Complete Parameterization of Piecewise-Polynomial Interpolation Kernels

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***Abstract*—Every now and then, a new design of an interpolation kernel shows up in the literature. While interesting results have emerged, the traditional design methodology proves laborious and is riddled with very large systems of linear equations that must be solved analytically. In this paper, we propose to ease this burden by providing an explicit formula that will generate every possible piecewise-polynomial kernel given its degree, its support, its regularity, and its order of approximation. This formula contains a set of coefficients that can be chosen freely and do not interfere with the four main design parameters; it is thus easy to tune the design to achieve any additional constraints that the designer may care for.**

***Index Terms*—Compactly supported kernels, order of approximation, piecewise polynomial approximation, sampling, spline functions, uniform samples.**

I. Introduction

INTERPOLATION is a standard operation in image processing. It is usually described by the following equation:

$$f_h(x) = \sum_{k\in\mathbb{Z}} s_k \varphi_{\text{int}}(x/h - k) \tag{1}$$

where f_h is a continuous function reconstructed from discrete samples $s_k = s(hk)$; h is the sampling step, and φ_{int} is the interpolation function. If quality is a key issue—better than commonplace linear interpolation—then the selection of an appropriate φ_{int} becomes very important. For practical reasons, this function is often chosen to be piecewise-polynomial of moderate degree and support, with uniform knots.

Over the years, a large body of work has been devoted to the design of interpolators that tend to be sinc-like while offering more practical benefits; in particular, a finite support. Beside the requirement that φ_{int} be interpolating (i.e., $\varphi_{\text{int}}(k) = \delta_k$), the aspects that have been emphasized are: 1) its degree N; 2) the width W of its support; 3) its regularity R; and, to some extent, 4) its order of approximation L. This search for adequate interpolators is still active today; recent contributions include those of Schaum [1], Appledorn [2], German [3], Dodgson [4], or Meijering [5]. Unfortunately, it appears that the improvements of each new proposal have been less and less substantial. Recently, we showed that one reason for this saturation of design is that the interpolation constraint is too strong; only by relaxing it altogether were we able to achieve significant gains in performance [6]. The corresponding generalized interpolation model is

$$f_h(x) = \sum_{k\in\mathbb{Z}} c_k \varphi(x/h - k) \tag{2}$$

where the function φ is not necessarily interpolating anymore. The coefficients c_k are determined from the samples s_k using a digital filtering technique [7], [8], which ensures that f_h fits the sample values exactly: $s_k = s(hk) = f_h(hk)$.

The traditional design of functions φ_{int} imposes the interpolation constraint from the start on, and thereafter builds on it. Here instead, we propose to let the designer proceed by first imposing the four other characteristics: degree N, support W, regularity R, and order L. The main contribution of this paper is to be able to express in a finite-dimension vector space the *complete* class of piecewise polynomials that satisfy these four characteristics. We also identify the subclass of symmetric piecewise polynomials. The designer may then freely select among them, or may perhaps throw in additional constraints for good measure, like the interpolation constraint if he so chooses.

Before proceeding further, let us define the relevant design parameters.

1) Degree: The maximal degree of a piecewise-polynomial function is, in some sense, an index of the complexity of what can be achieved with the function. In particular, a raise in the degree N results in more parameters—in this case, coefficients—to play with. To formulate our results, we shall extend the range of possible N to negative values in the following way: the Dirac distribution δ is considered a piecewise polynomial of degree -1, while its nth derivative[1] $\delta^{(n)}$ is a piecewise polynomial of degree $-(n+1)$. This will be required by our extension of piecewise polynomials, which is coherent with the property that, if f is piecewise polynomial of degree N, then $(\mathrm{d}/\mathrm{d}x)f(x)$ is piecewise polynomial of degree $(N-1)$.

2) Support: Without loss of generality, we consider that the support of φ is contained within $[0, W]$. Outside this interval, we have that $\varphi = 0$. The value of W is the most critical parameter to determine the computational cost of interpolation. In p dimensions, this cost grows like W^p. Distributions may have a support concentrated on the origin, with $W = 0$, e.g., the Dirac distribution and its derivatives.

3) Regularity: In general, a function f is said to be of regularity R—or to belong to C^R—if and only if it is R-times continuously differentiable. In the traditional design of interpolators, regularity has often been maximized so as to give the de-

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[1]Derivatives have to be understood in the sense of distributions (see Section II).

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signer a criterion to help him reject solutions, by want of better design criteria. In the context of image processing, less-than-maximal regularity is often sufficient, because only the image and its gradient need be continuously defined. Reclaiming degrees of freedom by reducing the requirement on the regularity of φ from $R_{\max}$ to $R < R_{\max}$ can be put to good use toward a better design.

To formulate our results, we shall extend to negative values the range to which R belongs: a piecewise-polynomial function u is said to be of regularity -1 if it is bounded; a Dirac distribution δ is said to be of regularity -2, while its nth derivative $\delta^{(n)}$ is of regularity $-(n+2)$. As for the degree (see above), this extension is motivated by the properties of piecewise-polynomial functions.

4) Order: One aspect often overlooked in the traditional design of a function φ is its order of approximation L, which is an essential index of its intrinsic quality [9], [10]. It is defined by the rate of decrease of the error between the original function s and the reconstructed function f_h when the sampling step h vanishes

$$\|s - f_h\|_{\mathbf{L}^2} \propto h^L \text{ as } h \to 0.$$

From approximation theory, we know that the order L can be determined from φ only, no matter what the sampled function s may be [11]—provided it is regular enough. The order of approximation is particularly relevant to image processing because the frequency content of most images is essentially low-pass, which is equivalent to say that the sampling step h is small relatively to the image content. Thus, the continuous image f_h reconstructed from the samples s_k will be closer to the original s when the order of approximation L associated to φ is high than when it is low. The importance of the order has been confirmed by all our experiments [6].

It should be noted that these four characteristics are all *inclusive*; in particular, if f is piecewise-polynomial of degree N, then we may more generally consider that f is piecewise-polynomial of degree $N' \geq N$; if f is supported in $[0, W]$, then it is more generally supported in $[0, W']$ with $W' \geq W$; if f is C^R, then it is $C^{R'}$ for all $R' \leq R$; and if its order of approximation is L, then we may also say that it is L' for whatever $L' \leq L$.

In this paper, we state and prove several theorems leading to explicit time-domain formulæ that express every possible piecewise-polynomial kernel φ of a given degree N, support W, regularity R, and order L. The first important result (Theorem 1) decomposes φ into two terms: a B-spline β^{L-1} that carries the totality of the desired order of approximation, and a distribution u that controls by how much one must *degrade* the other three relevant properties of the B-spline (minimal degree, minimal support, maximal regularity) so as to meet the design criteria. The next important result (Theorem 2) shows how to build the complete family of distributions that have a specific degree, support, and regularity, irrespective of the order of approximation. The distribution u is a member of this family. The combined result $\beta * u$ is expressed in Theorem 3 which acts as a recipe for constructing interpolation kernels; those are fully determined by a set of free coefficients $\{a_{k,l}, b_{k,l}, c_{k,l}\}$. We also include Corollary 3', for those practitioners who would like to impose the symmetry property. Finally, we help the designer to navigate the design parameters by providing him with a map that gives the explicit constraints that N, W, R, and L must satisfy for φ to exist at all, and that counts how many degrees of freedom result from any given choice of design parameters.

The organization of this paper is as follows. In Section II, we introduce the reader to the notations and concepts that will be used throughout the paper. In Section III, we identify the four most important characteristics of an interpolation kernel and relate them to properties in the Fourier domain. We develop the other theoretical aspects of this paper in Section IV, which the reader may skip at first reading, except for the statement of our main result in Section IV-C. Then, we present in Section V some examples of design where we show how to rederive some known kernels, and present some new ones as well.

II. Definitions

A shorthand notation for the relation $\limsup_{x\to 0} |f(x)/x^n| = 0$ is $f(x) = o(x^n)$. Similarly, the relation $\limsup_{x\to 0} |f(x)/x^n| < \infty$ indicates that the value at the origin is nonnecessarily vanishing; we summarize this by $f(x) = O(x^n)$.

When there is no ambiguity, we will also write $\int f$, short for $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$.

The Fourier transform of $f(x)$ is

$$\mathcal{F}\{f\}(\omega) = \hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-j\omega x}\,\mathrm{d}x.$$

The "Schwartz class" of functions $\mathcal{S}$ [12] is the set of infinitely differentiable functions that decrease faster than $|x|^{-n}, \forall n \in \mathbb{N}$, and their derivatives as well; e.g., $e^{-x^2} \in \mathcal{S}$.

1) Distributions: Instead of considering only plain functions, we also consider generalized functions known as tempered distributions [12]. This extension is essential in the work we are presenting here. Unlike a function which is defined pointwise, a tempered distribution is defined through its scalar product $\langle u, \psi \rangle$ with every function ψ of $\mathcal{S}$. There are two main advantages of tempered distributions that have special relevance to this paper: 1) they are infinitely differentiable since, by definition, $\langle (\mathrm{d}^n/\mathrm{d}x^n)u, \psi \rangle = (-1)^n \langle u, (\mathrm{d}^n/\mathrm{d}x^n)\psi \rangle, \forall \psi \in \mathcal{S}$; and 2) their Fourier transform is a distribution as well that is defined by $\langle \hat{u}, \psi \rangle = \langle u, \hat{\psi} \rangle, \forall \psi \in \mathcal{S}$. The power function $x^n \mathrm{sign}(x)$, the Dirac mass δ, or its nth-derivative $\delta^{(n)}$, and the rational function x^{-n} are examples of tempered distributions that are not square integrable. When a distribution turns out to be a function, we will emphasize this fact by saying that it is a *true* function.

Definition 1: A polynomial simple element (PSE) of degree $n \in \mathbb{Z}, \varsigma^n(x)$, is the distribution

$$\varsigma^n(x) = \begin{cases} \frac{x^n \mathrm{sign}(x)}{2(n!)} & \text{for } n \text{ integer} \geq 0 \\ \delta^{(|n|-1)}(x) & \text{for } n \text{ integer} < 0. \end{cases}$$

Its Fourier transform in the sense of distributions is

$$\hat{\varsigma}^n(\omega) = \frac{1}{(j\omega)^{n+1}}. \tag{3}$$

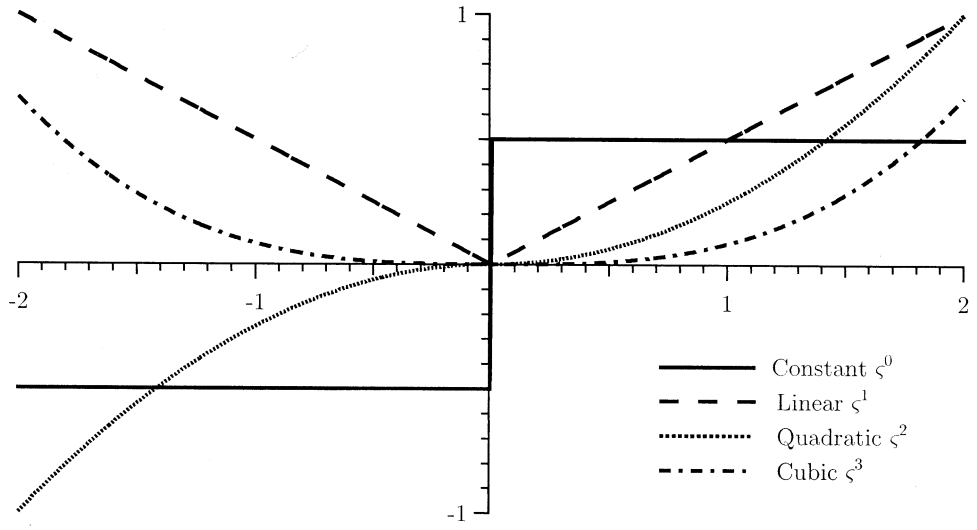

Fig. 1. Some polynomial simple elements. Solid line: piecewise constant; dashed line: piecewise linear; dotted line: piecewise quadratic; mixed line: piecewise cubic.

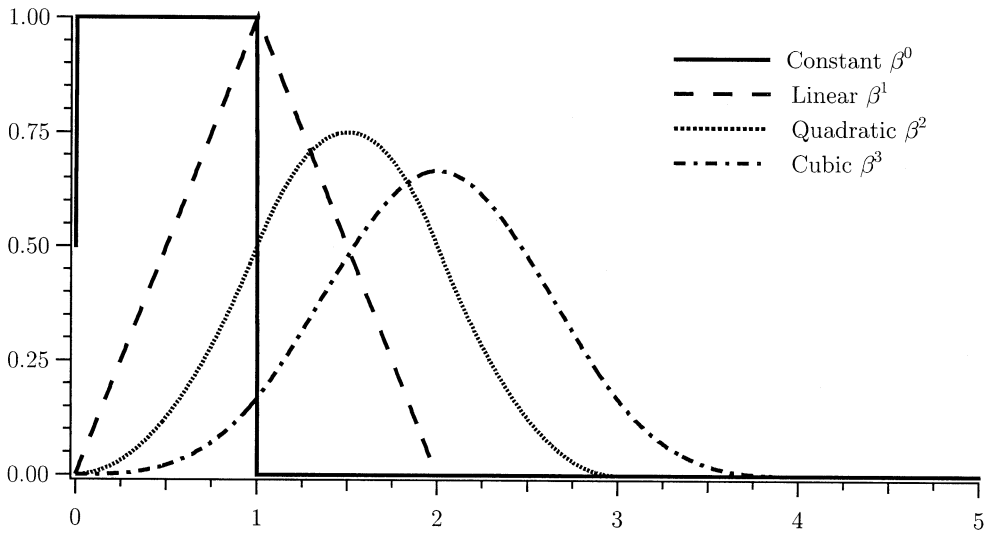

Fig. 2. Some causal B-splines. Solid line: piecewise constant; dashed line: piecewise linear; dotted line: piecewise quadratic; mixed line: piecewise cubic.

This function will play an important role in this paper.

2) PSE Properties: From Definition 1, a polynomial simple element is a power function for $0 \leq n$ and a Dirac mass (or one of its derivatives) for $n \leq -1$. We thus have the power property

$$\varsigma^{m+n}(x) = \frac{n!}{(m+n)!} x^m \varsigma^n(x) \quad \forall m, n \text{ integers} \geq 0. \tag{4}$$

Note also that the derivative of a PSE is the PSE of lower degree given by

$$\frac{\mathrm{d}^l}{\mathrm{d}x^l} \varsigma^n = \varsigma^{n-l}. \tag{5}$$

Fig. 1 shows the "constant," linear, quadratic, and cubic PSE.

Definition 2: The causal finite-difference operator Δ associates the function $\Delta f(x) = f(x) - f(x-1)$ to the function $f(x)$. The nth-order finite difference of $f(x)$ is

$$\Delta^n f(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x-k)$$

whose equivalent Fourier relation is $(1 - e^{-j\omega})^n \hat{f}(\omega)$. Note that the degree of a polynomial is reduced by one under the application of the finite-difference operator.

Definition 3: The causal B-spline of degree $n \geq 0$ is obtained by the $(n+1)$-times application of the finite-difference operator Δ on the PSE ς^n

$$\beta^n(x) = \Delta^{n+1} \varsigma^n(x)$$
$$\stackrel{\mathcal{F}}{\rightarrow} \hat{\beta}^n(\omega) = \left(\frac{1 - e^{-j\omega}}{j\omega} \right)^{n+1}. \tag{6}$$

3) B-Spline Properties: B-splines have numerous interesting properties such as positivity, symmetry, compact support, and maximal order of approximation [13], [14]. For example, differentiating a B-spline or taking finite differences is equivalent in the following sense: $(\mathrm{d}^l/\mathrm{d}x^l)\beta^n = \Delta^l \beta^{n-l}$. Another fundamental property is $\beta^m * \beta^n = \beta^{m+n+1}$. We show in Fig. 2 some B-splines of moderate degree.

Definition 4: A *function* is piecewise-polynomial (PP) of degree N and regularity R if and only if it is R-times continuously differentiable, and if its restriction to each interval[2] $]n, n+1[_{n\in\mathbb{Z}}$ is a polynomial of degree (at most) N. By extension, we say that a *distribution* is piecewise-polynomial of degree N and regularity R if and only if it is the rth derivative of a piecewise-polynomial *function* of degree $N+r$ and regularity $R+r$, for some integer differentiation depth r. For example, the PSE of Definition 1 is a PP distribution of degree n and regularity $(n-1)$, but it is a true function only for positive degrees. Another example is the causal B-spline of degree $n \geq 0$ which is of regularity $(n-1)$; additionally, it has the finite support $[0, n+1]$.

Definition 5: The order of approximation is defined as the exponent L such that the difference between any sufficiently regular function $f \in \mathbf{L}^2$ and its orthogonal projection f_h onto $\mathcal{V}_h = \mathrm{span}_{n\in\mathbb{Z}}\{\varphi((x/h)-n)\}$ tends to 0 with h^L; i.e., $\|f - f_h\|_{\mathbf{L}^2} \leq \mathrm{const} \times h^L$. For this property to hold, it is necessary to assume not only that $f \in \mathbf{L}^2$, but also that its Lth derivative belongs to $\mathbf{L}^2$. This definition cannot be directly extended to distributions because these are not necessarily square-integrable. However, the equivalence between the order of approximation and the Strang-Fix conditions [11] will allow us to do so: see Section III-B for details.

Definition 6: We will say that a PP distribution belongs to $\{N, W, R, L\}$ if and only if it is of degree N, support W, regularity R, and order L. For instance, the B-spline of degree $n, \beta^n(x)$, belongs to $\{n, n+1, n-1, n+1\}$.

III. Fourier Characterization of PP Kernels

A. Degree, Support, and Regularity

The following lemma shows that any PP distribution in $\{N, W, R, 0\}$ can be expressed as a finite sum of shifted PSE's of degree $\leq N$ and regularity $\geq R$. Conversely, every such expression is a PP distribution that belongs to $\{N, W, R, 0\}$, under a simple condition which is best expressed in Fourier variables.

Lemma 1: $\varphi(x)$ is a PP distribution of degree N, support W, and regularity R if and only if there exist $(N-R)$ polynomials $P_l(z) = \sum_{k=0}^{W} p_{k,l} z^k$ of degree at most W satisfying

$$\sum_{l=0}^{N-R-1} (j\omega)^l P_l(e^{-j\omega}) = O(\omega^{N+1}) \tag{7}$$

such that

$$\varphi(x) = \sum_{l=0}^{N-R-1} \sum_{k=0}^{W} p_{k,l} \varsigma^{N-l}(x-k). \tag{8}$$

Moreover, the parameters $p_{k,l}$ are unique.

There exists no nontrivial PP function for which $N-R-1 < 0$; thus, the existence condition is

$$N \geq R+1. \tag{9}$$

The proof of this lemma is given in Appendix A. As (8) only ensures that φ is piecewise polynomial of degree N and regularity R, we need Condition (7) to enforce the *compact support* property. A hint is to express (8) in Fourier variables

$$\hat{\varphi}(\omega) = \frac{1}{(j\omega)^{N+1}} \sum_{l=0}^{N-R-1} (j\omega)^l P_l(e^{-j\omega}) \tag{10}$$

which ensures that $\hat{\varphi}(\omega)$ be bounded near $\omega = 0$.

1) Application: This is an example of how to use Lemma 1. We will exhibit a collection of PP functions of degree n, support n, and regularity $-1 \ldots (n-2)$ that will prove useful in the sequel.

First, we define the coefficients $\lambda_{k,l}$ from the Mac-Laurin development of $(\log(1-t))^k$

$$(-\log(1-t))^k = \underbrace{\sum_{l=0}^{l_0} \lambda_{k,l} t^{k+l}}_{\stackrel{\mathrm{def}}{=} t^k \Lambda_k^{l_0}(t)} + o(t^{k+l_0}). \tag{11}$$

When $k=1$, this development is well-known and yields $\lambda_{1,l} = 1/(l+1)$. For higher values of k, the $\lambda_{k,l}$'s can be computed using the formula

$$\Lambda_k^{l_0}(t) \equiv \left(\Lambda_1^{l_0}(t)\right)^k \equiv \left(\sum_{l=0}^{l_0} \frac{t^l}{l+1}\right)^k \pmod{t^{l_0+1}}$$

which proves in particular that they are strictly positive. The first few values of these coefficients are shown in Table I.

TABLE I
Value of Some Coefficients $\lambda_{k,l}$

$k \backslash l$	0	1	2	3	4	5
1	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$
2	1	1	$\frac{11}{12}$	$\frac{5}{6}$	$\frac{137}{180}$	$\frac{7}{10}$
3	1	$\frac{3}{2}$	$\frac{7}{4}$	$\frac{15}{8}$	$\frac{29}{15}$	$\frac{469}{240}$
4	1	2	$\frac{17}{6}$	$\frac{7}{2}$	$\frac{967}{240}$	$\frac{89}{20}$
5	1	$\frac{5}{2}$	$\frac{25}{6}$	$\frac{35}{6}$	$\frac{1069}{144}$	$\frac{285}{32}$
6	1	3	$\frac{23}{4}$	9	$\frac{3013}{240}$	$\frac{781}{48}$

Proposition 1: For $1 \leq k \leq n$, the function

$$\gamma_k^n(x) = \frac{1}{\lambda_{k,n-k+1}} \left(\varsigma^{n-k}(x) - \sum_{l=0}^{n-k} \lambda_{k,l} \Delta^{k+l} \varsigma^n(x)\right)$$
$$\stackrel{\mathcal{F}}{\rightarrow} \hat{\gamma}_k^n(\omega) = \frac{(j\omega)^k - (1-e^{-j\omega})^k \Lambda_k^{n-k}(1-e^{-j\omega})}{\lambda_{k,n-k+1}(j\omega)^{n+1}} \tag{12}$$

is compactly supported in $[0, n]$ and its integral is unity. More precisely, we have that γ_k^n belongs to $\{n, n, n-k-1, 0\}$.

Proof: We observe that the Fourier transform (12) of γ_k^n takes the form (10) with $N = n, R = n-k-1, P_0(z) = -(1/\lambda_{k,n-k+1})(1-z)^k \Lambda_k^{n-k}(1-z)$, $P_k(z) = (1/\lambda_{k,n-k+1})$, and $P_l(z) = 0$ for all values of l different from 0 and k. If we

[2]We consider only uniform knots in this paper.

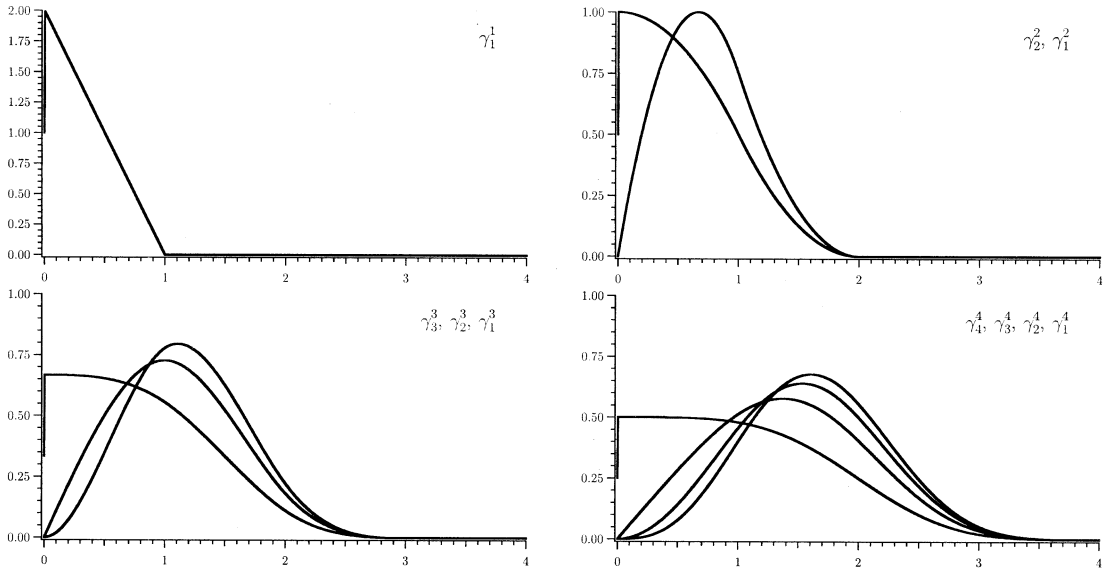

Fig. 3. Some functions γ_k^n. Top left: all functions of support 1. Top right: all functions of support 2. Bottom left: all functions of support 3. Bottom right: all functions of support 4.

set $t = 1 - z = 1 - e^{-j\omega}$ and if $\omega \in]-\pi, \pi[$, then we have $j\omega = -\log(1-t)$. Thus,

$$\begin{aligned}
&\sum_{l=0}^{N-R-1} (j\omega)^l P_l(e^{-j\omega}) \\
&= P_0(1-t) + (-\log(1-t))^k P_k(1-t) \\
&= \frac{-t^k \Lambda_k^{n-k}(t) + (-\log(1-t))^k}{\lambda_{k,n-k+1}} \\
&= t^{n+1} + O(t^{n+2}) \quad \text{because of (11)}
\end{aligned}$$

which is $O(\omega^{n+1})$ since $O(\omega) = O(t)$. Finally, the polynomials P_l satisfy (7). This proves two things: first, since the degree of the polynomials P_l is at most n, Lemma 1 tells us that the functions γ_k^n are compactly supported in $[0, n]$; second, since $(t^{n+1}/(j\omega)^{n+1}) = 1 + O(\omega)$, we have that $\int \gamma_k^n = 1$. ■

Some of these functions are plotted in Fig. 3. We observe that they are positive; indeed, we can prove more generally that $\gamma_k^n(x) > 0$ for all $n \geq 1, 1 \leq k \leq n$, and $x \in]0, n[$.

B. Order of Approximation

When $\varphi(x)$ is compactly supported, the theory of approximation tells us that the decrease rate of the approximation error is necessarily integer and finite [11], [15], [16]. More specifically, we will see (Theorem 1) that the support of φ must be at least of length L, for $\|f - f_h\|_{\mathbf{L}^2}$ to decrease at least with h^L.

To check the approximation order of $\mathcal{V}_h$ as given in Definition 5, Strang and Fix established in 1973 the equivalence between an Lth-order of approximation and the following conditions [11]:

$$\begin{cases} \hat{\varphi}(0) \neq 0 \\ \hat{\varphi}(2n\pi + \omega) = O(\omega^L) \quad \forall n \in \mathbb{Z}^*. \end{cases} \tag{13}$$

Another equivalent form of (13) is ([15], Proposition 4.4; see (14) at the bottom of the page) where $\mathcal{P}^{L-1}$ is the set of all polynomials of degree $\leq L-1$, and where a.e. means *almost everywhere*. In the rest of this paper, we will use Condition (14) as the most useful formulation of the approximation order.

IV. Decomposition of Polynomial Kernels

A. Decomposition With Respect to the Order of Approximation

The theorem that follows simplifies the design by dividing the task in two independent parts: first, find a function that fully satisfies the order L constraint, and second, find a distribution that satisfies a reduced version of the constraints for the three other parameters, with no regard to its own order of approximation. It turns out that the first function is a B-spline with degree $(L-1)$. The role of the remaining distribution is to allow for a potential raise of degree, extension of support, or change of regularity, without decreasing the order of approximation. Table II shows the corresponding properties in the case where φ is PP.

Theorem 1: Let $\varphi(x)$ be any approximation kernel (not necessarily piecewise-polynomial). Then, φ is of order L and

$$\begin{cases} \hat{\varphi}(0) \neq 0 \\ \forall A \in \mathcal{P}^{L-1}, \quad \exists C_A \in \mathbb{R}: \sum_{k\in\mathbb{Z}} A(x-k)\varphi(x-k) = C_A \text{ a.e.} \end{cases} \tag{14}$$

TABLE II
PROPERTIES THAT APPLY TO $\varphi = \beta^{L-1} * u$

	φ	β^{L-1}	u
Degree	N	$L-1$	$N-L$
Support	W	L	$W-L$
Regularity	R	$L-2$	$R-L$
Order	L	L	0, with $\int u \neq 0$

support W if and only if there exists a distribution u (with $\int u \neq 0$) supported in an interval of length $(W-L)$ such that

$$\varphi = \beta^{L-1} * u. \tag{15}$$

Additionally, φ is piecewise-polynomial of degree N and regularity R if and only if u is of degree $(N-L)$ and regularity $(R-L)$.

There exists no nontrivial PP function for which $W - L < 0$ or $N - L < -1$; thus, the existence conditions are

$$W \geq L \quad \text{and} \quad N \geq L-1. \tag{16}$$

The factorization (15) was first presented in [17] in a formalism that is, we believe, unfamiliar to a signal processing audience.[3] We rediscovered this result independently [10], [18]; for the sake of completeness, we give a direct proof in Appendix B. The corollary that ensues is new and further decomposes the distribution u into two parts: a true function ψ, and an irreducible distributional part. It turns out in the present case that the last distribution is necessarily a weighted sum of derivatives of the Dirac mass. Except for normalization, the corresponding set of weights $c_{k,l}$ is free of constraints: It is the first in a series of three similar sets of coefficients to which the designer may give arbitrary values, without fear of interference with N, W, R, or L.

Corollary 1': A true function φ belongs to $\{N, W, R, L\}$ if and only if it can be expressed as

$$\varphi(x) = (\beta^{L-1} * \psi)(x) + \sum_{l=0}^{L-R-2} \sum_{k=0}^{W-L} c_{k,l} \Delta^l \beta^{L-l-1}(x-k) \tag{17}$$

where ψ is a PP function that belongs to $\{N-L, W-L, \max(-1, R-L), 0\}$, and where the coefficients $c_{k,l}$ satisfy $\int \psi \neq -\sum_{k=0}^{W-L} c_{k,0}$. Moreover, ψ and the $c_{k,l}$ are unique.

Proof: We already know by Theorem 1 that it is equivalent to say that φ is in $\{N, W, R, L\}$ and that $\varphi = \beta^{L-1} * u$, where u is in $\{N-L, W-L, R-L, 0\}$ and $\int u \neq 0$. The characterization (8) of PP distributions shows that, if $L - R < 2$, then

$$u(x) = \underbrace{\sum_{l=R-L+1}^{N-L} \sum_{k=0}^{W-L} p_{k,l} \varsigma^l(x-k)}_{\psi(x)}$$

and, if $L - R \geq 2$, then

$$u(x) = \underbrace{\sum_{l=0}^{N-L} \sum_{k=0}^{W-L} p_{k,l} \varsigma^l(x-k)}_{\psi(x)} + \underbrace{\sum_{l=R-L+1}^{-1} \sum_{k=0}^{W-L} p_{k,l} \varsigma^l(x-k)}_{\sum_{l=0}^{L-R-2} \sum_{k=0}^{W-L} c_{k,l} \delta^{(l)}(x-k)}$$

where the $p_{k,l}$—hence ψ and the $c_{k,l}$—are unique. We observe that the second double sum (Dirac sums) on the right-hand side is supported in $[0, W-L]$, which is also the support of u. Hence, ψ is, in both cases, supported in $[0, W-L]$. By construction, ψ is a PP function of regularity given by $\max(-1, R-L)$ and of degree $(N-L)$. We immediately get (17) if we convolve this expression with β^{L-1} and take into account that $\beta^{L-1} * \delta^{(l)} = (\mathrm{d}^l/\mathrm{d}x^l)\beta^{L-1} = \Delta^l \beta^{L-1-l}$. Finally, the coefficients $c_{k,0}$ are related to $\int \psi$ through the condition $\int \psi + \sum_{k=0}^{W-L} c_{k,0} = \int u \neq 0$.

Conversely, the function defined by (17) can obviously be expressed as $\varphi = \beta^{L-1} * u$, where u belongs to $\{N-L, W-L, R-L, 0\}$ and satisfies $\int u \neq 0$. Thus, by Theorem 1, φ is in $\{N, W, R, L\}$. ■

B. Decomposition With Respect to Degree, Support, and Regularity

Thanks to the result of Section IV-A, we can proceed without regard to the order of approximation since the only function that is unspecified so far has none: it is $\psi \in \{n, w, r, 0\}$, with $n = N-L, w = W-L$, and $r = \max(-1, R-L)$. The task of the present section is then to characterize every possible piecewise polynomial of an arbitrary degree N, support W, and regularity R. For this, we state Theorem 2, which yields the desired decomposition, the proof of which is given in Appendix C.

Theorem 2: Let ψ be in $\{N, W, R, 0\}$. Then, there exists a unique set of coefficients $a_{k,l}$ and $b_{k,l}$ such that

$$\psi(x) = \sum_{l=1}^{N-R-1} \sum_{k=0}^{N-l} a_{k,l} \left(\beta^{k-1} * \gamma_l^{N-k}\right)(x) + \sum_{l=0}^{N-R-1} \sum_{k=0}^{W-N+l-1} b_{k,l} \beta^{N-l}(x-k) \tag{18}$$

where $\gamma_{l,N-k}$ is defined by (12). Conversely, any function ψ that takes the expression (18) belongs to $\{N, \max(W, N), R, 0\}$. Consequently, if $W \geq N$, it is equivalent to say that ψ is in $\{N, W, R, 0\}$ and that ψ can be expressed as (18).

The functions γ_k^n that appear in this theorem have been shown to be in $\{n, n, n-k-1, 0\}$ in Proposition 1, but we further need their convolution with B-splines. A straightforward Fourier computation using (6) and (12) shows that

$$\beta^m * \gamma_k^n = \frac{1}{\lambda_{k,n-k+1}} \left(\Delta^{m+1} \varsigma^{n-k+m+1} - \sum_{l=0}^{n-k} \lambda_{k,l} \Delta^{k+l+m+1} \varsigma^{n+m+1} \right) \tag{19}$$

[3] To be precise, [17] deals with the more general case of exponential splines which emerge from a generalization of the Strang-Fix conditions.

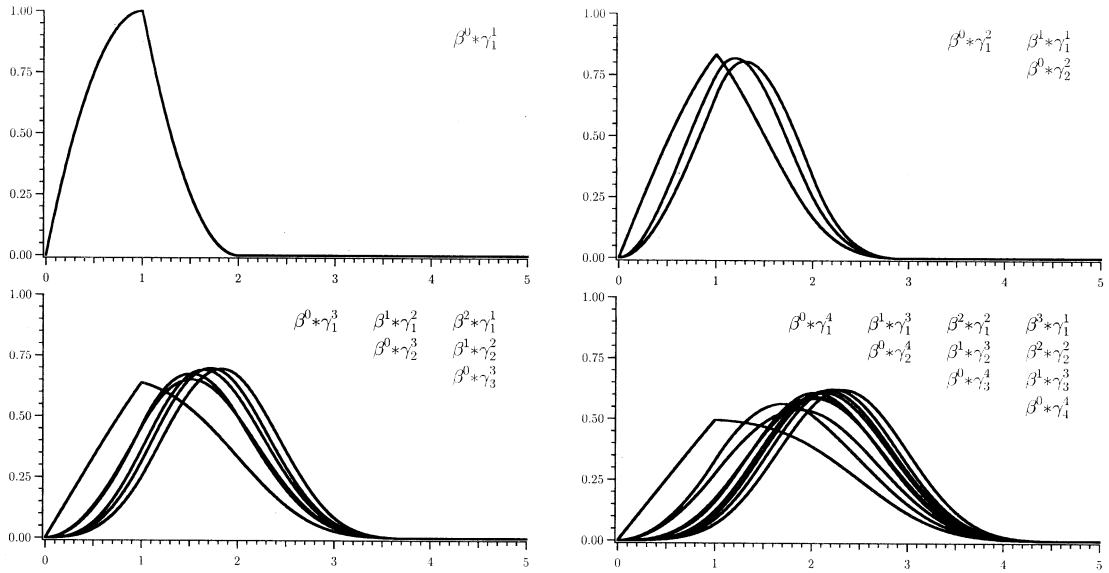

Fig. 4. Some convolutions $\beta^m * \gamma_k^n$. Top left: all functions of support 2. Top right: all functions of support 3. Bottom left: all functions of support 4. Bottom right: all functions of support 5.

which is in $\{n+m+1, n+m+1, n-k+m, m+1\}$ as a result of Theorem 1. Fig. 4 illustrates several examples of these functions. Note that, because β^m and γ_k^n are both positive, $\beta^m * \gamma_k^n$ is positive as well.

C. Final Decomposition

Our main result is a theorem that sews together the pieces of the puzzle that we have uncovered in the previous sections. Along with (19), it gives the explicit form for an interpolation kernel that satisfies the design constraints.

Theorem 3: Let φ belong to $\{N, W, R, L\}$. Then, there exists a unique set of coefficients $a_{k,l}, b_{k,l}$ and $c_{k,l}$ such that

$$\varphi(x) = \sum_{l=1}^{N-\max(R+1,L)} \sum_{k=0}^{N-L-l} a_{k,l} \left(\beta^{L+k-1} * \gamma_l^{N-L-k}\right)(x) + \sum_{l=0}^{N-\max(R+1,L)} \sum_{k=0}^{W-N+l-1} b_{k,l}\beta^{N-l}(x-k) + \sum_{l=0}^{L-R-2} \sum_{k=0}^{W-L} c_{k,l}\Delta^l \beta^{L-l-1}(x-k). \quad (20)$$

Conversely, a function φ that takes the form of Expression (20) is such that it is a PP function that belongs to $\{N, \max(W,N), R, L\}$; i.e., if $W \geq N$, φ belongs to $\{N, W, R, L\}$ if and only if it can be expressed as (20). The coefficients $a_{k,l}, b_{k,l}$, and $c_{k,l}$, are essentially free, but for the condition

$$\sum_{l=1}^{N-\max(R+1,L)} \sum_{k=0}^{N-L-l} a_{k,l} + \sum_{l=0}^{N-\max(R+1,L)} \sum_{k=0}^{W-N+l-1} b_{k,l} + \sum_{k=0}^{W-L} c_{k,0} \neq 0 \quad (21)$$

which ensures that $\int \varphi \neq 0$.

The two sets of coefficients $a_{k,l}$ and $b_{k,l}$ complement the set of coefficients $c_{k,l}$ that we encountered in Section IV-A. Altogether, they describe every possible piecewise-polynomial interpolation kernel in $\{N, W, R, L\}$. When the designer asks for a degree that is strictly larger than the support (i.e., $N > W$), the design coefficients $\{a_{k,l}, b_{k,l}, c_{k,l}\}$ are not completely free anymore. We note however that the existence of a basis function φ with such a constraint is often impossible because desirable external constraints (e.g., symmetry, interpolation) may be incompatible with such design parameters. To the best of our knowledge, no φ that would satisfy $N > W$ has ever been found useful in the literature. For this reason, from now on we concentrate on the case $N \leq W$.

A useful refinement of Theorem 3 allows one to include symmetry in addition to the $\{N, W, R, L\}$ constraints.

Corollary 3': Let φ belong to $\{N, W, R, L\}$ and satisfy the symmetry property $\varphi(x) = \varphi(W - x)$. Then, there exists a unique function φ_0 in $\{N, W, R, L\}$ such that

- the parameters $a_{k,l}, b_{k,l}$, and $c_{k,l}$ in (20) satisfy

$$\begin{aligned} a_{k,l} &= 0 \quad \text{if } k + l \equiv N - L \pmod 2 \\ b_{k,l} &= 0 \quad \text{if } k \geq \frac{W - N + l + 1}{2} \\ c_{k,l} &= 0 \quad \text{if } k \geq \frac{W - L + \varepsilon_l}{2} \end{aligned} \quad (22)$$

  where $\varepsilon_l = 0 \,|\, 1$ if l is odd $|$ even;
- φ can be expressed as

$$\varphi(x) = \varphi_0(x) + \varphi_0(W - x). \quad (23)$$

The attractiveness of Theorem 3 (and of Corollary 3') is that the designer may address at an early stage the aspects of the design that are the most important in the context of image processing—particularly, support and order—while he can defer to later stages the fulfilling of less important constraints. Especially relevant is the fact that the coefficients $\{a_{k,l}, b_{k,l}, c_{k,l}\}$ are essentially free (except when $N > W$), so that they do not interfere with the characteristics $\{N, W, R, L\}$.

TABLE III
PARTITION OF THE DEGREES OF FREEDOM

	$L \leq R+1$
$\mathrm{card}\{a_{k,l}\}$	$\frac{1}{2}(N-R-1)(N+R-2L+2)$
$\mathrm{card}\{b_{k,l}\}$	$\frac{1}{2}(N-R)(2W-(N+R)-1)$
$\mathrm{card}\{c_{k,l}\}$	0
	$R+2 \leq L$
$\mathrm{card}\{a_{k,l}\}$	$\frac{1}{2}(N-L+1)(N-L)$
$\mathrm{card}\{b_{k,l}\}$	$\frac{1}{2}(N-L+1)(2W-(L+N))$
$\mathrm{card}\{c_{k,l}\}$	$(L-R-1)(W-L+1)$

D. Navigating the Map of Constraints

On the one hand, Lemma 1 and Theorem 1 tell us that $N \geq R+1, W \geq L$, and $N \geq L-1$. Moreover, we assume that we are in the case $W \geq N$, for which the parameters $a_{k,l}, b_{k,l}$, and $c_{k,l}$, are free except for (21). In a more compact form, our constraints are

$$\begin{cases} W \geq \max(N, L) \\ N \geq \max(L-1, R+1). \end{cases} \tag{24}$$

On the other hand, a direct count based on (20) shows that there are exactly $P = (N-R)(W-L)+L-R-1$ parameters (see Table III). Taking (21) into account, this means that there are $(P-1)$ degrees of freedom. It turns out that the minimal requirement $P-1 \geq 0$ is automatically satisfied with (24): if $N \geq R+2$, we have

$$\begin{aligned} P-1 &= \underbrace{(N-R)}_{\geq 1}(W-L)+L-R-2 \\ &\geq W-\underbrace{(R+2)}_{\leq N} \geq W-N \geq 0 \end{aligned}$$

and if $N = R+1$, the solutions turn out to be splines (see Example V.A), for which the condition $P-1 \geq 0$ is satisfied.

When designing a symmetric PP function, the number of degrees of freedom are given by those of the function φ_0 defined in Corollary 3'.

1) Interpolation: If the PP function is interpolating and continuous, then the P parameters are related through the W ($W-1$, respectively) equations $\varphi((W/2)+n) = \delta_n$ when W is odd (even, respectively). Note that, when $L \geq 1$, the interpolation condition implies that (21) is automatically satisfied: if we choose $A(x) = 1$ in (14), we get $C_A = 1$ by setting $x = W/2$, which shows, by integrating (14) over $[0, 1]$, that $\int \varphi = 1$. This means that, in the case of interpolation, we only have to enforce the additional condition $P-W \geq 0$ ($P-W+1 \geq 0$, respectively) when W is odd (even, respectively).

V. EXAMPLES

A. Splines

As a first example, we look for the PP functions φ that belong to $\{N, W, N-1, 0\}$; i.e., that maximize the regularity ($R = N-1$), regardless of the other parameters. A direct application of Theorem 3 yields

$$\varphi(x) = \sum_{k=0}^{W-N-1} b_{k,0}\beta^N(x-k)$$

which shows that the most regular PP function, given its degree N, is a spline of same degree. Moreover, φ is automatically of order $L = N+1$ even though we were less ambitious and required only $L = 0$. Note however that, when $N \geq 2$, it is not possible to impose the interpolation constraint and, at the same time, to keep W finite [14].

B. Keys

We shall now rederive the popular "optimal" Keys interpolation kernel [19]. The goal is here to find a kernel that has degree $N = 3$, support $W = 4$, maximum regularity, and that is interpolating. Since the maximum degree $R = N-1$ yields splines of degree $N = 3$ whose interpolator is not of finite length, we can have at most $R = N-2 = 1$ instead. The interpolation condition shows that we can expect $P-W+1 = 3-L$ free parameters. Thus, if $L = 3$, only one solution is to be expected, with every parameter constrained.

From Table III and Theorem 3, we find that this function can be expressed as $\varphi(x) = b_{0,0}\beta^3(x) + c_{0,0}\beta^2(x) + c_{1,0}\beta^2(x-1)$, where the coefficients are linked through the interpolation constraint $\varphi(2+n) = \delta_n$

$$\begin{cases} 0 = b_{0,0}\underbrace{\beta^3(1)}_{=1/6} + c_{0,0}\underbrace{\beta^2(1)}_{=1/2} + c_{1,0}\underbrace{\beta^2(0)}_{=0} \\ 1 = b_{0,0}\underbrace{\beta^3(2)}_{=2/3} + c_{0,0}\underbrace{\beta^2(2)}_{=1/2} + c_{1,0}\underbrace{\beta^2(1)}_{=1/2} \\ 0 = b_{0,0}\underbrace{\beta^3(3)}_{=1/6} + c_{0,0}\underbrace{\beta^2(3)}_{=0} + c_{1,0}\underbrace{\beta^2(2)}_{=1/2} \end{cases}$$

which yields $b_{0,0} = 3$ and $c_{0,0} = -1$. Thus,

$$\varphi(x) = 3\beta^3(x) - (\beta^2(x) + \beta^2(x-1)).$$

This short, simple, yet complete derivation has to be contrasted with the original solution in [19], where essentially a linear system of eight equations in eight unknowns had to be solved and where an explicit foray into the Taylor expansion of φ was necessary.

C. German

We tackle now the interpolation kernel K4 that was introduced in [3], where the solution of an explicit system of no less than twenty linear equations was required, along with intricate Taylor considerations. The design parameters of this symmetric interpolating kernel are $\{N = 4, W = 6, R = 1, L = 5\}$.

From Table III, we identify the six free parameters as $\{c_{0,0}, c_{1,0}, c_{0,1}, c_{1,1}, c_{0,2}, c_{1,2}\}$ that characterize any function of $\{N = 4, W = 6, R = 1, L = 5\}$; by symmetry considerations (Corollary 3'), those immediately reduce to only three because we can write the general solution as $\varphi(x) = c_{0,0}(\beta^4(x)+\beta^4(x-1)) + c_{0,1}(\Delta\beta^3(x) - \Delta\beta^3(x-1)) + c_{0,2}(\Delta^2\beta^2(x) + \Delta^2\beta^2(x-1))$. The interpolation constraint is

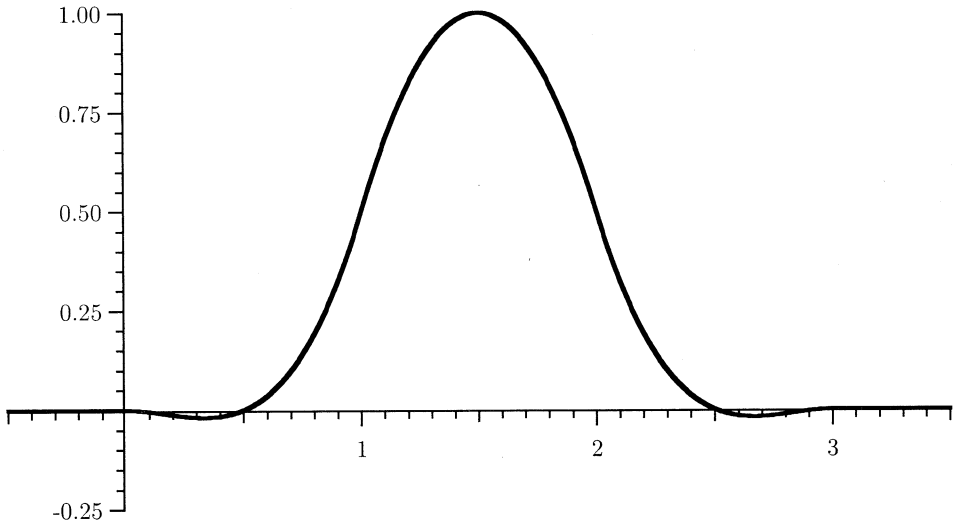

Fig. 5. Symmetric, continuously differentiable, interpolating cubic kernel with support $W = 3$.

satisfied if and only if (see the equation at the bottom of the page) which is easily solved and yields

$$\varphi(x) = \frac{1}{2}(\beta^4(x) + \beta^4(x-1)) - \frac{3}{4}(\Delta\beta^3(x) - \Delta\beta^3(x-1)) + \frac{5}{24}(\Delta^2\beta^2(x) + \Delta^2\beta^2(x-1)).$$

We were thus able to determine in just a few lines a kernel that proved particularly tiresome to derive in the original paper [3]. Moreover, we end up with a global expression in terms of known functions—to be compared with the piecewise expression.

D. MOMS

From the existence conditions (16), any PP function φ that has order L is such that $W \geq L$; thus, the kernels that satisfy $W = L$ minimize the support for a given order of approximation [9]. Those are called MOMS (Maximal Order Minimal Support) and are all members of the family characterized by $\{N, N+1, -1, N+1\}$

$$\varphi(x) = \sum_{l=0}^{N} c_{0,l}\Delta^l\beta^{N-l}(x)$$

where $c_{0,0} \neq 0$ as required by (21) (see [10]). This includes the B-spline of degree N and the interpolating functions discussed in [1].

Let us give an example. We fix $N = 3$ and require symmetry. After the normalization $\int \varphi_a = 1$, we get the one-parameter family of functions

$$\varphi_a(x) = \beta^3(x) + a\Delta^2\beta^1(x).$$

The parameter a may be adjusted to the needs of the designer: by optimizing the approximation properties of φ_a, we get the O-MOMS of degree 3, for which $a = (1/42)$; by asking that φ_a be interpolating, we get $a = -(1/6)$ instead [10]. It is also possible, like in [20], to consider the empirical SNR optimization of a over a collection of signals.

E. $\{W, W, W-2, 1\}$ Kernels

So far, we have given examples of known kernels only; it is now time to derive new ones, with the specific goal of illustrating the use of the functions $\beta^m * \gamma_k^n$ that first appeared in Theorem 2. We propose the design characterized by $\{W, W, W-2, 1\}$ with $W \geq 2$, which produces what we call the WWW symmetric interpolating kernels. It is easy to verify from the symmetry conditions (22) that this family of designs will result in $\lfloor (W-1/2) \rfloor$ free coefficients $a_{k,1}$, along with a unique free coefficient $b_{0,1}$, and no coefficient $c_{k,l}$. On the other hand, the (symmetric) interpolation condition removes $\lfloor (W+1/2) \rfloor$ degrees of freedom, thus leaving none of them.

Let us give an example. We fix $W = 3$ and require symmetry. After the normalization $\int \varphi_a = 1$, we get the one-parameter family of functions

$$\varphi_a(x) = a\left(\left(\beta^0 * \gamma_1^2\right)(x) + \left(\beta^0 * \gamma_1^2\right)(3-x)\right) + (1-2a)\beta^2(x).$$

The interpolation condition removes the remaining degree of freedom by imposing $a = -2$. We show this interpolating kernel in Fig. 5.

$$\begin{cases} 0 = c_{0,0}\underbrace{(\beta^4(1) + \beta^4(0))}_{=1/24} + c_{0,1}\underbrace{(\Delta\beta^3(1) - \Delta\beta^3(0))}_{=1/6} + c_{0,2}\underbrace{(\Delta^2\beta^2(1) + \Delta^2\beta^2(0))}_{=1/2} \\ 0 = c_{0,0}\underbrace{(\beta^4(2) + \beta^4(1))}_{=1/2} + c_{0,1}\underbrace{(\Delta\beta^3(2) - \Delta\beta^3(1))}_{=1/3} + c_{0,2}\underbrace{(\Delta^2\beta^2(2) + \Delta^2\beta^2(1))}_{=0} \\ 1 = c_{0,0}\underbrace{(\beta^4(3) + \beta^4(2))}_{=11/12} + c_{0,1}\underbrace{(\Delta\beta^3(3) - \Delta\beta^3(2))}_{=-1} + c_{0,2}\underbrace{(\Delta^2\beta^2(3) + \Delta^2\beta^2(2))}_{=-1} \end{cases}$$

The WWW kernel has a high degree of regularity R (the highest possible after splines); for example, φ_a is more regular than the kernel of same support presented in [4]. But since the order of approximation of a WWW kernel is so low, we expect that it will perform poorly in the context of image processing, where the order of approximation L is the single most important indicator of the quality of an interpolator. With $L = 1$, a WWW kernel should offer about the same level of performance as nearest-neighbor interpolation, no matter which regularity $R = W - 2$ is imposed. Note that, for the same support W, the members of the MOMS family of interpolators will gain $(W - 1)$ orders of approximation for a lesser computational cost, because of the reduction in the degree of the polynomials. In other words, we may wonder whether WWW is a waste of time.

VI. Conclusion

We have presented a methodology that will help the designer to roam the complete space of piecewise-polynomial interpolation kernels. We have stated and proved new theorems that result in an explicit formula for constructing any piecewise-polynomial interpolation kernel with specified degree, support, regularity, and order constraints. The advantages of our parameterization are the following:

- the four main design parameters are freely specified—within existence conditions;
- the final expression is a linear combination of simple, *positive* basis functions, based on B-splines, $\beta^n(x)$, and on new functions, $\gamma_k^n(x)$. They are provided *explicitly* in full functional form, and through their Fourier transforms as well—in contrast with the literature, where interpolation kernels are expressed in a piecewise polynomial form;
- the decomposition is *unique* and *complete*, implying that the free remaining parameters are independent. As a result, a design problem requires the solution of much fewer equations than before;
- symmetry is cared for by considering an adequate subset of the free parameters, resulting in a representation that is nonredundant and complete for symmetric kernels.

We have applied this formula to rederive several examples of known kernels and we have shown how to obtain new ones. This parameterization is also well-suited for the specification of design constraints in the frequency domain; this suggests using filter design techniques to optimize interpolation kernels for specific classes of signals.

Kernel design is not the only application of our decomposition theorem; the flexibility of this nonredundant representation is also potentially useful for a piecewise-polynomial description of measured data. An example is the retrieval of the probability density function that rules the randomness of observed data; in that case the positivity of the basis functions is a very desirable property that can simplify the fitting problem considerably.

Appendix
Proof of Lemma 1

We first consider PP functions; thus, $R \geq -1$. Because φ is in $\{N, W, R, 0\}$, $\varphi^{(R+1)}$ is in $\{N - R - 1, W, -1, 0\}$; in other words, $\varphi^{(R+1)}$ is a bounded function. Thus, Definition 4 tells us that, on each interval $]k, k+1[$, $\varphi^{(R+1)}(x) = A_k(x)$, where A_k is a polynomial of degree $(N - R - 1)$. It follows that $\varphi^{(R+1)}(x) = \sum_{k=0}^{W-1} A_k(x-k)\beta^0(x-k)$, where the sum is finite because $\varphi(x) = 0$ when $x \notin [0, W]$. If, according to Definition 3, we replace $\beta^0(x)$ by the expression $\varsigma^0(x) - \varsigma^0(x-1)$, then we can find $(W+1)$ polynomials B_k of degree at most $(N - R - 1)$ such that

$$\begin{aligned}\varphi^{(R+1)}(x) &= \sum_{k=0}^{W} B_k(x-k)\varsigma^0(x-k) \\ &= \sum_{k=0}^{W}\sum_{l=0}^{N-R-1} B_k^{(l)}(0)\varsigma^l(x-k).\end{aligned}$$

The second equality is obtained by making a (finite) Taylor expansion of $B_k(x)$; this expression is further simplified using the power property (4). This shows that $\varphi^{(R+1)}(x)$ is a finite sum of PSE's, $\varsigma^l(x-k)$, of degree at most $(N - R - 1)$. More specifically, if we let $p_{k,l} = B_k^{(N-R-1-l)}(0)$ and take the distributional Fourier transform of this expression, then we get (10) which is also equivalent to (8). Conversely, a function defined by (8) is clearly of degree N and of regularity R. Thanks to Definition 4, the extension from PP functions to PP distributions (i.e., $R \leq -2$) is straightforward. The coefficients $p_{k,l}$ are unique because the polynomials $A_k(x)$ are unique—and so are the B_k's.

However, a distribution defined by (8) is not necessarily compactly supported. More precisely, (8) implies that, when x lies outside $[0, W]$, $\varphi(x) = \mathrm{sign}(x)\Pi(x)$, where $\Pi(x)$ is the following polynomial:

$$\begin{aligned}\Pi(x) &= \sum_{k=0}^{W}\sum_{l=0}^{N-R-1} p_{k,l}\frac{(x-k)^{N-l}}{2(N-l)!} \\ &= \frac{x^N}{2N!} * \left(\sum_{k=0}^{W}\sum_{l=0}^{N-R-1} p_{k,l}\delta^{(l)}(x-k)\right) \\ \xrightarrow{\mathcal{F}}\ \hat{\Pi}(\omega) &= \frac{\pi j^N}{N!}\delta^{(N)}(\omega)\left(\sum_{l=0}^{N-R-1}(j\omega)^l P_l(e^{-j\omega})\right).\end{aligned}$$

As a result, φ is compactly supported in $[0, W]$ if and only if $\Pi(x)$—or equivalently, its Fourier transform—vanishes. This happens if and only if (7) is satisfied, because $f(\omega)\delta^{(N)}(\omega) = 0$ is equivalent to $f^{(n)}(0) = 0$ for $n = 0 \ldots N$.

A consequence of the characterization (8) is that it is necessary that $N \geq R+1$ in order for nontrivial $\{N, W, R, 0\}$ functions to exist. ■

VII. Proof of Theorem 1

Assume that φ is of order L—we use the characterization (14)—and that it is supported within $[0, W]$. Let us define the function $\psi(x)$ by

$$\psi(x) = \sum_{k\geq 0}\frac{\mathrm{d}}{\mathrm{d}x}\varphi(x-k) = -\sum_{k\leq -1}\frac{\mathrm{d}}{\mathrm{d}x}\varphi(x-k) \qquad (25)$$

where the differentiation is taken in the sense of distributions. The rightmost equality results from (14) for order 1, which is equivalent to $\sum_k (\mathrm{d}/\mathrm{d}x)\varphi(x-k) = 0$. Thus, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\varphi(x) = \psi(x) - \psi(x-1). \qquad (26)$$

Taking the Fourier transform of both sides yields $\hat{\varphi}(\omega) = \hat{\beta}^0(\omega)\hat{\psi}(\omega)$, which proves that φ can be expressed as

$$\varphi = \beta^0 * \psi. \tag{27}$$

Then, we have the following properties:

a) $\psi(x)$ *is compactly supported within* $[0, W-1]$.
   According to (25), the support of $\psi(x)$ is contained in both $[0, +\infty[$ and $]-\infty, W-1]$. Hence, we can say that $\text{support}(\psi) \subset [0, W-1]$.

b) $\psi(x)$ *satisfies the Strang-Fix conditions of order* $(L-1)$.
   To prove this, we differentiate (14). This yields $\sum_k (\mathrm{d}/\mathrm{d}x)A(x-k)\varphi(x-k) + \sum_k A(x-k)(\mathrm{d}/\mathrm{d}x)\varphi(x-k) = 0$. We replace the first term of this equation by $C_{(\mathrm{d}A/\mathrm{d}x)}$ (since $(\mathrm{d}A/\mathrm{d}x)$ is a polynomial of degree strictly smaller than $L-1$), and we replace $(\mathrm{d}/\mathrm{d}x)\varphi(x)$ in the second term by its expression (26). Since ψ is compactly supported, we easily obtain

$$\sum_{k\in\mathbb{Z}} \underbrace{(A(x-k) - A(x-k+1))}_{B(x-k)} \psi(x-k) = -C_{\frac{\mathrm{d}A}{\mathrm{d}x}}.$$

   Note that the polynomial $B(x)$ defined above as the finite difference of $A(x)$ is of degree exactly one less than $\deg(A)$. Now, if $A(x)$ spans the entire set of polynomials of degree $\leq L-1$, then $B(x)$ spans the entire set of polynomials of degree $\leq L-2$. This also means that, for any polynomial $B(x)$ of degree $\leq L-2$, there exists a constant C_B such that $\sum_{k\in\mathbb{Z}} B(x-k)\psi(x-k) = C_B$. In addition, we see that if $A(x) = x$, then $B(x) = -1$; thus, $-\sum_{k\in\mathbb{Z}} \psi(x-k) = -\sum_{k\in\mathbb{Z}} \varphi(x-k)$. Integrating this equation over $[0,1]$ leads to $\int \psi = \int \varphi \neq 0$. Thus, $\psi(x)$ satisfies the Strang-Fix conditions of order $(L-1)$;

c) *If* φ *is in* $\{N, W, R, L\}$, *then* $\psi(x)$ *is in* $\{N-1, W-1, R-1, L-1\}$.
   It suffices to replace φ by (8) in (25) to prove that the degree and regularity of ψ are decreased by 1; the other points have already been shown in the previous items.

Thanks to these properties, we can reason by induction on the order of approximation, setting $\varphi_L = \varphi$ and $\varphi_{L-1} = \psi$. This induction process yields a set of $(L+1)$ distributions $\{\varphi_{L-l}(x)\}_{0\leq l\leq L}$ that enjoy the following properties:

a) $\varphi_{L-l}(x)$ is compactly supported within $[0, W-l]$;
b) $\varphi_{L-l}(x)$ satisfies Strang-Fix conditions of order $(L-l)$ and, if $l = L$, then $\int \varphi_0 \neq 0$;
c) $\varphi_{L-l}(x)$ is linked to $\varphi(x)$ through the convolution $\varphi = \beta^{l-1} * \varphi_{L-l}$;
d) $\varphi_L \in \{N, W, R, L\} \Rightarrow \varphi_{L-l} \in \{N-l, W-l, R-l, L-l\}$.

We thus have found a distribution $u = \varphi_0$ with $\int u \neq 0$, that has a support of length $(W-L)$ and such that (15) is satisfied.

Conversely, let us take a distribution $u = \varphi_0$ with $\int u \neq 0$, and that is supported within $[0, W-L]$. Then, the function defined as $\varphi(x) = (\beta^{L-1} * u)(x)$ is compactly supported within $[0, W]$ and is of approximation order L. Moreover, if u is PP of degree $(N-L)$ and of regularity $(R-L)$, then the convolution with β^{L-1} is PP of degree N and regularity R, as can be easily checked using the Fourier characterization (10) of PP functions.

This shows that it is not possible to have $W - L < 0$. Moreover, if φ is PP of degree N and order L, then $N - L \leq -2$ implies that u is made only of (multiple) derivatives of the Dirac distribution whose integral vanish ($\int u = 0$), which is impossible. Hence, we must have $N - L \geq -1$. ■

VIII. Proof of Theorem 2

Assume that φ satisfies the hypotheses of Theorem 2. Then, according to Lemma 1, there exist $(N-R)$ polynomials $P_l(z)$ of degree W satisfying (7) such that $\hat{\varphi}$ can be expressed according to (10). Denoting by $B_l(z), B'_l(z)$, the quotient of $P_l(z)$ modulo $(1-z)^{N-l+1}$ and its remainder, respectively, we have that $P_l(z) = (1-z)^{N-l+1}B_l(z) + B'_l(z)$, where $\deg(B'_l) \leq N-l$ and $\deg(B_l) \leq W-N+l-1$. Therefore, we can rewrite (10) as

$$\hat{\varphi}(\omega) = \frac{1}{(j\omega)^{N+1}} \sum_{l=0}^{N-R-1} (j\omega)^l B'_l(e^{-j\omega}) + \sum_{l=0}^{N-R-1} \underbrace{\left(\frac{1-e^{-j\omega}}{j\omega}\right)^{N-l+1}}_{\hat{\beta}^{N-l}(\omega)} B_l(e^{-j\omega}). \tag{28}$$

We express $B'_l(z)$ as $\sum_{k=0}^{N-l} b'_{k,l}(1-z)^k$; we also use the Fourier expression (12) of γ_k^n with the substitution $k \to l$ and $n \to N-k$, which yields

$$(j\omega)^l = \lambda_{l,N-k-l+1}(j\omega)^{N-k+1}\hat{\gamma}_l^{N-k}(\omega) + (1-e^{-j\omega})^l \Lambda_l^{N-k-l}(1-e^{-j\omega})$$

and which is valid for $1 \leq l$ and $k \leq N-l$. Then, the first summation in (28) can be transformed into (see the equation at the bottom of the next page). Because of (28), the left-hand side of the above expression is $O(\omega^{N+1})$, which implies that the right-hand side is $O(\omega^{N+1})$ itself. Since the first term of the right-hand side is obviously $O(\omega^{N+1})$, we can claim that the second term, namely $G(e^{-j\omega})$, is $O(\omega^{N+1})$ as well. However, $G(z)$ is a polynomial of degree at most N, and we have just shown that it should cancel $(N+1)$ times; the only admissible polynomial is thus $G(z) = 0$, which leaves us with the equality

$$\sum_{l=0}^{N-R-1} (j\omega)^l B'_l(e^{-j\omega}) = (j\omega)^{N+1} \times \sum_{l=1}^{N-R-1} \sum_{k=0}^{N-l} b'_{k,l}\lambda_{l,N-k-l+1} \left(\frac{1-e^{-j\omega}}{j\omega}\right)^k \hat{\gamma}_l^{N-k}(\omega).$$

Finally, if we let $a_{k,l} = b'_{k,l}\lambda_{l,N-k-l+1}$, then (28) becomes

$$\hat{\varphi}(\omega) = \sum_{l=1}^{N-R-1} \sum_{k=0}^{N-l} a_{k,l}\hat{\beta}^{k-1}(\omega)\hat{\gamma}_l^{N-k}(\omega) + \sum_{l=0}^{N-R-1} \hat{\beta}^{N-l}(\omega)B_l(e^{-j\omega}).$$

This is exactly the Fourier transform of (18), if we express $B_l(z)$ as $\sum_{k=0}^{\deg(B_l)} b_{k,l}z^k$ and if we remember that $\deg(B_l) \leq W - N + l - 1$. This proves that any function of degree N, support W, and regularity R, can be expressed as (18). Conversely, because

the support of γ_k^n is n and that of β^n is $(n+1)$, the function φ defined by (18) is necessarily of support $\max(N, W)$. It is also obviously of degree N and regularity R.

Note that this decomposition is unique because of the unicity of the Euclidean division of $P_l(z)$ by $(1-z)^{N-l+1}$. ■

IX. Proof of Theorem 3

Behold! ■

X. Proof of Corollary 3'

We will first prove the result when $L = 0$, in which case we do not have the third sum in (20). Obviously, there are functions φ_0 in $\{N, W, R, L\}$ for which (23) holds (e.g., $\varphi_0 = (1/2)\varphi$). We show now that, if the conditions (22) are satisfied, then there is exactly one such function. This will be done in two steps: first, there is *at most* one φ_0; second, there is *at least* one φ_0.

1) Step 1: Unicity: We have to show that the function 0 can be reconstructed only with the coefficients $a_{k,l} = b_{k,l} = 0$. According to Lemma 1, $\hat{\varphi}_0$ can be expressed as (10). For $l \geq 1$, we let $P_l(z) = B'_l(z) + (1-z)^{N-l+1}B_l(z)$ where B'_l is the (unique) remainder of the Euclidean division of $P_l(z)$ by $(1-z)^{N-l+1}$. We have

$$B'_l(z) = \sum_{k=0}^{N-l} \frac{a_{k,l}}{\lambda_{l,N-k-l+1}}(1-z)^k$$
$$B_l(z) = \sum_{k=0}^{W-N+l-1} b_{k,l}z^k.$$

The equality $\varphi_0(x) + \varphi_0(W-x) = 0$ implies that $P_l(z) + (-1)^{N-l+1}z^W P_l(z^{-1}) = 0$ for $l = 0 \ldots N-R-1$. A consequence is that, for $l = 1 \ldots N-R-1$

$$B'_l(z) + (-1)^{N-l+1}z^W B'_l(z^{-1}) \equiv 0 \pmod{(1-z)^{N-l+1}}. \tag{29}$$

Two cases arise:

- $(N-l)$ *is odd.* Then, (29) implies that $B'_l(1) = 0$. In addition, due to (22), we have that $(1-z)^2$ divides B'_l. Thus, $(B'_l(z)/(1-z)^2) + z^{W-2}(B'_l(z^{-1})/(1-z^{-1})^2) \equiv 0 \pmod{(1-z)^{N-l-1}}$, and we can do the same reasoning again, which shows that $(1-z)^2$ divides $(B'_l(z)/(1-z)^2)$. By induction, this shows that $(1-z)^{N-l+2}$ divides $B'_l(z)$, i.e., $B'_l = 0$.
- $(N-l)$ *is even.* Then, (22) tells us that $(1-z)$ divides $B'_l(z)$. Using (29), we get that $(B'_l(z)/1-z) + z^{W-1}(B'_l(z^{-1})/1-z^{-1}) \equiv 0 \pmod{(1-z)^{N-l}}$, which shows that $B'_l(z) = 0$, using the same reasoning as when $(N-l)$ is odd.

Since $B'_l(z) = 0$ for $l = 1 \ldots N-R-1$, we are left with $B_l(z) + (-1)^{N-l+1}z^W B_l(z^{-1}) = 0$, which trivially implies $B_l(z) = 0$, thanks to (22). Finally we have $P_0 = B_0$, which must satisfy $B_0(z) + (-1)^{N+1}z^W B_0(z^{-1}) = 0$. Together with (22), this implies that $B_0(z) = 0$;

2) Step 2: Completeness: We count the number of free parameters for a symmetric $\{N, W, R, 0\}$ kernel, and verify that it coincides with the number of parameters implied by (22). According to Lemma 1, a kernel in $\{N, W, R, 0\}$ is characterized by $(N-R)$ polynomials $P_l(z)$ of degree W satisfying (7). Symmetry is equivalent to $P_l(z) = (-1)^{N-l+1}z^W P_l(z^{-1})$.

If we pick $l \neq 0$, then symmetry implies that $P_l(z)$ is specified using $\lfloor (W+1/2) \rfloor$ independent coefficients when $(N-l)$ is even, and using $(1+\lfloor (W/2) \rfloor)$ independent coefficients when $(N-l)$ is odd. On the other hand, (22) indicates that φ_0 is specified by $\lfloor (W-N+l+1/2) \rfloor + \lfloor (N-l+1/2) \rfloor$, which matches the dimensionality of $P_l(z)$ exactly.

If $l = 0$, then (7) is equivalent to stating that the remainder of the Euclidean division of $P_0(z)$ by $(1-z)^{N+1}$ is given from the knowledge of the P_l's, for $l \neq 0$. Denoting by $K_0(z)$ this remainder, the symmetry property on $\{P_l(z)\}_{l \neq 0}$ automatically enforces $K_0(z) \equiv (-1)^{N+1}z^W K_0(z^{-1}) \pmod{(1-z)^{N+1}}$. Thus, we can let $P_0(z) = (1/2)(K_0(z) + (-1)^N z^W K_0(z^{-1})) + (z-1)^{N+1}K_1(z)$, where $K_1(z)$ is some polynomial of degree $(W-N-1)$—if $W \leq N$, then $K_1(z) = 0$. The symmetry property on $P_0(z)$ leads to $K_1(z) = z^{W-N-1}K_1(z^{-1})$, which implies that the number of free parameters of $P_0(z)$ is $\lfloor (W-N+1/2) \rfloor$. This number matches the number of nonzero $b_{k,0}$ in (22), exactly. We can thus say that a symmetric

---

$$\begin{aligned}
\sum_{l=0}^{N-R-1} (j\omega)^l B'_l(e^{-j\omega}) &= B'_0(e^{-j\omega}) + \sum_{l=1}^{N-R-1}\sum_{k=0}^{N-l} b'_{k,l}(j\omega)^l(1-e^{-j\omega})^k \\
&= B'_0(e^{-j\omega}) + \sum_{l=1}^{N-R-1}\sum_{k=0}^{N-l} b'_{k,l}\left[\lambda_{l,N-k-l+1}(j\omega)^{N-k+1}\hat{\gamma}_l^{N-k}(\omega)\right. \\
&\quad \left. + (1-e^{-j\omega})^l \Lambda_l^{N-k-l}(1-e^{-j\omega})\right](1-e^{-j\omega})^k \\
&= (j\omega)^{N+1}\sum_{l=1}^{N-R-1}\sum_{k=0}^{N-l} b'_{k,l}\lambda_{l,N-k-l+1}\left(\frac{1-e^{-j\omega}}{j\omega}\right)^k \hat{\gamma}_l^{N-k}(\omega) \\
&\quad \underbrace{+ B'_0(e^{-j\omega}) + \sum_{l=0}^{N-R-1}\sum_{k=0}^{N-l} b'_{k,l}(1-e^{-j\omega})^{k+l}\Lambda_l^{N-k-l}(1-e^{-j\omega})}_{G(e^{-j\omega})}
\end{aligned}$$

$\{N, W, R, 0\}$ kernel requires as many parameters as the representation characterized by Conditions (22). Because of the linear independence of this representation (see Step 1: unicity), we conclude that the representation is complete.

We have proven the corollary for $L = 0$. If $L \neq 0$, we apply Theorem 1. The resulting distribution, u, is symmetric if φ is symmetric. Moreover, we can decompose u according to Corollary 1' as the sum of a function $\psi \in \{N - L, W - L, \max(-1, R - L), 0\}$, and of a sum of derivatives of Diracs. Since u is symmetric, so is ψ, which allows us to apply Theorem 3' for $L = 0$. The Dirac part is symmetric as well, and it is a simple matter—because δ is symmetric—to show that the $c_{k,l}$ part of (22) constitutes a complete and unique representation of it. ■

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