(1+|\cdot|^2)^{\tau/2}\hat{f}\right\} \in L_p. \tag{40}$$

Moreover, we say $f \in W^{\tau}_{p,\mathrm{loc}}$ if, for any compactly supported smooth function φ, $f\varphi \in W^{\tau}_{p}$.

By setting $p = 2$, we recover the L_2-Sobolev regularity, while $p = \infty$ corresponds to the Hölder regularity [64]. Fractional Sobolev spaces are Banach spaces; for any fixed $p \ge 1$ and $\tau_1 \le \tau_2$, we have the continuous embedding $W^{\tau_2}_{p,\mathrm{loc}} \subseteq W^{\tau_1}_{p,\mathrm{loc}}$.

Following [65], we characterize the regularity properties of a generalized (random) function $f \in S'$ via its critical smoothness function, defined for $1 \le p \le \infty$ by

$$\tau_f(p) = \sup\{\tau \in \mathbb{R},\ f \in W^{\tau}_{p,\mathrm{loc}}\}. \tag{41}$$

The critical smoothness precisely tells us in which Sobolev spaces a given function lies; more precisely, $f \in W^{\tau}_{p,\mathrm{loc}}$ for any $\tau < \tau_f(p)$ and $f \notin W^{\tau}_{p,\mathrm{loc}}$ for $\tau > \tau_f(p)$. For instance, it is well-known that the Brownian motion is $(1/2-\epsilon)$-Hölder continuous for any $\epsilon > 0$, but it is not $1/2$-Hölder continuous almost surely [66, Corollary 1.20]. The Hölder regularity corresponds to fractional Sobolev spaces with $p = \infty$. Hence, we have that

$$\tau_B(\infty) = 1/2. \tag{42}$$

Similar results are known for various classes of self-similar random processes. We characterize the fractional Sobolev regularity of self-similar stable processes by expressing the critical smoothness in Theorem 3.15.

Theorem 3.15. *Under the conditions of Proposition 3.4, the fractional stable process $S^{\gamma,\alpha}_{a,b}$ has the following properties.*

- *If $\alpha = 2$ (Gaussian case), then $\tau_{S^{\gamma,\alpha}_{a,b}}(p) = \left(\mathrm{Re}(\gamma)\ -\frac{1}{2}\right)$.*
- *If $\alpha < 2$, then $\tau_{S^{\gamma,\alpha}_{a,b}}(p) = \left(\mathrm{Re}(\gamma) + \frac{1}{\max(p,\alpha)} - 1\right)$.*

Proof. The proof follows from the combination of the two facts. We first recall some known results on the Sobolev regularity of stable white noises. The critical Sobolev smoothness $\tau_{W_\alpha}(p)$ of a SαS white noise has been fully characterized in the Gaussian case in [67] and in the general case in a series of papers [40,68,69]. For any $p \ge 1$, the results are available in [69, Theorem 1] as

$$\tau_{W_2}(p) = -1/2 \tag{43}$$

for the Gaussian ($\alpha = 2$) case and

$$\tau_{W_\alpha}(p) = \frac{1}{\max(p,\alpha)} - 1 \tag{44}$$

for the non-Gaussian ($\alpha < 2$) case.

Second, we show that the operators $\mathrm{D}^{\gamma}_{a,b}$ induce a systematic decrease of the Sobolev smoothness in the sense that $f \in W^{\tau}_{p,\mathrm{loc}}$ if and only if $\mathrm{D}^{\gamma}_{a,b} f \in W^{\tau-\mathrm{Re}(\gamma)}_{p,\mathrm{loc}}$ for any $p \ge 1$ and $\tau \in \mathbb{R}$. This is proven by applying the criterion of [31, Theorem 2] to this setting.[4] Indeed, denoting by $m(\omega) = \frac{h^{\gamma}_{a,b}(\omega)}{|\omega|^{\mathrm{Re}(\gamma)}}$, we observe that $|m(\omega)| = |a|$ if $\omega > 0$ and $|m(\omega)| = |b|$ for $\omega < 0$ and the relation [31, Eq. (28)] is readily satisfied.

Then, the integration operator $\mathrm{I}^{\gamma;k(\gamma,\alpha)}_{a,b}$, which is a right-inverse of $\mathrm{D}^{\gamma}_{a,b}$, satisfies the converse relation that $f \in W^{\tau}_{p,\mathrm{loc}}$ if and only if $g = \mathrm{I}^{\gamma;k(\gamma,\alpha)}_{a,b} f \in W^{\tau+\mathrm{Re}(\gamma)}_{p,\mathrm{loc}}$ (using that $\mathrm{D}^{\gamma}_{a,b} g = f$). This shows that the critical smoothness of $S^{\gamma,\alpha}_{a,b} = \mathrm{I}^{\gamma;k(\gamma,\alpha)}_{a,b} W_\alpha$ is such that

$$\tau_{S^{\gamma,\alpha}_{a,b}}(p) = \tau_{W_\alpha}(p) + \mathrm{Re}(\gamma), \tag{45}$$

and the result follows from the stable white noise case in (43) and (44). □

[4] We observe that [31] deals with periodic random processes. The results easily apply to our setting since we consider the *local* regularity of the proposed self-similar stable processes.

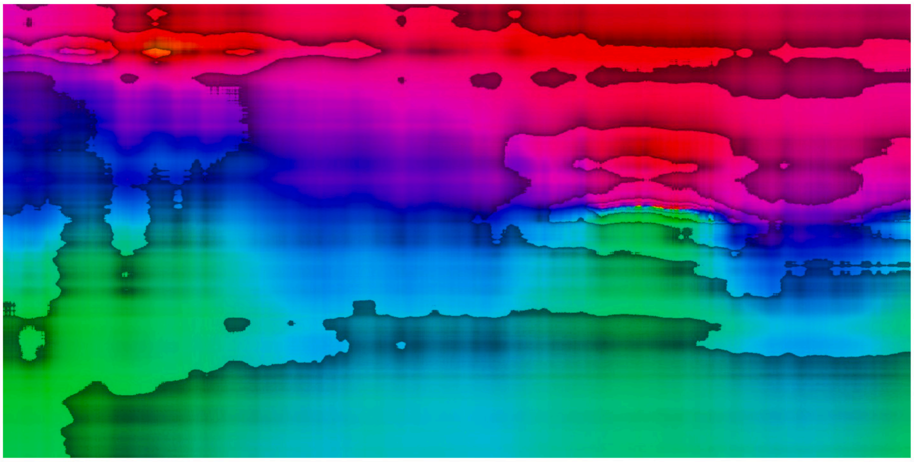

Fig. 2. A realization of the $S^{\gamma,\alpha}_{a,b}$ process with $\alpha = 2$ (Gaussian distribution), $a = 1$, $b = -1$ and $\gamma = 1.3 - 0.7\mathrm{i}$.

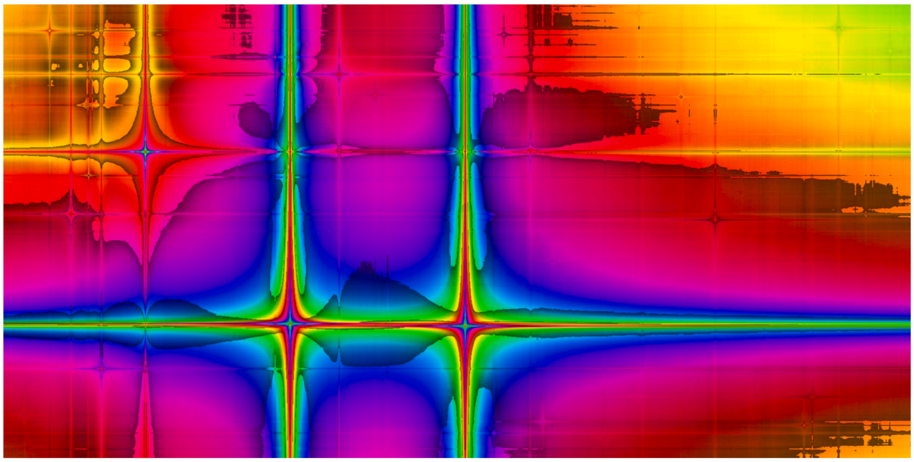

Fig. 3. A realization of the $S^{\gamma,\alpha}_{a,b}$ process with $\alpha = 1$ (Cauchy distribution), $a = 1$, $b = 1$ and $\gamma = 0.7 + \mathrm{i}$.

Remark 3.16. The critical L_2-Sobolev regularity, corresponding to $p = 2$, does not depend on α and is equal to $\tau_{S^{\gamma,\alpha}_{a,b}}(2) = \left(\mathrm{Re}(\gamma) - \frac{1}{2}\right)$ for any fractional stable process. Moreover, the critical Hölder regularity is $\left(\mathrm{Re}(\gamma) - \frac{1}{2}\right)$ for $\alpha = 2$ and $\left(\mathrm{Re}(\gamma) - 1\right)$ otherwise. This last result is coherent with [54] for $\gamma \geq 1$ and generalizes the result for new fractional stable random processes. Likewise, $S^{\gamma,\alpha}_{a,b}$ has negative Sobolev regularity for $\left(\mathrm{Re}(\gamma) + 1 - \frac{1}{\alpha}\right) < 0$ due to Theorem 3.15. This situation corresponds to singular self-similar random processes (see Remarks 3.11 and 3.13).

3.5. Simulations

The $S^{\gamma,\alpha}_{a,b}$ processes introduced in this work are complex-valued. Hence, to plot sample realizations we need to show the real and imaginary parts separately. But, for more compelling visual illustrations, we have applied our framework to the generalization of the processes in 2D where complex values can be more conveniently shown by colors; we apply separable 2D scale-invariant operators (multiplication of a horizontally and a vertically scale-invariant operator) to 2D white-noise processes. In Figs. 2 and 3 we can see two sample realizations. For generating these figures, we have first generated a 2D fine-grid discretization of the white noise process in form of an array of i.i.d. random variables. Then, we have applied the integration operator $\mathrm{I}^{\gamma;k}_{a,b}$ to the white noise (the 2D array) both vertically and horizontally using the impulse response expressions in (23). For Fig. 2 we have used $\alpha = 2$ (Gaussian distribution) with $a = 1$, $b = (-1)$ and $\gamma = (1.3 - 0.7\mathrm{i})$; this case requires $k = \lfloor \frac{1}{2} + 1.3 \rfloor = 1$. Similarly, we have set $\alpha = 1$ (Cauchy distribution) with $a = b = 1$ and $\gamma = 0.7 + \mathrm{i}$ in Fig. 3 (again requiring $k = \lfloor \frac{1}{1} + 0.7 \rfloor = 1$). The color-coding in these figures follows the standard approach for showing complex numbers, where the intensity of the pixels reflects the modulus of the complex numbers, while their phase is encoded in the hue.

It is interesting to mention that the 2D plots of these processes were useful in designing face masks during the COVID-19 pandemic; in Fig. 4, we see a face mask over which a fractionally integrated complex-valued process is printed. The white noise process in this

Fig. 4. A face mask during the COVID-19 pandemic showing complex-order integration applied to a 2D Poisson white noise.

case is Poisson with Gaussian jumps; although this process is not self-similar, it converges to the Gaussian white noise when the density of jumps goes to infinity [70]. Therefore, we expect the results generated by this Poisson white noise to resemble that of the Gaussian white noise with large enough jump densities.

4. Useful lemmas

Lemma 4.1. *Let $h^{\gamma}_{a,b}(\cdot)$ be a homogeneous generalized function of degree $\gamma\in\mathbb{C}$ with $\mathrm{Re}(\gamma)>-1$ as in (9), and let $n\in\{1,2,\dots,\lceil\mathrm{Re}(\gamma)\rceil+1\}$. If $\widehat{\phi}:\mathbb{R}\mapsto\mathbb{C}$ is n-times continuously differentiable such that $|\widehat{\phi}^{(j)}(\omega)|(1+|\omega|^{r+j})$ is bounded for all $1\le j\le n$ and some $r>\mathrm{Re}(\gamma)-n+1$, then,*

$$\forall\, 1<T,\qquad \Big\|\frac{\mathrm{d}^n}{\mathrm{d}\omega^n}\Big(\big(\widehat{\phi}(\omega)-\widehat{\phi}(\tfrac{\omega}{T})\big)h^{\gamma}_{a,b}(\omega)\Big)\Big\|_1\le c\,T^{\mathrm{Re}(\gamma)+2-n}, \tag{46}$$

where

$$c=\tfrac{8\,(r+1)\,(|\mathrm{Re}(\gamma)|+2)\,\max(|a|,|b|)}{(\mathrm{Re}(\gamma)-n+2)\,(r+n-\mathrm{Re}(\gamma)-1)}\Bigg(\sum_{k=1}^{n}\binom{n}{k}\sup_{\omega}\Big(\Big|\widehat{\phi}^{(k)}(\omega)\Big|(1+|\omega|^{r+k})\Big)\Bigg). \tag{47}$$

Proof. Intuitively, $\widehat{\phi}(\omega)-\widehat{\phi}(\frac{\omega}{T})$ behaves similar to $(1-\frac{1}{T})\omega\widehat{\phi}^{(1)}(\omega)$ around $\omega=0$ (a rigorous statement will be presented soon). Thus, $\big(\widehat{\phi}(\omega)-\widehat{\phi}(\frac{\omega}{T})\big)h^{\gamma}_{a,b}(\omega)$ has a zero of order at least $\mathrm{Re}(\gamma)+1$ at $\omega=0$. This suggests that $\frac{\mathrm{d}^n}{\mathrm{d}\omega^n}\big(\widehat{\phi}(\omega)-\widehat{\phi}(\frac{\omega}{T})\big)$ is either bounded at $\omega=0$ (if $n\le\lceil\mathrm{Re}(\gamma)\rceil$) or is at least locally integrable around $\omega=0$ (if $n=\lceil\mathrm{Re}(\gamma)\rceil+1$). We also employ the boundedness property of $\big\{|\widehat{\phi}^{(j)}(\omega)|(1+|\omega|^{\gamma+j})\big\}_i$ to show that $\frac{\mathrm{d}^n}{\mathrm{d}\omega^n}\big(\widehat{\phi}(\omega)-\widehat{\phi}(\frac{\omega}{T})\big)$ decays asymptotically no slower than $\frac{1}{1+|\omega|^{1+\epsilon}}$ for some $\epsilon>0$ that is determined by r. As a result, we conclude that the L_1-norm of $\frac{\mathrm{d}^n}{\mathrm{d}\omega^n}\big(\widehat{\phi}(\omega)-\widehat{\phi}(\frac{\omega}{T})\big)$ is finite ($\widehat{\phi}$ is not even required to be asymptotically decaying). However, the main point in Lemma 4.1 is the scaling of the L_1-norm in terms of T.

To start our rigorous arguments, let us define

$$\underline{\widehat{\phi}}(\omega)=\sup_{\substack{\tau\\|\tau|\ge|\omega|}}\Big|\frac{\mathrm{d}}{\mathrm{d}\tau}\widehat{\phi}(\tau)\Big|. \tag{48}$$

Based on the assumptions, $|\widehat{\phi}^{(1)}(\omega)|(1+|\omega|^{r+1})$ is bounded. Moreover, $r+1>\mathrm{Re}(\gamma)-n+2>\mathrm{Re}(\gamma)-\lceil\mathrm{Re}(\gamma)\rceil+1>0$. This shows that $|\widehat{\phi}^{(1)}(\omega)|$ is bounded and asymptotically decaying. Hence, $\underline{\widehat{\phi}}(\omega)$ is a bounded, even, and non-negative-valued function that is non-increasing in terms of $|\omega|$. Since $r+1>0$, we conclude that $\underline{\widehat{\phi}}(\omega)(1+|\omega|^{r+1})$ is also bounded:

$$\begin{aligned}\underline{\widehat{\phi}}(\omega)(1+|\omega|^{r+1})&=\sup_{\substack{\tau\\|\tau|\ge|\omega|}}\Big|\widehat{\phi}^{(1)}(\tau)\Big|(1+|\omega|^{r+1})\le\sup_{\substack{\tau\\|\tau|\ge|\omega|}}\Big|\widehat{\phi}^{(1)}(\tau)\Big|(1+|\tau|^{r+1})\\&\le\underbrace{\sup_{\tau}\Big|\widehat{\phi}^{(1)}(\tau)\Big|(1+|\tau|^{r+1})}_{\text{a finite constant}}.\end{aligned} \tag{49}$$

Next, we bound the difference $\widehat{\phi}(\omega) - \widehat{\phi}(\frac{\omega}{T})$ via $\underline{\widehat{\phi}}(\omega)$. Because our approach is based on the Taylor's series, we need to initially separate the real and imaginary parts of $\widehat{\phi}$. Note that $\mathrm{Re}\left(\widehat{\phi}(\omega)\right)$ and $\mathrm{Im}\left(\widehat{\phi}(\omega)\right)$ are both continuously differentiable functions for which we can write the Lagrange form of the Taylor's series as

$$\begin{aligned}
&\mathrm{Re}\left(\widehat{\phi}(\omega)\right) - \mathrm{Re}\left(\widehat{\phi}(\tfrac{\omega}{T})\right) = (\omega - \tfrac{\omega}{T})\tfrac{\mathrm{d}}{\mathrm{d}\omega}\mathrm{Re}\left(\widehat{\phi}(\omega)\right)\Big|_{\omega=\zeta_r} = \tfrac{T-1}{T}\omega\,\mathrm{Re}\left(\widehat{\phi}^{(1)}(\zeta_r)\right),\\
&\mathrm{Im}\left(\widehat{\phi}(\omega)\right) - \mathrm{Im}\left(\widehat{\phi}(\tfrac{\omega}{T})\right) = (\omega - \tfrac{\omega}{T})\tfrac{\mathrm{d}}{\mathrm{d}\omega}\mathrm{Im}\left(\widehat{\phi}(\omega)\right)\Big|_{\omega=\zeta_i} = \tfrac{T-1}{T}\omega\,\mathrm{Im}\left(\widehat{\phi}^{(1)}(\zeta_i)\right),
\end{aligned} \tag{50}$$

where ζ_r, ζ_i are real numbers between $\frac{\omega}{T}$ and ω. Therefore, if $\mathcal{I}_{\omega,T}$ represents the closed interval between $\frac{\omega}{T}$ and ω, we have that

$$\begin{aligned}
\left|\mathrm{Re}\left(\widehat{\phi}(\omega) - \widehat{\phi}(\tfrac{\omega}{T})\right)\right| &\le \tfrac{T-1}{T}|\omega| \sup_{\tau\in\mathcal{I}_{\omega,T}} \left|\mathrm{Re}\left(\phi^{(1)}(\tau)\right)\right| \le |\omega| \sup_{\tau\in\mathcal{I}_{\omega,T}} \left|\phi^{(1)}(\tau)\right|\\
&\le |\omega|\,\underline{\widehat{\phi}}(\tfrac{\omega}{T}).
\end{aligned} \tag{51}$$

Similarly, we can show that

$$\left|\mathrm{Im}\left(\widehat{\phi}(\omega) - \widehat{\phi}(\tfrac{\omega}{T})\right)\right| \le |\omega|\,\underline{\widehat{\phi}}(\tfrac{\omega}{T}). \tag{52}$$

By combining (51) and (52), we achieve

$$\left|\widehat{\phi}(\omega) - \widehat{\phi}(\tfrac{\omega}{T})\right| \le \left|\mathrm{Re}\left(\widehat{\phi}(\omega) - \widehat{\phi}(\tfrac{\omega}{T})\right)\right| + \left|\mathrm{Im}\left(\widehat{\phi}(\omega) - \widehat{\phi}(\tfrac{\omega}{T})\right)\right| \le 2|\omega|\,\underline{\widehat{\phi}}(\tfrac{\omega}{T}). \tag{53}$$

Now, we are equipped to consider the main claim:

$$\begin{aligned}
&\left|\tfrac{\mathrm{d}^n}{\mathrm{d}\omega^n}\left(\left(\widehat{\phi}(\omega) - \widehat{\phi}(\tfrac{\omega}{T})\right)h^{\gamma}_{a,b}(\omega)\right)\right| = \left|\sum_{k=0}^{n}\binom{n}{k}\left(\tfrac{\mathrm{d}^k}{\mathrm{d}\omega^k}\left(\widehat{\phi}(\omega) - \widehat{\phi}(\tfrac{\omega}{T})\right)\right)\left(\tfrac{\mathrm{d}^{n-k}}{\mathrm{d}\omega^{n-k}}h^{\gamma}_{a,b}(\omega)\right)\right|\\
&= \left|\left(\widehat{\phi}(\omega) - \widehat{\phi}(\tfrac{\omega}{T})\right)h^{\gamma,(n)}_{a,b}(\omega) + \sum_{k=1}^{n}\binom{n}{k}\left(\widehat{\phi}^{(k)}(\omega) - \tfrac{1}{T^k}\widehat{\phi}^{(k)}(\tfrac{\omega}{T})\right)h^{\gamma,(n-k)}_{a,b}(\omega)\right|\\
&\le 2\underline{\widehat{\phi}}(\tfrac{\omega}{T})\,|\omega\, h^{\gamma,(n)}_{a,b}(\omega)| + \sum_{k=1}^{n}\binom{n}{k}\left(\left|\widehat{\phi}^{(k)}(\omega)\right| + \tfrac{1}{T^k}\left|\widehat{\phi}^{(k)}(\tfrac{\omega}{T})\right|\right)|h^{\gamma,(n-k)}_{a,b}(\omega)|
\end{aligned} \tag{54}$$

which implies

$$\begin{aligned}
\left\|\tfrac{\mathrm{d}^n}{\mathrm{d}\omega^n}\left(\left(\widehat{\phi}(\omega) - \widehat{\phi}(\tfrac{\omega}{T})\right)h^{\gamma}_{a,b}(\omega)\right)\right\|_1 &\le 2\left\|\underline{\widehat{\phi}}(\tfrac{\omega}{T})\,\omega\, h^{\gamma,(n)}_{a,b}(\omega)\right\|_1\\
&+ \sum_{k=1}^{n}\binom{n}{k}\left(\left\|\widehat{\phi}^{(k)}(\omega)h^{\gamma,(n-k)}_{a,b}(\omega)\right\|_1 + \tfrac{1}{T^k}\left\|\widehat{\phi}^{(k)}(\tfrac{\omega}{T})h^{\gamma,(n-k)}_{a,b}(\omega)\right\|_1\right).
\end{aligned} \tag{55}$$

To simplify the upperbound, note that if $l(\omega)$ is a homogeneous generalized function of degree s, then, for a generic function $g(\omega)$ we have that

$$\begin{aligned}
\left\|g(\tfrac{\omega}{T})l(\omega)\right\|_1 &= \int_{\mathbb{R}}\left|g(\underbrace{\tfrac{\omega}{T}}_{\nu})l(\omega)\right|\mathrm{d}\omega = T\int_{\mathbb{R}}\left|g(\nu)l(T\,\nu)\right|\mathrm{d}\nu\\
&= T^{\mathrm{Re}(s)+1}\int_{\mathbb{R}}\left|g(\nu)l(\nu)\right|\mathrm{d}\nu = T^{\mathrm{Re}(s)+1}\left\|g(\omega)l(\omega)\right\|_1.
\end{aligned} \tag{56}$$

Since $\omega\, h^{\gamma,(n)}_{a,b}(\omega)$ and $h^{\gamma,(n-k)}_{a,b}(\omega)$ are both homogeneous with degrees $\gamma - n + 1$ and $\gamma - n + k$, respectively, we can simplify (55) as

$$\begin{aligned}
\left\|\tfrac{\mathrm{d}^n}{\mathrm{d}\omega^n}\left(\left(\widehat{\phi}(\omega) - \widehat{\phi}(\tfrac{\omega}{T})\right)h^{\gamma}_{a,b}(\omega)\right)\right\|_1 &\le\ 2T^{\mathrm{Re}(\gamma)+2-n}\left\|\omega\,\underline{\widehat{\phi}}(\omega)\,h^{\gamma,(n)}_{a,b}(\omega)\right\|_1\\
&+ \left(T^{\mathrm{Re}(\gamma)+1-n} + 1\right)\sum_{k=1}^{n}\binom{n}{k}\left\|\widehat{\phi}^{(k)}(\omega)h^{\gamma,(n-k)}_{a,b}(\omega)\right\|_1\\
&< \underbrace{\left(2\left\|\omega\,\underline{\widehat{\phi}}(\omega)\,h^{\gamma,(n)}_{a,b}(\omega)\right\|_1 + 2\sum_{k=1}^{n}\binom{n}{k}\left\|\widehat{\phi}^{(k)}(\omega)h^{\gamma,(n-k)}_{a,b}(\omega)\right\|_1\right)}_{c}T^{\mathrm{Re}(\gamma)+2-n}
\end{aligned} \tag{57}$$

which proves the claim. However, we still need to show that $\left\|\omega\,\underline{\widehat{\phi}}(\omega)\,h^{\gamma,(n)}_{a,b}(\omega)\right\|_1$ and $\left\|\widehat{\phi}^{(k)}(\omega)h^{\gamma,(n-k)}_{a,b}(\omega)\right\|_1$ are all well-defined and finite. To that end, we first observe that

$$\forall\, 0 \le k \le n: \quad \left| h_{a,b}^{\gamma,(k)}(\omega) \right| \le \mathbb{b}\, |\operatorname{Re}(\gamma) - k| \cdot |\omega|^{\operatorname{Re}(\gamma)-k},$$

where $\mathbb{b} = \max\big(|a|, |b|\big)$. Based on the assumption, the value $\hat{\phi}_k = \sup_\omega \left|\hat{\phi}^{(k)}(\omega)\right|(1+|\omega|^{r+k})$ is finite for all $1 \le k \le n$. Moreover, we have previously shown in (49) that $\underline{\hat{\phi}}(\omega)(1+|\omega|^{r+1})$ is upper-bounded by $\hat{\phi}_1$. Therefore,

$$\begin{aligned}
\left\| \omega\, \underline{\hat{\phi}}(\omega)\, h_{a,b}^{\gamma,(n)}(\omega) \right\|_1 &\le 2|\operatorname{Re}(\gamma) - n|\, \mathbb{b}\, \hat{\phi}_1 \int_0^\infty \frac{\omega^{\operatorname{Re}(\gamma)-n+1}}{1+\omega^{r+1}} \mathrm{d}\omega \\
&\le 2|\operatorname{Re}(\gamma) - n|\, \mathbb{b}\, \hat{\phi}_1 \left(\int_0^1 \omega^{\operatorname{Re}(\gamma)-n+1} \mathrm{d}\omega + \int_1^\infty \omega^{\operatorname{Re}(\gamma)-n-r} \mathrm{d}\omega \right) \\
&= 2|\operatorname{Re}(\gamma) - n|\, \mathbb{b}\, \hat{\phi}_1 \left(\frac{\omega^{\operatorname{Re}(\gamma)-n+2}}{\operatorname{Re}(\gamma)-n+2}\bigg|_0^1 + \frac{\omega^{\operatorname{Re}(\gamma)-n-r+1}}{\operatorname{Re}(\gamma)-n-r+1}\bigg|_1^\infty \right).
\end{aligned}$$

The upper-bound is finite because

$$\begin{aligned}
\operatorname{Re}(\gamma) - n + 2 &\ge \operatorname{Re}(\gamma) - (\lceil \operatorname{Re}(\gamma) \rceil + 1) + 2 = \operatorname{Re}(\gamma) - \lceil \operatorname{Re}(\gamma) \rceil + 1 > 0, \\
\operatorname{Re}(\gamma) - n - r + 1 &< \operatorname{Re}(\gamma) - n - (\operatorname{Re}(\gamma) - n + 1) + 1 = 0.
\end{aligned} \tag{58}$$

For the existence of $\left\| \hat{\phi}^{(k)}(\omega) h_{a,b}^{\gamma,(n-k)}(\omega) \right\|_1$, we employ a similar technique:

$$\begin{aligned}
\left\| \hat{\phi}^{(k)}(\omega) h_{a,b}^{\gamma,(n-k)}(\omega) \right\|_1 &\le 2|\operatorname{Re}(\gamma) - n + k|\, \mathbb{b}\, \hat{\phi}_k \int_0^\infty \frac{\omega^{\operatorname{Re}(\gamma)-n+k}}{1+\omega^{r+k}} \mathrm{d}\omega \\
&\le 2|\operatorname{Re}(\gamma) - n + k|\, \mathbb{b}\, \hat{\phi}_k \left(\int_0^1 \omega^{\operatorname{Re}(\gamma)-n+k} \mathrm{d}\omega + \int_1^\infty \omega^{\operatorname{Re}(\gamma)-n-r} \mathrm{d}\omega \right) \\
&= 2|\operatorname{Re}(\gamma) - n + k|\, \mathbb{b}\, \hat{\phi}_k \left(\frac{\omega^{\operatorname{Re}(\gamma)-n+k+1}}{\operatorname{Re}(\gamma)-n+k+1}\bigg|_0^1 + \frac{\omega^{\operatorname{Re}(\gamma)-n-r+1}}{\operatorname{Re}(\gamma)-n-r+1}\bigg|_1^\infty \right).
\end{aligned}$$

The finiteness of the upper-bound follows from (58) by considering $k \ge 1$. ■

Lemma 4.2. *If $\phi(x) : \mathbb{R} \setminus \{0\} \mapsto \mathbb{C}$ satisfies*

$$\forall x \in \mathbb{R} \setminus \{0\},\ \forall T \in [2, 4[: \quad \left| \phi(x) - T^\gamma \phi(T\, x) \right| \le \frac{c}{|x|^m},$$

where $c \in \mathbb{R}^+$, $m \in \mathbb{R}$ and $\gamma \in \mathbb{C}$ are constants, then, there is a constant $\bar{c}$ such that

$$\forall x \in \mathbb{R} \setminus \{0\}: \quad \left| \phi(x) \right| \le \frac{\bar{c}}{\min\big(|x|^{\operatorname{Re}(\gamma)}, |x|^m\big)},$$

if $m \ne \operatorname{Re}(\gamma)$, and

$$\forall x \in \mathbb{R} \setminus \{0\}: \quad \left| \phi(x) \right| \le \bar{c}\, \frac{1 + \left| \log|x| \right|}{|x|^{\operatorname{Re}(\gamma)}},$$

if $m = \operatorname{Re}(\gamma)$.

Proof. Let τ be an arbitrary positive real. Any real number x with $2\tau \le |x|$ or $0 < |x| \le \frac{\tau}{2}$ can be uniquely written in the form of $\pm T^n\, \tau$ with $n \in \{\pm 1, \pm 2, \dots, \pm 2^i, \dots\}$ and $T \in [2, 4[$.[5] For the sake of simplicity, we continue the proof only for $0 < x = \tau T^n$ (the negative case is similar). We define

$$S_n = \begin{cases} \{0, 1, \dots, n-1\}, & n \ge 1, \\ \{-1, -2, \dots, n\}, & n \le -1. \end{cases}$$

Now, we have that

$$\left| \tau^\gamma \phi(\tau) - \big(T^n \tau\big)^\gamma \phi(T^n\, \tau) \right| \le \sum_{k \in S_n} \left| \big(T^k \tau\big)^\gamma \phi(T^k\, \tau) - \big(T^{k+1} \tau\big)^\gamma \phi(T^{k+1}\, \tau) \right|$$

[5] This can be achieved by setting $n = \operatorname{sign}(\theta)\, 2^{\lfloor \log_2 |\theta| \rfloor}$ and $T = 2^{2^{\{\log_2 \theta\}}}$, where $\theta = \log_2 \left| \frac{x}{\tau} \right|$.

$$= \sum_{k\in S_n} \left(T^k\tau\right)^{\mathrm{Re}(\gamma)} \left|\phi(T^k\tau) - T^{\gamma}\phi(T^{k+1}\tau)\right|$$
$$\leq \sum_{k\in S_n} \frac{c\left(T^k\tau\right)^{\mathrm{Re}(\gamma)}}{(T^k\tau)^m} = c\,\tau^{\mathrm{Re}(\gamma)-m} \sum_{k\in S_n} \left(T^{\mathrm{Re}(\gamma)-m}\right)^k. \tag{59}$$

We consider the two cases of $m \neq \mathrm{Re}(\gamma)$ and $m = \mathrm{Re}(\gamma)$ separately.

1. $m \neq \mathrm{Re}(\gamma)$. Thus, $\sum_{k\in S_n} T^{k(\mathrm{Re}(\gamma)-m)} = \left|\frac{T^{n(\mathrm{Re}(\gamma)-m)}-1}{T^{\mathrm{Re}(\gamma)-m}-1}\right|$ and (59) can be written as

$$\left|\tau^{\gamma}\phi(\tau) - \left(T^n\tau\right)^{\gamma}\phi(T^n\tau)\right| \leq \tfrac{c}{|T^{\mathrm{Re}(\gamma)-m}-1|}\left(T^n\tau\right)^{\mathrm{Re}(\gamma)-m} + \tfrac{c\,\tau^{\mathrm{Re}(\gamma)-m}}{|T^{\mathrm{Re}(\gamma)-m}-1|}$$
$$\Rightarrow \quad \left|\left(T^n\tau\right)^{\gamma}\phi(T^n\tau)\right| \leq \tfrac{c(T^n\tau)^{\mathrm{Re}(\gamma)-m}}{|T^{\mathrm{Re}(\gamma)-m}-1|} + \tfrac{c\,\tau^{\mathrm{Re}(\gamma)-m}}{|T^{\mathrm{Re}(\gamma)-m}-1|} + \left|\tau^{\gamma}\phi(\tau)\right|$$
$$\leq \left(c\,\tfrac{\tau^{\mathrm{Re}(\gamma)-m}+1}{|T^{\mathrm{Re}(\gamma)-m}-1|} + \left|\tau^{\gamma}\phi(\tau)\right|\right)\ \max\left((T^n\tau)^{\mathrm{Re}(\gamma)-m},\,1\right)$$
$$\leq \underbrace{\left(c\,\tfrac{\tau^{\mathrm{Re}(\gamma)-m}+1}{\min(|2^{\mathrm{Re}(\gamma)-m}-1|,\,|4^{\mathrm{Re}(\gamma)-m}-1|)} + \left|\tau^{\gamma}\phi(\tau)\right|\right)}_{\bar{c}_\tau}\ \max\left((T^n\tau)^{\mathrm{Re}(\gamma)-m},\,1\right), \tag{60}$$

where $\bar{c}_\tau$ is a constant that depends neither on n nor on T. As explained earlier, for any positive $x \notin]\frac{\tau}{2}, 2\tau[$ we can find suitable n and T, such that $x = T^n\tau$. By substituting $x = T^n\tau$ in (60) we obtain

$$\forall\, 0 < x \notin]\tfrac{\tau}{2}, 2\tau[: \quad \left|\phi(x)\right| \leq \frac{\bar{c}_\tau}{\min\left(x^{\mathrm{Re}(\gamma)},\, x^m\right)}.$$

2. $m = \mathrm{Re}\ z$. This implies that $\sum_{k\in S_n} T^{k(\mathrm{Re}(\gamma)-m)} = |n|$. Now (59) simplifies to

$$\left|\tau^{\gamma}\phi(\tau) - \left(T^n\tau\right)^{\gamma}\phi(T^n\tau)\right| \leq c\,\tau^{\mathrm{Re}(\gamma)-m}\,|n| = c\,\tau^{\mathrm{Re}(\gamma)-m}\left|\tfrac{\log(T^n\tau)-\log(\tau)}{\log T}\right|$$
$$\Rightarrow \left|\left(T^n\tau\right)^{\gamma}\phi(T^n\tau)\right| \leq \underbrace{\left(c\,\tfrac{\tau^{\mathrm{Re}(\gamma)-m}(1+|\log\tau|)}{\log 2} + |\tau^{\gamma}\phi(\tau)|\right)}_{\bar{c}_\tau}\left(1 + |\log(T^n\tau)|\right), \tag{61}$$

where $\bar{c}_\tau$ is again a constant that depends neither on n nor on T. Thus, similar to the previous case, we conclude that

$$\forall\, 0 < x \notin]\tfrac{\tau}{2}, 2\tau[: \quad \left|\phi(x)\right| \leq \bar{c}_\tau\,\frac{1 + |\log x|}{x^{\mathrm{Re}(\gamma)}}.$$

The above arguments prove the claim, except for $x \in]\frac{\tau}{2}, 2\tau[$. However, the choice of τ was arbitrary; thus, all the above results also hold for $\tau' = 4\tau$, if $\bar{c}_\tau$ is replaced with $\bar{c}_{\tau'}$. In addition, $]\frac{\tau}{2}, 2\tau[$ and $]\frac{\tau'}{2}, 2\tau'[$ have empty intersection. Thus, if we use $\bar{c} = \max(\bar{c}_\tau, \bar{c}_{\tau'})$ as the constant, the inequalities hold for all $x \neq 0$. ■

Lemma 4.3. *If $\varphi : \mathbb{R} \to \mathbb{C}$ is $k+n$ times continuously differentiable ($k \geq 1$ and $n \geq 0$), then, the function $\phi(x) = \frac{\varphi(x) - \sum_{j=0}^{k-1}\frac{x^j}{j!}\varphi^{(j)}(0)}{x^k}$ is n times continuously differentiable. In addition, if $\frac{|\varphi^{(j)}(x)|}{1+|x|^{k-1-j}}$ is bounded for all $0 \leq j < k$, then, $|\phi^{(n)}(x)|(1+|x|^{n+1})$ is also bounded.*

Proof. For notational convenience, we define

$$f(x) = \sum_{i=0}^{k-1} \tfrac{x^i}{i!}\varphi^{(i)}(0) \qquad , \qquad g(x) = \sum_{i=0}^{k+n-1} \tfrac{x^i}{i!}\varphi^{(i)}(0).$$

Since φ is $k+n$ times continuously differentiable, $\varphi^{(j)}$ is also $k+n-j$ times continuously differentiable. By applying the Lagrange's form of the Taylor series on the real and imaginary parts of $\varphi^{(j)}(x)$ separately for $0 \leq j \leq n$, and then combining them, we have that

$$\varphi^{(j)}(x) = \sum_{i=0}^{k+n-j-1} \tfrac{x^i}{i!}\varphi^{(j+i)}(0) + \tfrac{x^{k+n-j}}{(k+n-j)!}\Big(\mathrm{Re}\left(\varphi^{(k+n)}(\zeta_{x,R}^{(j)})\right) + \mathrm{i}\,\mathrm{Im}\left(\varphi^{(k+n)}(\zeta_{x,I}^{(j)})\right)\Big)$$
$$= \tfrac{\mathrm{d}^j}{\mathrm{d}x^j}g(x) + \tfrac{x^{k+n-j}}{(k+n-j)!}\Big(\mathrm{Re}\left(\varphi^{(k+n)}(\zeta_{x,R}^{(j)})\right) + \mathrm{i}\,\mathrm{Im}\left(\varphi^{(k+n)}(\zeta_{x,I}^{(j)})\right)\Big), \tag{62}$$

where $\zeta_{x,R}^{(j)}$ and $\zeta_{x,I}^{(j)}$ are real numbers between 0 and x. Let us now evaluate the mth ($0 \leq m \leq n$) derivative of ϕ:

$$\tfrac{\mathrm{d}^m}{\mathrm{d}x^m}\phi(x) = \tfrac{\mathrm{d}^m}{\mathrm{d}x^m}\left(\frac{\varphi(x) - f(x)}{x^k}\right) = \sum_{j=0}^{m}\binom{m}{j}\left(\varphi^{(j)}(x) - f^{(j)}(x)\right)\,\tfrac{\mathrm{d}^{m-j}}{\mathrm{d}x^{m-j}}\left(\tfrac{1}{x^k}\right). \tag{63}$$

As $\{\varphi^{(j)}\}_j$ are continuous for all $0 \le j \le m$, it is straightforward to see that $\phi^{(m)}(x)$ exists and is continuous at $x \neq 0$. To investigate the behavior of $\phi^{(m)}(x)$ around $x = 0$, we rewrite (63) by using (62) as

$$\begin{aligned}
\tfrac{\mathrm{d}^m}{\mathrm{d}x^m}\phi(x) = \sum_{j=0}^{m}\binom{m}{j}\Big(g^{(j)}(x) - f^{(j)}(x) + \tfrac{x^{k+n-j}}{(k+n-j)!}\Big(\operatorname{Re}\big(\varphi^{(k+n)}(\zeta^{(j)}_{x,R})\big) \\
+ \mathrm{i}\operatorname{Im}\big(\varphi^{(k+n)}(\zeta^{(j)}_{x,I})\big)\Big)\Big)\tfrac{\mathrm{d}^{m-j}}{\mathrm{d}x^{m-j}}\big(\tfrac{1}{x^k}\big) \\
= \sum_{j=0}^{m}\binom{m}{j}\big(g^{(j)}(x) - f^{(j)}(x)\big)\ \tfrac{\mathrm{d}^{m-j}}{\mathrm{d}x^{m-j}}\big(\tfrac{1}{x^k}\big) \\
+ \sum_{j=0}^{m}\binom{m}{j}\tfrac{x^{k+n-j}}{(k+n-j)!}\Big(\operatorname{Re}\big(\varphi^{(k+n)}(\zeta^{(j)}_{x,R})\big) + \mathrm{i}\operatorname{Im}\big(\varphi^{(k+n)}(\zeta^{(j)}_{x,I})\big)\Big)\tfrac{\mathrm{d}^{m-j}}{\mathrm{d}x^{m-j}}\big(\tfrac{1}{x^k}\big) \\
= \tfrac{\mathrm{d}^m}{\mathrm{d}x^m}\Big(\underbrace{\tfrac{g(x)-f(x)}{x^k}}_{P(x)}\Big) + \tfrac{x^{n-m}}{(k-1)!\,(n+1-m)!}\sum_{j=0}^{m}\tfrac{(-1)^{m-j}\binom{m}{j}}{\binom{k+n-j}{k+m-j-1}}\Big(\operatorname{Re}\big(\varphi^{(k+n)}(\zeta^{(j)}_{x,R})\big) \\
+ \mathrm{i}\operatorname{Im}\big(\varphi^{(k+n)}(\zeta^{(j)}_{x,I})\big)\Big),
\end{aligned}$$

where $P(x)$ is a polynomial of degree no more than $n-1$. Since $\{\zeta^{(j)}_{x,R}, \zeta^{(j)}_{x,I}\}_j$ are all between 0 and x, they all approach 0 when $x \to 0$. Furthermore, $\varphi^{(k+n)}$ is a continuous function by assumption. Thus

$$\lim_{x\to 0}\tfrac{\mathrm{d}^m}{\mathrm{d}x^m}\phi(x) = \begin{cases} P^{(m)}(0), & 0 \le m < n, \\ \tfrac{n!}{(k+n)!}\ \varphi^{(k+n)}(0), & m = n. \end{cases}$$

Now that we have shown $\phi^{(n)}(x)$ is continuous, we turn to the claimed decay result. Because of the continuity of $\phi^{(n)}$, it is sufficient to show that $|\phi^{(n)}(x)|(1+|x|^{n+1})$ is bounded only for $|x| \ge 1$. As $f^{(j)}(x)$ is a polynomial of degree at most $\max(0, k-j-1)$, it is evident that $\frac{f^{(j)}(x)}{1+|x|^{k-j-1}}$ is bounded. This result, in addition to the assumption in the lemma, implies that

$$\frac{\varphi^{(j)}(x) - f^{(j)}(x)}{1+|x|^{k-j-1}}$$

is also bounded. Further, we have that

$$\frac{(1+|x|^{n+1})(1+|x|^{k-j-1})}{|x|^{k+n-j}} \overset{|x|\ge 1}{\le} \frac{(|x|^{n+1}+|x|^{n+1})(|x|^{k-j-1}+|x|^{k-j-1})}{|x|^{k+n-j}} = 4.$$

Now, recalling (63), we obtain

$$\begin{aligned}
|\phi^{(n)}(x)|(1+|x|^{n+1}) &\le (1+|x|^{n+1})\sum_{j=0}^{n}\binom{n}{j}\tfrac{(k+n-j-1)!}{(k-1)!\,|x|^{k+n-j}}\Big|\varphi^{(j)}(x) - f^{(j)}(x)\Big| \\
&= \sum_{j=0}^{n}\tfrac{\binom{n}{j}(k+n-j-1)!}{(k-1)!}\ \tfrac{|\varphi^{(j)}(x)-f^{(j)}(x)|}{1+|x|^{k-j-1}}\ \tfrac{(1+|x|^{n+1})(1+|x|^{k-j-1})}{|x|^{k+n-j}}.
\end{aligned}$$

The upper-bound consists of finitely many bounded terms for $|x| \ge 1$, and is therefore, bounded. ■

Corollary 4.4. *For $\varphi \in \mathcal{S}$, we know that $\widehat{\varphi}$ (Fourier transform of φ) is infinitely differentiable and $\frac{|\widehat{\varphi}^{(j)}(\omega)|}{1+|\omega|^m}$ is bounded for all $0 \le j, m$. Using Lemma 4.3, we conclude that $\widehat{\phi}(\omega) = \frac{\widehat{\varphi}(\omega) - \sum_{i=0}^{k}\frac{\omega^i}{i!}\widehat{\varphi}^{(i)}(0)}{\omega^{k+1}}$ is infinitely differentiable and $\widehat{\phi}^{(j)} \in L_1$ for all $j \ge 1$. This implies that $(1+|x|^m)\phi(x)$ (where ϕ is the inverse Fourier of $\widehat{\phi}$) is bounded for all $m \in \mathbb{N}$. Nevertheless, since $\widehat{\phi} \notin L_1$, $\phi(x)$ is not continuous.*

Lemma 4.5. *Let $h^{\gamma}_{a,b}(\cdot)$ be a homogeneous function of degree $\gamma \in \mathbb{C}$ as in (9) and let $k \ge \max\big(1, \lfloor \mathrm{Re}(\gamma)\rfloor\big)$. For a k-times continuously differentiable $\widehat{\varphi}$, define*

$$\widehat{\varphi}^{\gamma;k}_{a,b}(\omega) = \frac{\widehat{\varphi}(\omega) - \sum_{i=0}^{k-1}\frac{\widehat{\varphi}^{(i)}(0)}{i!}\omega^i}{h^{\gamma}_{a,b}(\omega)}.$$

If $\frac{|\hat{\varphi}(\omega)|}{1+|\omega|^{k-1}}$ *and* $\left\{\frac{|\hat{\varphi}^{(j)}(\omega)|}{1+|\omega|^{k-j}}\right\}_{j=1}^{k}$ *are all bounded, then, the inverse Fourier of* $\hat{\varphi}_{a,b}^{\gamma;k}$ *denoted by* $\varphi_{a,b}^{\gamma;k}(x)$ *exists as a function for* $x \neq 0$*, and*

(i) *when* $\mathrm{Re}(\gamma) \notin \mathbb{Z}$ *and* $k = \lfloor \mathrm{Re}(\gamma) \rfloor$*, the function* $\varphi_{a,b}^{\gamma;k}(x)$ *is continuous everywhere including* $x = 0$*;*

(ii) *when* $\mathrm{Re}(\gamma) \in \mathbb{Z}$*,* $\mathrm{Im}(\gamma) \neq 0$ *and* $k = \lfloor \mathrm{Re}(\gamma) \rfloor$*, the function* $\varphi_{a,b}^{\gamma;k}(x)$ *is continuous at* $x \neq 0$ *and bounded around* $x = 0$*;*

(iii) *when* $\gamma \in \mathbb{Z}$*,* $h_{a,b}^{\gamma}(\omega) \equiv c\,\omega^{\gamma}$ *for some* $c \in \mathbb{C}$*, and* $k = \lfloor \mathrm{Re}(\gamma) \rfloor$*, the function* $\varphi_{a,b}^{\gamma;k}(x)$ *is continuous at* $x \neq 0$ *and bounded around* $x = 0$*;*

(iv) *when* $\gamma \in \mathbb{Z}$*,* $h_{a,b}^{\gamma}(\omega) \not\equiv c\,\omega^{\gamma}$*, and* $k = \lfloor \mathrm{Re}(\gamma) \rfloor$*, the function* $\varphi_{a,b}^{\gamma;k}(x)$ *is continuous at* $x \neq 0$*, and* $\frac{|\varphi_{a,b}^{\gamma;k}(x)|}{\log|x|}$ *is bounded around* $x = 0$*;*

(v) *when* $k > \lfloor \mathrm{Re}(\gamma) \rfloor$*, the function* $\varphi_{a,b}^{\gamma;k}(x)$ *is continuous at* $x \neq 0$*, and* $|x|^{k-\mathrm{Re}(\gamma)}|\varphi_{a,b}^{\gamma;k}(x)|$ *is bounded at* $x \in [-1, 1]$*.*

Proof. We first rewrite $\hat{\varphi}_{a,b}^{\gamma;k}$ as

$$\hat{\varphi}_{a,b}^{\gamma;k}(\omega) = \underbrace{\frac{\hat{\varphi}(\omega) - \sum_{i=0}^{k-1} \frac{\hat{\varphi}^{(i)}(0)}{i!}\omega^i}{\omega^k}}_{\hat{\phi}(\omega)} \underbrace{\frac{\omega^k}{h_{a,b}^{\gamma}(\omega)}}_{\underset{\sim}{h}(\omega)} = \hat{\phi}(\omega)\underset{\sim}{h}(\omega). \tag{64}$$

Lemma 4.3 implies that $\hat{\phi}$ is continuous and $\hat{\phi}(\omega)(1+|\omega|)$ is bounded. In addition, $\underset{\sim}{h}(\omega) = h_{a',b'}^{\gamma'}(\omega)$ with $a' = \frac{1}{a}$, $b' = \frac{(-1)^k}{b}$, $\gamma' = k - \gamma$ is a homogeneous function of degree $\underset{\sim}{\gamma} = k - \gamma$. To simplify, let $\lambda = \mathrm{Re}(\underset{\sim}{\gamma})$; one can check that $-1 < \lambda$. As $|\underset{\sim}{h}(\omega)|$ is proportional to $|\omega|^{\lambda}$, we know that $\underset{\sim}{h}(\omega)$ is locally integrable ($-1 < \lambda$). Here, we continue as

$$\begin{aligned}
\hat{\varphi}_{a,b}^{\gamma;k}(\omega) =& \Big(\underbrace{\hat{\phi}(\omega) + \tfrac{\hat{\varphi}^{(k-1)}(0)}{(k-1)!\,\omega}\mathbb{1}_{|\omega|>1}}_{\underset{\sim}{\hat{\phi}}(\omega)} - \tfrac{\hat{\varphi}^{(k-1)}(0)}{(k-1)!\,\omega}\mathbb{1}_{|\omega|>1} \Big)\underset{\sim}{h}(\omega) \\
=& \underset{\sim}{\hat{\phi}}(\omega)\underset{\sim}{\hat{h}}(\omega) - \tfrac{\hat{\varphi}^{(k-1)}(0)}{(k-1)!}|\omega|^{\lambda-1+\mathrm{i}\,\mathrm{Im}(\gamma)}\Big(\tfrac{\mathbb{1}_{\omega>1}}{a} + \tfrac{(-1)^{k-1}\mathbb{1}_{\omega<-1}}{b}\Big),
\end{aligned} \tag{65}$$

where $\mathbb{1}_I$ is the indicator function for the set I. The function $\underset{\sim}{\hat{\phi}}(\omega)$ is essentially equal to $\frac{\hat{\varphi}(\omega) - \sum_{i=0}^{k-2}\frac{\hat{\varphi}^{(i)}(0)}{i!}\omega^i}{\omega^k}$ for $|\omega| > 1$, and is bounded for $|\omega| < 1$. By invoking Lemma 4.3 it is not difficult to see that $|\underset{\sim}{\hat{\phi}}(\omega)|(1+\omega^2)$ is bounded. Thus, $|\underset{\sim}{\hat{\phi}}(\omega)\underset{\sim}{\hat{h}}(\omega)| \leq c\frac{|\omega|^{\lambda}}{1+\omega^2} \in L_1$; this implies that $\underset{\sim}{\hat{\phi}}(\omega)\underset{\sim}{\hat{h}}(\omega)$ has a continuous and bounded inverse Fourier. Next, we show that the inverse Fourier of $\omega^{\beta}\mathbb{1}_{\omega>1}$ is given by

$$\begin{aligned}
\mathcal{F}^{-1}\Big\{\omega^{\beta}\mathbb{1}_{\omega>1}\Big\}(x) =& \tfrac{1}{2\pi}\frac{\Gamma(\beta+1, -\mathrm{i}x)}{(-\mathrm{i}x)^{\beta+1}} \\
=& \begin{cases} \frac{1}{2\pi}\frac{\Gamma(\beta+1)}{(-\mathrm{i}x)^{\beta+1}} - \frac{1}{2\pi}\sum_{m=0}^{\infty}\frac{(-1)^m(-\mathrm{i}x)^m}{m!(\beta+1+m)}, & \beta \in \mathbb{C}\setminus\mathbb{Z}^-, \\ \frac{-1}{2\pi}\Big(\gamma + \ln|x| - \mathrm{i}\frac{\pi}{2}\mathrm{sign}(x) + \sum_{m=1}^{\infty}\frac{(-\mathrm{i}x)^m}{m\,m!}\Big), & \beta = -1, \end{cases}
\end{aligned} \tag{66}$$

where $\Gamma(\cdot,\cdot)$ is the upper branch of the incomplete gamma function, and γ is the Euler-Mascheroni constant. We first consider the case $\beta \notin \mathbb{Z}^-$. Since $\omega^{\beta}\mathbb{1}_{\omega>1}$ is not necessarily in L_1, we need to apply some techniques common for deriving the Fourier transform of tempered generalized function:

$$\begin{aligned}
\mathcal{F}^{-1}\Big\{\omega^{\beta}\mathbb{1}_{\omega>1}\Big\}(x) =& \tfrac{1}{2\pi}\int_1^{\infty}\omega^{\beta}\mathrm{e}^{\mathrm{i}\omega x}\mathrm{d}\omega = \tfrac{1}{2\pi}\lim_{\mu\to 0^+}\int_1^{\infty}\omega^{\beta}\mathrm{e}^{-(\mu-\mathrm{i}x)\omega}\mathrm{d}\omega \\
=& \tfrac{1}{2\pi}\lim_{\mu\to 0^+}\lim_{M\to+\infty}\int_1^{M}\omega^{\beta}\mathrm{e}^{-(\mu-\mathrm{i}x)\omega}\mathrm{d}\omega \\
=& \tfrac{1}{2\pi}\lim_{\mu\to 0^+}\lim_{M\to+\infty}\frac{1}{(\mu-\mathrm{i}x)^{\beta+1}}\int_{\overrightarrow{\mathcal{C}}}\tau^{\beta}\mathrm{e}^{-\tau}\mathrm{d}\tau,
\end{aligned} \tag{67}$$

where $\overrightarrow{\mathcal{C}}$ stands for the finite line that connects the two points $\mu - \mathrm{i}x$ and $(\mu - \mathrm{i}x)M$ in the complex plane, and the latter integral is interpreted as a contour integration. Note that $\overrightarrow{\mathcal{C}}$ lies strictly on the right side of the imaginary axis, and $z^{\beta}\mathrm{e}^{-z}$ is analytic in this region. Further, $\gamma(\beta+1, z)$ (the lower branch of the incomplete gamma function) is an anti-derivative for this function ($\beta \notin \mathbb{Z}^-$). Thus, (67) can be rewritten as

$$\mathcal{F}^{-1}\left\{\omega^{\beta}\mathbb{1}_{\omega>1}\right\}(x)=\tfrac{1}{2\pi}\lim_{\mu\to0^+}\lim_{M\to+\infty}\frac{\gamma\big(\beta+1,(\mu-\mathrm{i}x)M\big)-\gamma\big(\beta+1,\mu-\mathrm{i}x\big)}{(\mu-\mathrm{i}x)^{\beta+1}}. \tag{68}$$

It is known that $\lim_{|\zeta|\to\infty}\gamma(s,\zeta)=\Gamma(s)$, given that $|\measuredangle\zeta|<\frac{\pi}{2}$ while $|\zeta|\to\infty$. Indeed, this is the case for $\zeta=(\mu-\mathrm{i}x)M$ as $M\to+\infty$. Therefore,

$$\begin{aligned}\mathcal{F}^{-1}\left\{\omega^{\beta}\mathbb{1}_{\omega>1}\right\}(x)&=\tfrac{1}{2\pi}\lim_{\mu\to0^+}\frac{\Gamma(\beta+1)-\gamma\big(\beta+1,\mu-\mathrm{i}x\big)}{(\mu-\mathrm{i}x)^{\beta+1}}\\&=\tfrac{1}{2\pi}\frac{\Gamma(\beta+1)-\gamma(\beta+1,-\mathrm{i}x)}{(-\mathrm{i}x)^{\beta+1}}\\&=\tfrac{1}{2\pi}\frac{\Gamma(\beta+1)}{(-\mathrm{i}x)^{\beta+1}}-\tfrac{1}{2\pi}\sum_{m=0}^{\infty}\frac{(-1)^m(-\mathrm{i}x)^m}{m!(\beta+1+m)},\end{aligned} \tag{69}$$

which proves the first part of the claim in (66). Note that $\beta=-1$ does not satisfy many of the results used in the above argument. Hence, we consider this case separately:

$$\mathcal{F}^{-1}\left\{\tfrac{\mathbb{1}_{\omega>1}}{\omega}\right\}(x)=\tfrac{1}{2\pi}\int_1^{\infty}\frac{\mathrm{e}^{\mathrm{i}\omega t}}{\omega}\mathrm{d}\omega=\tfrac{1}{2\pi}E_1(-\mathrm{i}t),$$

where E_1 is the analytic continuation of the exponential integral function. The expression in (66) is in fact a well-known series expansion of $E_1(\zeta)$ at $\zeta=-\mathrm{i}t$.

The inverse Fourier transform of $|\omega|^{\beta}\mathbb{1}_{\omega<-1}$ could also be obtained from (66) via

$$\mathcal{F}^{-1}\left\{|\omega|^{\beta}\mathbb{1}_{\omega<-1}\right\}(x)=\mathcal{F}^{-1}\left\{\omega^{\beta}\mathbb{1}_{\omega>1}\right\}(-x), \tag{70}$$

which is valid due to the axis-flipping property of the Fourier transform ($\omega\to-\omega$).

We now use (66) to characterize the inverse Fourier of $\widehat{\varphi}^{\gamma;k}_{a,b}(\omega)$ in (65). Except the term $\widehat{\underset{\sim}{\phi}}(\omega)\widehat{\underset{\sim}{h}}(\omega)$ which has a bounded and continuous inverse Fourier, we have that

$$\begin{aligned}&\mathcal{F}^{-1}\left\{\tfrac{\widehat{\varphi}^{(k-1)}(0)}{(k-1)!}|\omega|^{\lambda-1+\mathrm{i}\,\mathrm{Im}(\gamma)}\left(\tfrac{\mathbb{1}_{\omega>1}}{a}+\tfrac{(-1)^{k-1}\mathbb{1}_{\omega<-1}}{b}\right)\right\}(x)\\&\quad=\tfrac{\widehat{\varphi}^{(k-1)}(0)}{2\pi\,(k-1)!}\Bigg(\underbrace{\frac{\Gamma(\lambda+\mathrm{i}\,\mathrm{Im}(\gamma),-\mathrm{i}x)}{a\,(-\mathrm{i}x)^{\lambda+\mathrm{i}\,\mathrm{Im}(\gamma)}}+\frac{(-1)^{k-1}\Gamma(\lambda+\mathrm{i}\,\mathrm{Im}(\gamma),\mathrm{i}x)}{b\,(\mathrm{i}x)^{\lambda+\mathrm{i}\,\mathrm{Im}(\gamma)}}}_{\Xi}\Bigg),\end{aligned} \tag{71}$$

where for $\lambda\neq0$ or $\mathrm{Im}(\gamma)\neq0$ (equivalently $\underset{\sim}{\gamma}\neq0$), Ξ can be expressed as

$$\tfrac{\Gamma(\lambda+\mathrm{i}\,\mathrm{Im}(\gamma))}{a(-\mathrm{i}x)^{\lambda+\mathrm{i}\,\mathrm{Im}(\gamma)}}+\tfrac{(-1)^{k-1}\Gamma(\lambda+\mathrm{i}\,\mathrm{Im}(\gamma))}{b(\mathrm{i}x)^{\lambda+\mathrm{i}\,\mathrm{Im}(\gamma)}}-\sum_{m=0}^{\infty}(-1)^m\tfrac{b(-\mathrm{i}x)^m+(-1)^{k-1}a(\mathrm{i}x)^m}{a\,b\,m!\,(\lambda+\mathrm{i}\,\mathrm{Im}(\gamma)+m)}, \tag{72}$$

and for $\lambda=\mathrm{Im}(\gamma)=0$ (equivalently, $\underset{\sim}{\gamma}=0$) as

$$-\tfrac{(b+a(-1)^{k-1})(\gamma+\ln|x|)}{a\,b}+\tfrac{\mathrm{i}\pi\,\mathrm{sign}(x)(b-(-1)^{k-1}a)}{2ab}-\sum_{m=1}^{\infty}(-1)^m\tfrac{b(-\mathrm{i}x)^m+a(-1)^{k-1}(\mathrm{i}x)^m}{a\,b\,m\,m!}. \tag{73}$$

Because of the fact that

$$\left|\sum_{m=1}^{\infty}(-1)^m\tfrac{b(-\mathrm{i}x)^m+(-1)^{k-1}a(\mathrm{i}x)^m}{a\,b\,m!\,(\lambda+\mathrm{i}\,\mathrm{Im}(\gamma)+m)}\right|\le\tfrac{|a|+|b|}{|a\,b\,(\lambda+1+\mathrm{i}\,\mathrm{Im}(\gamma))|}\underbrace{\sum_{m=1}^{\infty}\tfrac{|x|^m}{m!}}_{\mathrm{e}^{|x|}-1},$$

we conclude that $\sum_{m=0}^{\infty}(-1)^m\frac{b(-\mathrm{i}x)^m+(-1)^{k-1}a(\mathrm{i}x)^m}{a\,b\,m!\,(\lambda+\mathrm{i}\,\mathrm{Im}(\gamma)+m)}$ converges to a continuous function. Further, since $\pm\mathrm{i}x=\exp\big(\log|x|\pm\mathrm{i}\frac{\pi}{2}\mathrm{sign}(x)\big)$, we know that

$$\tfrac{1}{(\pm\mathrm{i}x)^{\lambda+\mathrm{i}\,\mathrm{Im}(\gamma)}}=\exp\Big(-\lambda\log|x|\pm\tfrac{\pi}{2}\,\mathrm{Im}(\gamma)\,\mathrm{sign}(x)-\mathrm{i}\big(\mathrm{Im}(\gamma)\,\log|x|\pm\tfrac{\pi}{2}\,\lambda\,\mathrm{sign}(x)\big)\Big) \tag{74}$$

is also a continuous function except possibly at $x=0$. Overall, we conclude that $\varphi^{\gamma;k}_{a,b}(x)$ in (65) (the inverse Fourier of $\widehat{\varphi}^{\gamma;k}_{a,b}$) is well-defined as a function and is continuous everywhere except possibly at $x=0$; around $x=0$, the behavior of $\varphi^{\gamma;k}_{a,b}(x)$ is determined by the $\frac{1}{(\pm\mathrm{i}x)^{\lambda+\mathrm{i}\,\mathrm{Im}(\gamma)}}$ terms for $\underset{\sim}{\gamma}\neq0$, and $-\frac{b+a(-1)^{k-1}}{a\,b}\ln|x|+\frac{\mathrm{i}\pi(b-(-1)^{k-1}a)}{2ab}\mathrm{sign}(x)$ term for $\underset{\sim}{\gamma}=0$. To proceed, we check different cases separately.

(i) $\mathrm{Re}(\gamma) \notin \mathbb{Z}$ and $k = \lfloor \mathrm{Re}(\gamma) \rfloor$. Thus, $-1 < \lambda < 0$. In this case,

$$\lim_{|x|\to 0} \left| \frac{1}{(\pm \mathrm{i}x)^{\lambda + \mathrm{i}\,\mathrm{Im}(\gamma)}} \right| = 0,$$

which results in

$$\lim_{x\to 0} \frac{\Gamma(\lambda + \mathrm{i}\,\mathrm{Im}(\gamma), \pm \mathrm{i}x)}{(\pm \mathrm{i}x)^{\lambda + \mathrm{i}\,\mathrm{Im}(\gamma)}} = \frac{-1}{\lambda + \mathrm{i}\,\mathrm{Im}(\gamma)}.$$

Consequently, $\frac{\Gamma(\lambda+\mathrm{i}\,\mathrm{Im}(\gamma),\pm \mathrm{i}x)}{(\pm \mathrm{i}x)^{\lambda+\mathrm{i}\,\mathrm{Im}(\gamma)}}$ is continuous everywhere including at $x = 0$. This proves claim (i).

(ii) $\mathrm{Re}(\gamma) \in \mathbb{Z}$, $\mathrm{Im}(\gamma) \neq 0$, and $k = \lfloor \mathrm{Re}(\gamma) \rfloor$. Thus, $\lambda = 0$ but $\underset{\sim}{\gamma} \neq 0$. Recalling (74), we observe that

$$\left| \frac{1}{(\mathrm{i}x)^{\mathrm{i}\,\mathrm{Im}(\gamma)}} \right| \leq \mathrm{e}^{\frac{\pi}{2}|\mathrm{Im}(\gamma)|}.$$

Consequently, $\frac{\Gamma(\mathrm{i}\,\mathrm{Im}(\gamma),\mathrm{i}x)}{(\mathrm{i}x)^{\mathrm{i}\,\mathrm{Im}(\gamma)}}$ (and in turn, $\varphi^{\gamma;k}_{a,b}$) is bounded around $x = 0$. However, the oscillating nature of $\frac{\Gamma(\mathrm{i}\,\mathrm{Im}(\gamma),\mathrm{i}x)}{(\mathrm{i}x)^{\mathrm{i}\,\mathrm{Im}(\gamma)}}$ around $x = 0$ makes it discontinuous.

(iii) $\gamma \in \mathbb{Z}$, $h^{\gamma}_{a,b}(\omega) \equiv c\,\omega^{\gamma}$, and $k = \lfloor \mathrm{Re}(\gamma) \rfloor$. This implies that $\underset{\sim}{\gamma} = 0$ and $b = c\,(-1)^k = (-1)^k a$. According to (73), the term $\ln|x|$ vanishes. Therefore, the inverse Fourier remains bounded around $x = 0$; however, due to the existence of the $\mathrm{sign}(x)$ term, it will be discontinuous at $x = 0$.

(iv) $\gamma \in \mathbb{Z}$, $h^{\gamma}_{a,b}(\omega) \not\equiv c\,\omega^{\gamma}$, and $k = \lfloor \mathrm{Re}(\gamma) \rfloor$. This case is very similar to the previous case, except that $b \neq (-1)^k a$ and the $\log|x|$ term remains. Thus, $\varphi^{\gamma;k}_{a,b}(x)$ is proportional to $\log|x|$ around $x = 0$.

(v) $k > \lfloor \mathrm{Re}(\gamma) \rfloor$. As a result $0 < \lambda = k - \mathrm{Re}(\gamma)$ and obviously $\underset{\sim}{\gamma} \neq 0$. This implies that $\frac{1}{(\mp \mathrm{i}x)^{\lambda+\mathrm{i}\,\mathrm{Im}(\gamma)}}$, and as a result $\varphi^{\gamma;k}_{a,b}(x)$, are singular at the origin such that $|x|^{\lambda}|\varphi^{\gamma;k}_{a,b}(x)|$ is bounded and discontinuous around $x = 0$. Note that, there is no choice of a, b such that the two singularities in (72) completely cancel each other out (the cancellation can happen either for $x > 0$ or for $x < 0$). ■

Lemma 4.6. *Let $\gamma \in \mathbb{C} \setminus \mathbb{Z}$, then, for all $k \geq 1$ we have that*

$$2\pi \mathcal{F}^{-1}\left\{ \omega_+^{\gamma-k}\left(\mathrm{e}^{\mathrm{i}\omega t} - \sum_{j=0}^{k-1} \frac{(\mathrm{i}\omega t)^j}{j!} \right) \right\}(x) = \frac{\Gamma(\gamma-k+1)}{\big(-\mathrm{i}(x+t)\big)^{\gamma-k+1}} - \sum_{j=0}^{k-1} \frac{(\mathrm{i}t)^{k-j-1}\Gamma(\gamma-j)}{(k-j-1)!(-\mathrm{i}x)^{\gamma-j}}. \tag{75}$$

Proof. For non-integer γ, it is known that [71,55]

$$2\pi \mathcal{F}^{-1}\{ w_+^{\gamma} \}(x) = \frac{\Gamma(\gamma+1)}{(-\mathrm{i}x)^{\gamma+1}}. \tag{76}$$

The above result is the key to prove the claim in Lemma 4.6:

$$\begin{aligned}
&\frac{\Gamma(\gamma-k+1)}{\big(-\mathrm{i}(x+t)\big)^{\gamma-k+1}} - \sum_{j=0}^{k-1} \frac{(\mathrm{i}t)^{k-j-1}\Gamma(\gamma-j)}{(k-j-1)!(-\mathrm{i}x)^{\gamma-j}} \\
&= 2\pi \mathcal{F}^{-1}\{ w_+^{\gamma-k} \}(x+t) - 2\pi \sum_{j=0}^{k-1} \frac{(\mathrm{i}t)^{k-j-1}}{(k-j-1)!} \mathcal{F}^{-1}\{ w_+^{\gamma-j-1} \}(x) \\
&= 2\pi \mathcal{F}^{-1}\{ w_+^{\gamma-k} \mathrm{e}^{\mathrm{i}\omega t} \}(x) - 2\pi \sum_{j=0}^{k-1} \mathcal{F}^{-1}\left\{ \frac{(\mathrm{i}t)^{k-j-1}}{(k-j-1)!} w_+^{\gamma-j-1} \right\}(x) \\
&= 2\pi \mathcal{F}^{-1}\left\{ w_+^{\gamma-k} \mathrm{e}^{\mathrm{i}\omega t} - \sum_{j=0}^{k-1} \frac{(\mathrm{i}t)^{k-j-1}}{(k-j-1)!} w_+^{\gamma-j-1} \right\}(x) \\
&= 2\pi \mathcal{F}^{-1}\left\{ w_+^{\gamma-k}\left(\mathrm{e}^{\mathrm{i}\omega t} - \sum_{j=0}^{k-1} \frac{(\mathrm{i}\omega t)^{k-j-1}}{(k-j-1)!} \right) \right\}(x),
\end{aligned}$$

which confirms the claim. ■

5. Proofs

In this section, we prove the results stated in Section 2. In the proofs we make use of the lemmas in Section 4.

5.1. Proof of Theorem 2.3

As $\mathrm{D}^{\gamma}_{a,b}$ corresponds to a Fourier multiplier, it is a convolutional operator. Since $\varphi \in \mathcal{S}$ is infinitely differentiable, this also carries over to the convolution of φ with any generalized function. Thus, $\big(\mathrm{D}^{\gamma}_{a,b}\varphi\big)(x)$ is infinitely differentiable.

The statement (ii) is a classical result which simply follows from the definition of Schwartz functions. As we take advantage of Lemma 4.2 later on, let $T \in [2,4[$ and consider the following system input:

$$\varphi(x) - T\,\varphi(Tx) \overset{\mathrm{D}^{\gamma}_{a,b}}{\longmapsto} \varphi^{\gamma}_{a,b}(x) - T^{\gamma+1}\varphi^{\gamma}_{a,b}(Tx),$$

where $\varphi^{\gamma}_{a,b}(x) = \big(\mathrm{D}^{\gamma}_{a,b}\,\varphi\big)(x)$. Using the Fourier representation we have that

$$\varphi^{\gamma}_{a,b}(x) - T^{\gamma+1}\varphi^{\gamma}_{a,b}(Tx) = \tfrac{1}{2\pi}\int_{\mathbb{R}} \big(\widehat{\varphi}(\omega) - \widehat{\varphi}(\tfrac{\omega}{T})\big) h^{\gamma}_{a,b}(\omega) \mathrm{e}^{\mathrm{i}\omega x} \mathrm{d}\omega. \tag{77}$$

Since $\varphi \in \mathcal{S}$, it is well-known that $\widehat{\varphi} \in \mathcal{S}$. Hence, $\widehat{\varphi}$ is infinitely differentiable and $\Big|\widehat{\varphi}^{(j)}(\omega)\Big|\big(1 + |\omega|^{r+j}\big)$ is bounded for all $1 \le j, r$. According to Lemma 4.1, $\big(\widehat{\varphi}(\omega) - \widehat{\varphi}(\tfrac{\omega}{T})\big) h^{\gamma}_{a,b}(\omega)$ is $n_\gamma = \lceil \mathrm{Re}(\gamma)\rceil + 1$ times continuously differentiable and $\frac{\mathrm{d}^m}{\mathrm{d}\omega^m}\Big(\big(\widehat{\varphi}(\omega) - \widehat{\varphi}(\tfrac{\omega}{T})\big) h^{\gamma}_{a,b}(\omega)\Big)$ has finite L_1 norm for all $1 \le m \le n_\gamma$ and $T \in [2,4[$.[6] Hence, by integration by parts, we can rewrite (77) as

$$\begin{aligned}\varphi^{\gamma}_{a,b}(x) - T^{\gamma+1}\varphi^{\gamma}_{a,b}(Tx) &= \tfrac{1}{2\pi}(\tfrac{-1}{x})^{n_\gamma}\int_{\mathbb{R}} \underbrace{\frac{\mathrm{d}^{n_\gamma}}{\mathrm{d}\omega^{n_\gamma}}\Big(\big(\widehat{\varphi}(\omega) - \widehat{\varphi}(\tfrac{\omega}{T})\big) h^{\gamma}_{a,b}(\omega)\Big)}_{\widehat{\psi}_T(\omega)} \mathrm{e}^{\mathrm{i}\omega x}\mathrm{d}\omega \\ &= \tfrac{1}{2\pi}(\tfrac{-1}{x})^{n_\gamma}\int_{\mathbb{R}} \widehat{\psi}_T(\omega)\mathrm{e}^{\mathrm{i}\omega x}\mathrm{d}\omega = (\tfrac{-1}{x})^{n_\gamma}\psi_T(x).\end{aligned} \tag{78}$$

Again based on Lemma 4.1, we have that $\|\widehat{\psi}_T\|_1 \le c4^{\mathrm{Re}(\gamma)+1-\lceil \mathrm{Re}(\gamma)\rceil}$. Thus,

$$\Big|\varphi^{\gamma}_{a,b}(x) - T^{\gamma+1}\varphi^{\gamma}_{a,b}(Tx)\Big| \le \frac{\|\psi_T\|_\infty}{|x|^{n_\gamma}} = \frac{\|\widehat{\psi}_T\|_1}{|x|^{n_\gamma}} < \frac{c\,4^{\mathrm{Re}(\gamma)+1-\lceil \mathrm{Re}(\gamma)\rceil}}{|x|^{n_\gamma}}.$$

This implies that $\varphi^{\gamma}_{a,b}$ satisfies the constraint of Lemma 4.2 with γ and m in Lemma 4.2 being replaced with $\gamma+1$ and n_γ, respectively. Thus, $\bar{c} \in \mathbb{R}^+$ exists such that

$$\forall x \in \mathbb{R}\setminus\{0\}: \quad \Big|\varphi^{\gamma}_{a,b}(x)\Big| \le \begin{cases} \frac{\bar{c}}{|x|^{\mathrm{Re}(\gamma+1)}}, & n_\gamma \ne \mathrm{Re}(\gamma+1), \\ \bar{c}\,\frac{1+\big|\log|x|\big|}{|x|^{\mathrm{Re}(\gamma+1)}}, & n_\gamma = \mathrm{Re}(\gamma+1), \end{cases} \tag{79}$$

where we used the fact that $\min(|x|^{\mathrm{Re}(\gamma+1)}, |x|^{n_\gamma}) = |x|^{\mathrm{Re}(\gamma+1)}$, as $n_\gamma = \lceil \mathrm{Re}(\gamma)\rceil + 1 \ge \mathrm{Re}(\gamma+1)$. We also recall that $\mathrm{Re}(\gamma) > -1$; thus, $\widehat{\varphi}(\omega)h^{\gamma}_{a,b}(\omega)$ is locally integrable. Further, $\widehat{\varphi} \in \mathcal{S}$ has rapid decay as $|\omega| \to \infty$. These two properties imply that $\widehat{\varphi}(\omega)h^{\gamma}_{a,b}(\omega) \in L_1$. Consequently, $\varphi^{\gamma}_{a,b}(x)$ (the inverse Fourier of $\widehat{\varphi}(\omega)h^{\gamma}_{a,b}(\omega)$) is both bounded and continuous:

$$\exists\, M_\varphi \in \mathbb{R}^+,\ \forall x \in \mathbb{R}: \quad \Big|\varphi^{\gamma}_{a,b}(x)\Big| \le M_\varphi. \tag{80}$$

For $\mathrm{Re}(\gamma) \notin \mathbb{Z}$, we have that $n_\gamma \ne \mathrm{Re}(\gamma+1)$, and can combine (79) and (80) as

$$\begin{aligned}\Big|\varphi^{\gamma}_{a,b}(x)\Big| &\le \min\Big(\tfrac{\bar{c}}{|x|^{\mathrm{Re}(\gamma+1)}}, M_\varphi\Big) \le \frac{2}{\frac{|x|^{\mathrm{Re}(\gamma+1)}}{\bar{c}} + \frac{1}{M_\varphi}} = \frac{2\bar{c}}{|x|^{\mathrm{Re}(\gamma+1)} + \frac{\bar{c}}{M_\varphi}} \\ &\le \frac{2(\bar{c}+M_\varphi)}{1+|x|^{\mathrm{Re}(\gamma+1)}},\end{aligned} \tag{81}$$

which is the same bound as claimed in statement (i). For $\mathrm{Re}(\gamma) \in \mathbb{Z}$ which coincides with $n_\gamma = \mathrm{Re}(\gamma+1)$, we have that:

$$\Big|\varphi^{\gamma}_{a,b}(x)\Big| \le \min\Big(\bar{c}\,\frac{1+\big|\log|x|\big|}{|x|^{\mathrm{Re}(\gamma+1)}}, M_\varphi\Big) \le \frac{2(\bar{c}+M_\varphi)\log(1+|x|)}{1+|x|^{\mathrm{Re}(\gamma+1)}}, \tag{82}$$

which is again the required bound in statement (iii). Here (82) holds because

[6] Note that when $h^{\gamma}_{a,b}(\omega) = c\omega^n$ for some $n \in \mathbb{N}$ and $c \in \mathbb{C}$, $\big(\widehat{\varphi}(\omega) - \widehat{\varphi}(\tfrac{\omega}{T})\big) h^{\gamma}_{a,b}(\omega)$ is infinitely differentiable, and these are the only cases with this property.

$$\frac{2(\bar{c}+M_\varphi)\log(1+|x|)}{1+|x|^{\mathrm{Re}(\gamma+1)}} \geq \begin{cases} M_\varphi, & |x|\leq 1,\\ \bar{c}\,\frac{1+\left|\log|x|\right|}{|x|^{\mathrm{Re}(\gamma+1)}}, & |x|\geq 1.\end{cases}$$

Based on (81) and (82), it is now easy to show that $\varphi^{\gamma}_{a,b}\in L_p$ for $p>\frac{1}{\mathrm{Re}(\gamma)+1}$. The continuity of the mapping from $\mathcal{S}$ to L_p also follows from linearity and boundedness of the mapping. ■

5.2. Proof of Proposition 2.5

We first show that $\mathrm{I}^{\gamma;k}_{a,b}$ is well-defined for $\varphi\in\mathcal{S}$. For this purpose, we express $\big(\mathrm{I}^{\gamma;k}_{a,b}\varphi\big)(x)$ as

$$\begin{aligned}\big(\mathrm{I}^{\gamma;k}_{a,b}\varphi\big)(x) &= \frac{1}{2\pi}\int_{\mathbb{R}}\widehat{\varphi}(\omega)\Big(\mathrm{e}^{\mathrm{i}\omega x}-\sum_{j=0}^{k-1}\frac{(\mathrm{i}x)^j}{j!}\omega^j\Big)\frac{1}{h^{\gamma}_{a,b}(\omega)}\,\mathrm{d}\omega\\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\widehat{\varphi}(\omega)\underbrace{\frac{\mathrm{e}^{\mathrm{i}\omega x}-\sum_{j=0}^{k-1}\frac{(\mathrm{i}x)^j}{j!}\omega^j}{\omega^k}}_{\widehat{\phi}(\omega)}\;\underbrace{\frac{\omega^k}{h^{\gamma}_{a,b}(\omega)}}_{\underset{\sim}{h}(\omega)}\,\mathrm{d}\omega.\end{aligned}\tag{83}$$

Lemma 4.3 implies that $\widehat{\phi}(\omega)$ is continuous and decays no slower than $\frac{1}{1+|\omega|}$. As $\varphi\in\mathcal{S}$, $\widehat{\varphi}$ is also in $\mathcal{S}$. Specially, $\widehat{\varphi}$ is continuous everywhere and asymptotically decays faster than $\frac{1}{1+|\omega|^r}$ for any $r>0$. It is easy to check that $\underset{\sim}{h}(\omega)=h^{\underset{\sim}{\gamma}}_{\underset{\sim}{a},\underset{\sim}{b}}(\omega)$ is homogeneous of degree $\underset{\sim}{\gamma}=k-\gamma$ with $-1<\mathrm{Re}(\underset{\sim}{\gamma})$, and $\underset{\sim}{a}=\frac{1}{a}$, $\underset{\sim}{b}=\frac{(-1)^k}{b}$. Hence, $\big|\widehat{\varphi}(\omega)\widehat{\phi}(\omega)\underset{\sim}{h}(\omega)\big|$ could be upper-bounded by $c|\omega|^{\mathrm{Re}(\underset{\sim}{\gamma})}$ around $\omega=0$, and by $c|\omega|^{-2}$ as $|\omega|\to\infty$ ($\widehat{\varphi}$ has a super-polynomial decay rate). Consequently, $\big|\widehat{\varphi}(\omega)\widehat{\phi}(\omega)\underset{\sim}{h}(\omega)\big|$ is dominated by $c\min\big(|\omega|^{\mathrm{Re}(\underset{\sim}{\gamma})},|\omega|^{-2}\big)$ when $c\in\mathbb{R}^+$ is large enough. The latter has a finite integral over the real line, which shows that the integral in (83) is well-defined. To see that $\mathrm{I}^{\gamma;k}_{a,b}$ is scale-invariant, pick any $T>0$ and consider

$$\begin{aligned}\varphi(Tx)\overset{\mathrm{I}^{\gamma;k}_{a,b}}{\longmapsto}\;&\frac{1}{2\pi}\int_{\mathbb{R}}\frac{1}{T}\widehat{\varphi}\big(\tfrac{\omega}{T}\big)\frac{\mathrm{e}^{\mathrm{i}\omega x}-\Big(\sum_{j=0}^{k-1}\frac{(\mathrm{i}x)^j}{j!}\omega^j\Big)}{h^{\gamma}_{a,b}(\omega)}\,\mathrm{d}\omega\\ &=\frac{1}{2\pi}\int_{\mathbb{R}}\widehat{\varphi}(\zeta)\frac{\mathrm{e}^{\mathrm{i}\zeta Tx}-\Big(\sum_{j=0}^{k-1}\frac{(\mathrm{i}x)^j}{j!}(T\zeta)^j\Big)}{h^{\gamma}_{a,b}(T\zeta)}\,\mathrm{d}\zeta\\ &=\frac{T^{-\gamma}}{2\pi}\int_{\mathbb{R}}\widehat{\varphi}(\zeta)\frac{\mathrm{e}^{\mathrm{i}\zeta Tx}-\Big(\sum_{j=0}^{k-1}\frac{(\mathrm{i}Tx)^j}{j!}\zeta^j\Big)}{h^{\gamma}_{a,b}(T\zeta)}\,\mathrm{d}\zeta=T^{-\gamma}\big(\mathrm{I}^{\gamma;k}_{a,b}\varphi\big)(Tx).\end{aligned}$$

Next, we show that $\mathrm{I}^{\gamma;k}_{a,b}$ is the right-inverse of the LSI operator $\mathrm{D}^{\gamma}_{a,b}$ when $k=\lfloor\mathrm{Re}(\gamma)\rfloor$ or $k=\lceil\mathrm{Re}(\gamma)\rceil$:

$$\begin{aligned}\big(\mathrm{D}^{\gamma}_{a,b}\mathrm{I}^{\gamma;k}_{a,b}\varphi\big)(x) &=\mathcal{F}^{-1}_{\omega}\Big\{h^{\gamma}_{a,b}(\omega)\mathcal{F}_{\tau}\Big\{\big(\mathrm{I}^{\gamma;k}_{a,b}\varphi\big)(\tau)\Big\}(\omega)\Big\}(x)\\ &=\mathcal{F}^{-1}_{\omega}\Big\{h^{\gamma}_{a,b}(\omega)\mathcal{F}_{\tau}\Big\{\frac{1}{2\pi}\int_{\mathbb{R}}\widehat{\varphi}(\zeta)\frac{\mathrm{e}^{\mathrm{i}\zeta\tau}-\sum_{j=0}^{k-1}\frac{(\mathrm{i}\tau)^j}{j!}\zeta^j}{h^{\gamma}_{a,b}(\zeta)}\mathrm{d}\zeta\Big\}(\omega)\Big\}(x)\\ &=\frac{1}{2\pi}\mathcal{F}^{-1}_{\omega}\Big\{h^{\gamma}_{a,b}(\omega)\int_{\mathbb{R}}\frac{\widehat{\varphi}(\zeta)}{h^{\gamma}_{a,b}(\zeta)}\mathcal{F}_{\tau}\Big\{\mathrm{e}^{\mathrm{i}\zeta\tau}-\sum_{j=0}^{k-1}\frac{(\mathrm{i}\tau)^j}{j!}\zeta^j\Big\}(\omega)\mathrm{d}\zeta\Big\}(x)\\ &=\mathcal{F}^{-1}_{\omega}\Big\{h^{\gamma}_{a,b}(\omega)\int_{\mathbb{R}}\frac{\widehat{\varphi}(\zeta)}{h^{\gamma}_{a,b}(\zeta)}\Big(\delta(\omega-\zeta)-\sum_{j=0}^{k-1}\frac{1}{j!}\zeta^j\delta^{(j)}(\omega)\Big)\mathrm{d}\zeta\Big\}(x)\\ &=\mathcal{F}^{-1}_{\omega}\Big\{\int_{\mathbb{R}}\frac{\widehat{\varphi}(\zeta)}{h^{\gamma}_{a,b}(\zeta)}\Big(h^{\gamma}_{a,b}(\omega)\delta(\omega-\zeta)-\sum_{j=0}^{k-1}\frac{1}{j!}\zeta^j h^{\gamma}_{a,b}(\omega)\delta^{(j)}(\omega)\Big)\mathrm{d}\zeta\Big\}(x).\end{aligned}\tag{84}$$

We know that $\frac{\mathrm{d}^j}{\mathrm{d}\omega^j}h^{\gamma}_{a,b}(\omega)\Big|_{\omega=0}=0$ for all $0\leq j<\mathrm{Re}(\gamma)$; this means that $h^{\gamma}_{a,b}(\omega)\delta^{(j)}(\omega)\equiv 0$ for $0\leq j<\mathrm{Re}(\gamma)$. In case $k=\lfloor\mathrm{Re}(\gamma)\rfloor$ or $k=\lceil\mathrm{Re}(\gamma)\rceil$, the interval $[0,\mathrm{Re}(\gamma)[$ covers the full range of required j values in the summation in (84). Hence, we drop this summation from (84) and establish that

$$\begin{aligned}\big(\mathrm{D}^{\gamma}_{a,b}\,\mathrm{I}^{\gamma;k}_{a,b}\,\varphi\big)(x)&=\mathcal{F}^{-1}_{\omega}\Big\{\int_{\mathbb{R}}\frac{\widehat{\varphi}(\zeta)}{h^{\gamma}_{a,b}(\zeta)}h^{\gamma}_{a,b}(\omega)\delta(\omega-\zeta)\mathrm{d}\zeta\Big\}(x)\\&=\mathcal{F}^{-1}_{\omega}\Big\{\widehat{\varphi}(\omega)\frac{h^{\gamma}_{a,b}(\omega)}{h^{\gamma}_{a,b}(\omega)}\Big\}(x)=\varphi(x),\end{aligned}\tag{85}$$

which proves that $\mathrm{I}^{\gamma;k}_{a,b}$ is the right-inverse of $\mathrm{D}^{\gamma}_{a,b}$.

Our last task is to find the adjoint operator of $\mathrm{I}^{\gamma;k}_{a,b}$. Let $\varphi,\psi\in\mathcal{S}$. We can write

$$\begin{aligned}\big\langle\big(\mathrm{I}^{\gamma;k}_{a,b}\,\varphi\big)(x)\,,\,\psi(x)\big\rangle&=\int_{\mathbb{R}}\big(\mathrm{I}^{\gamma;k}_{a,b}\,\varphi\big)(x)\,\psi(x)\mathrm{d}x\\&=\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\widehat{\varphi}(\omega)\widehat{h}(\omega)\frac{\mathrm{e}^{\mathrm{i}\omega x}-\sum_{j=0}^{k-1}\frac{(\mathrm{i}x)^j}{j!}\omega^j}{h^{\gamma}_{a,b}(\omega)}\psi(x)\mathrm{d}\omega\mathrm{d}x\\&=\frac{1}{2\pi}\int_{\mathbb{R}}\frac{\widehat{\varphi}(\omega)}{h^{\gamma}_{a,b}(\omega)}\Big(\int_{\mathbb{R}}\Big(\mathrm{e}^{\mathrm{i}\omega x}-\sum_{j=0}^{k-1}\frac{(\mathrm{i}x)^j}{j!}\omega^j\Big)\psi(x)\mathrm{d}x\Big)\mathrm{d}\omega.\end{aligned}\tag{86}$$

We use Parseval's theorem to obtain

$$\begin{aligned}\int_{\mathbb{R}}\Big(\mathrm{e}^{\mathrm{i}\omega x}-\sum_{j=0}^{k-1}\frac{(\mathrm{i}x)^j}{j!}\omega^j\Big)\psi(x)\mathrm{d}x&=\frac{1}{2\pi}\int_{\mathbb{R}}\widehat{\psi}(\zeta)\mathcal{F}_x\Big\{\mathrm{e}^{\mathrm{i}\omega x}-\sum_{j=0}^{k-1}\frac{(\mathrm{i}x)^j}{j!}\omega^j\Big\}(-\zeta)\mathrm{d}\zeta\\&=\int_{\mathbb{R}}\widehat{\psi}(\zeta)\Big(\delta(-\zeta-\omega)-\sum_{j=0}^{k-1}\frac{\omega^j}{j!}\delta^{(j)}(\zeta)\Big)\mathrm{d}\zeta\\&=\widehat{\psi}(-\omega)-\sum_{j=0}^{k-1}\frac{(-\omega)^j}{j!}\widehat{\psi}^{(j)}(0).\end{aligned}\tag{87}$$

By rewriting (86) based on (87), we arrive at

$$\begin{aligned}\big\langle\big(\mathrm{I}^{\gamma;k}_{a,b}\,\varphi\big)(x)\,,\,\psi(x)\big\rangle&=\frac{1}{2\pi}\int_{\mathbb{R}}\frac{\widehat{\varphi}(\omega)}{h^{\gamma}_{a,b}(\omega)}\Big(\widehat{\psi}(-\omega)-\sum_{j=0}^{k-1}\frac{(-\omega)^j}{j!}\widehat{\psi}^{(j)}(0)\Big)\mathrm{d}\omega\\&=\int_{\mathbb{R}}\varphi(x)\mathcal{F}^{-1}_{\omega}\Big\{\frac{\widehat{\psi}(-\omega)-\sum_{j=0}^{k-1}\frac{(-\omega)^j}{j!}\widehat{\psi}^{(j)}(0)}{h^{\gamma}_{a,b}(\omega)}\Big\}(-x)\mathrm{d}x\\&=\int_{\mathbb{R}}\varphi(x)\mathcal{F}^{-1}_{\omega}\Big\{\frac{\widehat{\psi}(\omega)-\sum_{j=0}^{k-1}\frac{\omega^j}{j!}\widehat{\psi}^{(j)}(0)}{h^{\gamma}_{a,b}(-\omega)}\Big\}(x)\mathrm{d}x\\&=\big\langle\varphi(x)\,,\,\big(\mathrm{I}^{(\gamma;k)*}_{a,b}\,\psi\big)(x)\big\rangle,\end{aligned}\tag{88}$$

where $\big(\mathrm{I}^{(\gamma;k)*}_{a,b}\,\psi\big)(x)=\mathcal{F}^{-1}_{\omega}\Big\{\frac{\widehat{\psi}(\omega)-\sum_{j=0}^{k-1}\frac{\omega^j}{j!}\widehat{\psi}^{(j)}(0)}{h^{\gamma}_{a,b}(-\omega)}\Big\}(x)$. ■

5.3. Proof of Theorem 2.7

First note that Lemma 4.5 implies that for $\varphi\in\mathcal{S}$, the output $\big(\mathrm{I}^{(\gamma;k)*}_{a,b}\varphi\big)(x)$ is well-defined and continuous at $x\neq0$; further, this lemma describes the behavior of $\big(\mathrm{I}^{(\gamma;k)*}_{a,b}\varphi\big)(x)$ around $x=0$ the same way as claimed in Theorem 2.7. Thus, to complete the proof of Theorem 2.7, we need to investigate the decay properties of $\big(\mathrm{I}^{(\gamma;k)*}_{a,b}\varphi\big)(x)$ and the L_p spaces to which this function belongs.

For $k=\lfloor\mathrm{Re}(\gamma)\rfloor$ (Parts (iii)-(i) of Theorem 2.7), Lemma 4.5 shows that $\Big|\big(\mathrm{I}^{(\gamma;k)*}_{a,b}\varphi\big)(x)\Big|$ can be upper-bounded by $\nu+\kappa\Big|\log|x|\Big|$ at $|x|\leq1$ for some $\nu,\kappa\in\mathbb{R}^+\cup\{0\}$. Thus,

$$\int_{-1}^{1}\Big|\big(\mathrm{I}^{(\gamma;k)*}_{a,b}\varphi\big)(x)\Big|^p\mathrm{d}x\leq2\,\kappa^p\int_{0}^{1}\big(\tfrac{\nu}{\kappa}+|\log x|\big)^p\mathrm{d}x=2\,\kappa^p\mathrm{e}^{\frac{\nu}{\kappa}}\,\Gamma(p+1\,,\,\tfrac{\nu}{\kappa})<\infty,\tag{89}$$

for any $p>0$.

For $k > \lfloor \mathrm{Re}(\gamma) \rfloor$ (Parts (iv)-(v) of Theorem 2.7), Lemma 4.5 implies that $|x|^{k-\mathrm{Re}(\gamma)} \left| \left(\mathrm{I}_{a,b}^{(\gamma;k)*} \varphi \right)(x) \right|$ is bounded. Thus,

$$\int_{-1}^{1} \left| \left(\mathrm{I}_{a,b}^{(\gamma;k)*} \varphi \right)(x) \right|^p \mathrm{d}x \leq 2 \int_{0}^{1} \frac{c}{|x|^{p(k-\mathrm{Re}(\gamma))}} \mathrm{d}x < \infty, \tag{90}$$

for $p < \frac{1}{k-\mathrm{Re}(\gamma)}$.

We continue with a similar technique as in the proof of Lemma 4.5 and obtain that

$$\mathcal{F}\left\{ \left(\mathrm{I}_{a,b}^{(\gamma;k)*} \varphi \right)(x) \right\}(\omega) = \underbrace{\frac{\hat{\varphi}(\omega) - \sum_{j=0}^{k-1} \frac{\hat{\varphi}^{(j)}(0)}{j!} \omega^j}{\omega^k}}_{\hat{\phi}(\omega)} \underbrace{\frac{\omega^k}{h_{a,b}^{\gamma}(-\omega)}}_{\underset{\sim}{h}(\omega)} = \hat{\phi}(\omega) \underset{\sim}{h}(\omega), \tag{91}$$

where we know that $\underset{\sim}{h}(\omega) = h_{a',b'}^{\underset{\sim}{\gamma}}(\omega)$ is a homogeneous function of degree $\underset{\sim}{\gamma} = k - \gamma$ with $-1 < \mathrm{Re}(\underset{\sim}{\gamma})$ (and $a' = \frac{1}{b}$, $b' = \frac{(-1)^k}{a}$). Because of $\varphi \in \mathcal{S}$, $\hat{\varphi}$ is also in the Schwartz space and is infinitely differentiable; moreover, $\frac{\hat{\varphi}^{(j)}(\omega)}{1+|\omega|^m}$ is bounded for all $0 \leq j, m$. As a result, Lemma 4.3 implies that $\hat{\phi}$ is infinitely differentiable, and $\hat{\phi}^{(n)}(\omega)(1 + |\omega|^{n+1})$ is bounded for all $n \geq 0$.

Next, we consider the result of $\mathrm{I}_{a,b}^{(\gamma;k)*}$ to the input $T^{k+1}\varphi(Tx)$, where $T \in [2, 4[$, yielding

$$\begin{aligned}
T^{k+1}\varphi(Tx) \quad &\overset{\mathrm{I}_{a,b}^{(\gamma;k)*}}{\longmapsto} \quad \mathcal{F}^{-1}\left\{ T^k \frac{\hat{\varphi}(\frac{\omega}{T}) - \sum_{j=0}^{k-1} \frac{\hat{\varphi}^{(j)}(0)}{T^j} \frac{\omega^j}{j!}}{h_{a,b}^{\gamma}(-\omega)} \right\}(x) \\
&= \mathcal{F}^{-1}\left\{ \frac{\hat{\varphi}(\frac{\omega}{T}) - \sum_{j=0}^{k-1} \frac{\hat{\varphi}^{(j)}(0)}{j!} (\frac{\omega}{T})^j}{(\frac{\omega}{T})^k} \frac{\omega^k}{h_{a,b}^{\gamma}(-\omega)} \right\}(x) \\
&= \mathcal{F}^{-1}\left\{ \hat{\phi}(\tfrac{\omega}{T}) \underset{\sim}{h}(\omega) \right\}(x).
\end{aligned}$$

Since $\mathrm{I}_{a,b}^{(\gamma;k)*}$ is scale-invariant of order $-\gamma$, we shall have

$$\begin{aligned}
\varphi(x) - T^{k+1}\varphi(Tx) \quad &\overset{\mathrm{I}_{a,b}^{(\gamma;k)*}}{\longmapsto} \quad \left(\mathrm{I}_{a,b}^{(\gamma;k)*} \varphi \right)(x) - T^{\underset{\sim}{\gamma}+1} \left(\mathrm{I}_{a,b}^{(\gamma;k)*} \varphi \right)(Tx) \\
&= \mathcal{F}^{-1}\left\{ \left(\hat{\phi}(\omega) - \hat{\phi}(\tfrac{\omega}{T}) \right) \underset{\sim}{h}(\omega) \right\}(x) \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \left(\hat{\phi}(\omega) - \hat{\phi}(\tfrac{\omega}{T}) \right) \underset{\sim}{h}(\omega) \mathrm{e}^{\mathrm{i}\omega x} \mathrm{d}\omega \\
&= \frac{\mathrm{i}}{2\pi x} \int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}\omega} \Big(\left(\hat{\phi}(\omega) - \hat{\phi}(\tfrac{\omega}{T}) \right) \underset{\sim}{h}(\omega) \Big) \mathrm{e}^{\mathrm{i}\omega x} \mathrm{d}\omega \\
&= \cdots = \frac{\mathrm{i}^n}{2\pi x^n} \int_{\mathbb{R}} \frac{\mathrm{d}^n}{\mathrm{d}\omega^n} \Big(\left(\hat{\phi}(\omega) - \hat{\phi}(\tfrac{\omega}{T}) \right) \underset{\sim}{h}(\omega) \Big) \mathrm{e}^{\mathrm{i}\omega x} \mathrm{d}\omega \\
&= \frac{\mathrm{i}^n}{x^n} \mathcal{F}^{-1}\left\{ \frac{\mathrm{d}^n}{\mathrm{d}\omega^n} \Big(\left(\hat{\phi}(\omega) - \hat{\phi}(\tfrac{\omega}{T}) \right) \underset{\sim}{h}(\omega) \Big) \right\}(x),
\end{aligned}$$

where $n = \lceil \mathrm{Re}(\underset{\sim}{\gamma}) \rceil + 1 = k + 1 - \lfloor \mathrm{Re}(\gamma) \rfloor$. Recalling Lemma 4.1, we have that $\| \frac{\mathrm{d}^n}{\mathrm{d}\omega^n} \left(\hat{\phi}(\omega) - \hat{\phi}(\frac{\omega}{T}) \right) \underset{\sim}{h}(\omega) \|_1 \leq c\, T^{\mathrm{Re}(\underset{\sim}{\gamma})+2-n} < c\, 4^{1-\{\mathrm{Re}(\gamma)\}}$. Therefore,

$$\begin{aligned}
\left| \left(\mathrm{I}_{a,b}^{(\gamma;k)*} \varphi \right)(x) - T^{\underset{\sim}{\gamma}+1} \left(\mathrm{I}_{a,b}^{(\gamma;k)*} \varphi \right)(Tx) \right| &\leq \frac{\left\| \mathcal{F}^{-1}\left\{ \frac{\mathrm{d}^n}{\mathrm{d}\omega^n} \Big(\left(\hat{\phi}(\omega) - \hat{\phi}(\frac{\omega}{T}) \right) \underset{\sim}{h}(\omega) \Big) \right\}(x) \right\|_\infty}{|x|^n} \\
&\leq \frac{\left\| \frac{\mathrm{d}^n}{\mathrm{d}\omega^n} \Big(\left(\hat{\phi}(\omega) - \hat{\phi}(\frac{\omega}{T}) \right) \underset{\sim}{h}(\omega) \Big) \right\|_1}{|x|^n} \\
&\leq \frac{c\, 4^{1-\{\mathrm{Re}(\gamma)\}}}{|x|^n}.
\end{aligned}$$

By setting γ and m parameters of Lemma 4.2 as $\underset{\sim}{\gamma} + 1$ and n, respectively, we observe that

$$\exists\, \bar{c} > 0,\ \forall\, 1 \leq |x| : \quad \left| \left(\mathrm{I}_{a,b}^{\gamma;k*} \varphi \right)(x) \right| \leq \begin{cases} \frac{\bar{c}}{|x|^{k+1-\mathrm{Re}(\gamma)}}, & \mathrm{Re}(\gamma) \notin \mathbb{N}, \\ \bar{c}\, \frac{1+\log|x|}{|x|^{k+1-\mathrm{Re}(\gamma)}}, & \mathrm{Re}(\gamma) \in \mathbb{N}. \end{cases} \tag{92}$$

Equation (92) proves all the decay results stated in Theorem 2.7. The above result further shows that

$$\int_{|x|\geq 1} \left|\left(\mathrm{I}_{a,b}^{(\gamma;k)*}\varphi\right)(x)\right|^p \mathrm{d}x < \infty, \tag{93}$$

for $p > \frac{1}{k+1-\mathrm{Re}(\gamma)}$, whether $\mathrm{Re}(\gamma) \in \mathbb{N}$ or $\mathrm{Re}(\gamma) \notin \mathbb{N}$. Now, it is easy to combine (89), (90) and (93) to conclude that $\mathrm{I}_{a,b}^{\gamma;k*}\varphi$ is in L_p for $p > \frac{1}{k+1-\mathrm{Re}(\gamma)}$ if $k = \lfloor \mathrm{Re}(\gamma) \rfloor$, and in L_p for $\frac{1}{k+1-\mathrm{Re}(\gamma)} < p < \frac{1}{k-\mathrm{Re}(\gamma)}$ if $k > \lfloor \mathrm{Re}(\gamma) \rfloor$. ■

5.4. Proof of Theorem 2.9

As the operators are primarily defined in the Fourier domain, we use the fact that $\mathcal{F}\{\delta(\cdot - \tau)\}(\omega) = \mathrm{e}^{-\mathrm{i}\omega\tau}$. With this, the result regarding $\mathrm{D}_{a,b}^{\gamma}$ directly follows from (76):

$$\begin{aligned}
\left(\mathrm{D}_{a,b}^{\gamma}\delta(\cdot - \tau)\right)(x) &= \mathcal{F}^{-1}\left\{\mathrm{e}^{-\mathrm{i}\omega\tau} h_{a,b}^{\gamma}(\omega)\right\}(x) = \mathcal{F}^{-1}\left\{h_{a,b}^{\gamma}(\omega)\right\}(x-\tau) \\
&= a\mathcal{F}^{-1}\left\{\omega_+^{\gamma}\right\}(x-\tau) + b\mathcal{F}^{-1}\left\{\omega_-^{\gamma}\right\}(x-\tau) \\
&= a\mathcal{F}^{-1}\left\{\omega_+^{\gamma}\right\}(x-\tau) + b\mathcal{F}^{-1}\left\{\omega_+^{\gamma}\right\}(\tau-x) \\
&= \frac{\Gamma(\gamma+1)}{2\pi}\left(\frac{a}{(\mathrm{i}\tau - \mathrm{i}x)^{\gamma+1}} + \frac{a}{(\mathrm{i}x - \mathrm{i}\tau)^{\gamma+1}}\right).
\end{aligned}$$

The claims for both $\mathrm{I}_{a,b}^{\gamma;k}$ and $\mathrm{I}_{a,b}^{(\gamma;k)*}$ are obtained via Lemma 4.6. We first show the result for $\mathrm{I}_{a,b}^{\gamma;k}$ and then, proceed with $\mathrm{I}_{a,b}^{(\gamma;k)*}$:

$$\begin{aligned}
&\left(\mathrm{I}_{a,b}^{\gamma;k}\delta(\cdot - \tau)\right)(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\omega\tau}\frac{\mathrm{e}^{\mathrm{i}\omega x} - \sum_{j=0}^{k-1}\frac{(\mathrm{i}\omega x)^j}{j!}}{h_{a,b}^{\gamma}(\omega)}\mathrm{d}\omega \\
&= \frac{1}{2a\pi}\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\omega\tau} w_+^{-\gamma}\left(\mathrm{e}^{\mathrm{i}\omega x} - \sum_{j=0}^{k-1}\frac{(\mathrm{i}\omega x)^j}{j!}\right)\mathrm{d}\omega + \frac{1}{2b\pi}\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\omega\tau} w_-^{-\gamma}\left(\mathrm{e}^{\mathrm{i}\omega x} - \sum_{j=0}^{k-1}\frac{(\mathrm{i}\omega x)^j}{j!}\right)\mathrm{d}\omega \\
&= \frac{1}{a}\mathcal{F}^{-1}\left\{w_+^{-\gamma}\left(\mathrm{e}^{\mathrm{i}\omega x} - \sum_{j=0}^{k-1}\frac{(\mathrm{i}\omega x)^j}{j!}\right)\right\}(-\tau) + \frac{1}{b}\mathcal{F}^{-1}\left\{w_-^{-\gamma}\left(\mathrm{e}^{\mathrm{i}\omega x} - \sum_{j=0}^{k-1}\frac{(\mathrm{i}\omega x)^j}{j!}\right)\right\}(-\tau) \\
&= \frac{\mathcal{F}^{-1}\left\{w_+^{\underset{\sim}{\gamma}-k}\left(\mathrm{e}^{\mathrm{i}\omega x} - \sum_{j=0}^{k-1}\frac{(\mathrm{i}\omega x)^j}{j!}\right)\right\}(-\tau)}{a} + \frac{\mathcal{F}^{-1}\left\{w_+^{\underset{\sim}{\gamma}-k}\left(\mathrm{e}^{-\mathrm{i}\omega x} - \sum_{j=0}^{k-1}\frac{(-\mathrm{i}\omega x)^j}{j!}\right)\right\}(\tau)}{b},
\end{aligned} \tag{94}$$

where $\underset{\sim}{\gamma} = k - \gamma$ and $\mathrm{Re}(\underset{\sim}{\gamma}) > -1$. Now, the two inverse-Fourier terms in (94) are evaluated via Lemma 4.6. The desired claimed form then follows with simple modifications.

For operator $\mathrm{I}_{a,b}^{(\gamma;k)*}$, we follow a similar approach and obtain that

$$\begin{aligned}
&\left(\mathrm{I}_{a,b}^{(\gamma;k)*}\delta(\cdot - \tau)\right)(x) = \mathcal{F}^{-1}\left\{\frac{\mathrm{e}^{-\mathrm{i}\omega\tau} - \sum_{j=0}^{k-1}\frac{(-\mathrm{i}\omega\tau)^j}{j!}}{h_{a,b}^{\gamma}(-\omega)}\right\}(x) \\
&= \frac{\mathcal{F}^{-1}\left\{w_+^{-\gamma}\left(\mathrm{e}^{-\mathrm{i}\omega\tau} - \sum_{j=0}^{k-1}\frac{(-\mathrm{i}\omega\tau)^j}{j!}\right)\right\}(x)}{b} + \frac{\mathcal{F}^{-1}\left\{w_-^{-\gamma}\left(\mathrm{e}^{-\mathrm{i}\omega\tau} - \sum_{j=0}^{k-1}\frac{(-\mathrm{i}\omega\tau)^j}{j!}\right)\right\}(x)}{a} \\
&= \frac{\mathcal{F}^{-1}\left\{w_+^{\underset{\sim}{\gamma}-k}\left(\mathrm{e}^{-\mathrm{i}\omega\tau} - \sum_{j=0}^{k-1}\frac{(-\mathrm{i}\omega\tau)^j}{j!}\right)\right\}(x)}{b} + \frac{\mathcal{F}^{-1}\left\{w_+^{\underset{\sim}{\gamma}-k}\left(\mathrm{e}^{\mathrm{i}\omega\tau} - \sum_{j=0}^{k-1}\frac{(\mathrm{i}\omega\tau)^j}{j!}\right)\right\}(-x)}{a}.
\end{aligned} \tag{95}$$

Again, the two inverse-Fourier terms in (95) are evaluated via Lemma 4.6, and simplified to yield the claimed form.

6. Conclusion

In this paper, we initially studied complex-order fractional operators. We showed that, by applying certain complex-order fractional-integration operators to real-valued symmetric α-stable white-noise processes, we can generate self-similar stable processes with complex-valued Hurst exponent. Some of the introduced processes can be whitened by the application of complex-order fractional-derivative operators. As such, they can be described as solutions of fractional complex-order stochastic differential equations. While we proved the existence of the random processes using characteristic functionals, we also provided tools for the numerical approximation of such processes. We further studied the smoothness properties of the processes and showed that they have stationary increments of large-enough orders.

Data availability

No data was used for the research described in the article.

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