A Theoretical Analysis of the Projection Error onto Discrete Wavelet Subspaces

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ABSTRACT

A filterbank decomposition can be seen as a series of projections onto several discrete wavelet subspaces. In this presentation, we analyze the projection onto one of them—the low-pass one, since many signals tend to be low-pass. We prove a general but simple formula that allows the computation of the ℓ^2-error made by approximating the signal by its projection. This result provides a norm for evaluating the accuracy of a complete decimation/interpolation branch for arbitrary analysis and synthesis filters; such a norm could be useful for the joint design of an analysis and synthesis filter, especially in the non-orthonormal case. As an example, we use our framework to compare the efficiency of different wavelet filters, such as Daubechies' or splines. In particular, we prove that the error made by using a Daubechies' filter downsampled by 2 is of the same order as the error using an orthonormal spline filter downsampled by 6. This proof is valid asymptotically as the number of regularity factors tends to infinity, and for a signal that is essentially low-pass. This implies that splines bring an additional compression gain of at least 3 over Daubechies' filters, asymptotically.

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1. INTRODUCTION

Filterbanks have become a standard way to analyze or to compress nonstationary signals. Beause of the strong interpretation—frequency separation of the channels—given to the analysis it is customary to optimize the analysis filters so that they get as close as possible to the ideal principal component filters.[1,2] In this approach, the filters are most of the time assumed to be orthonormal, so that the synthesis side has exactly the same properties as the analysis side. When it came to the biorthonormal generalization of filterbanks, researchers have seeked to remain as close as possible to the orthonormal case by requiring that the synthesis filters be close to the analysis filters, at least as far as frequency behavior was concerned.[3]

We maintain however that this measure is not the most natural for evaluating the accuracy of a filterbank. Instead, we propose here a new measure that is directly linked to the accuracy of the analysis-synthesis reconstruction when we keep only the most significant branch of the filterbank. This idea arises from the observation that most signals that can be well compressed have their frequency content concentrated in a small fraction of the sampling frequency interval: this is for example the case for images, which have a very lowpass behavior.

This measure has the advantage of putting together the analysis and the synthesis filters into a single expression, and thus allows for more general biorthonormal designs than those previously available. The natural advantage of this measure is that it directly provides a signal-to-noise ratio (SNR) of the whole scheme: the measure can be interpreted as the SNR resulting from a large quantization step in the other bands of the filterbank.

The quartic form we are proposing can be easily computed. We give a simple induction relation to obtain the quantities of interest, if iterated filters are to be used.

Since we are concentrating on lowpass signals, we have also evaluated the limit behavior of a filterbank when the bandwidth of the signal tends to zero. This gives rise to an extension of what is known as "approximation order" in approximation theory. Similarly, we extend to the discrete case the well-known Strang-Fix conditions[4] and obtain the asymptotic constants that characterize the lowpass behavior of a filterbank.

Finally, we give some examples of the accuracy that can be obtained, for example by using spline filters instead of Daubechies orthonormal filters. It is interesting to note that the gain can be quite large. As a matter of fact, for large approximation order, the sampling gain spline/Daubechies is between 3 and π in terms of the number of iterations.

1.1. Notations

Discrete real sequences$\{x[n]\}_{n\in\mathbb{Z}}$ will be designated without reference to the running index n, e.g. x here. Their associated z-transform $\sum_n x[k]z^{-k}$ will be denoted using upper case letters; e.g. $X(z)$.

We also introduce a notation for the shifted version of a sequence by m samples: $x_m[n] = x[n-m]$ for all $n \in \mathbb{Z}$. Analogously, f_μ denotes the function f shifted by the real number μ; i.e., $f_\mu(t) = f(t-\mu)$.

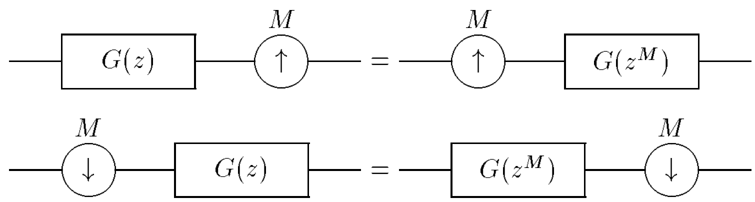
Usual operations on discrete data are depicted graphically as follows

- Upsampling operator $x \longrightarrow \overset{M}{\textcircled{\uparrow}} \longrightarrow y$, which maps x into y with $y[Mn+n_0] = x[n]\delta[n_0]$ for all $n \in \mathbb{Z}$ and $n_0 = 0 \ldots M-1$.

- Downsampling operator $x \longrightarrow \overset{M}{\textcircled{\downarrow}} \longrightarrow y$, which maps x into y with $y[n] = x[Mn]$ for all $n \in \mathbb{Z}$.

- Filter $x \longrightarrow \boxed{G(z)} \longrightarrow y$, which maps x into y with $y[n] = \sum_{k\in\mathbb{Z}} g[n-k]x[k]$ for all $n \in \mathbb{Z}$.

We also recall the two standard exchange rules, widely known as the "noble relations"[5,6]:



2. ANALYSIS-SYNTHESIS FILTERBANKS

An analysis filterbank is a Single Input Multiple Output (SIMO) linear system that maps an input signal x into multiple output signals $y^{(j)}$ as shown in Fig. 1.

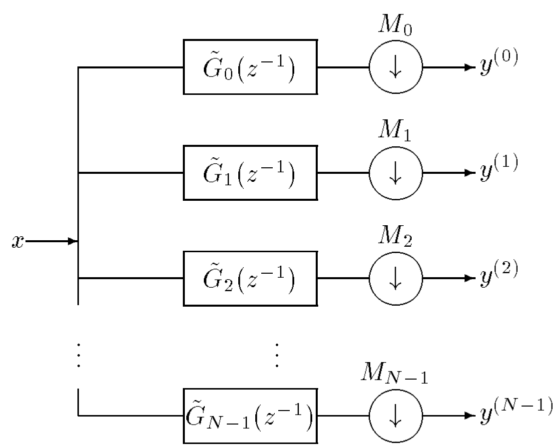


Figure 1. Analysis filterbank

Similarly, a synthesis filterbank is a Multiple Input Single Output (MISO) linear system that maps multiple input signals $y^{(j)}$ into a single output signal x as shown in Fig. 2.

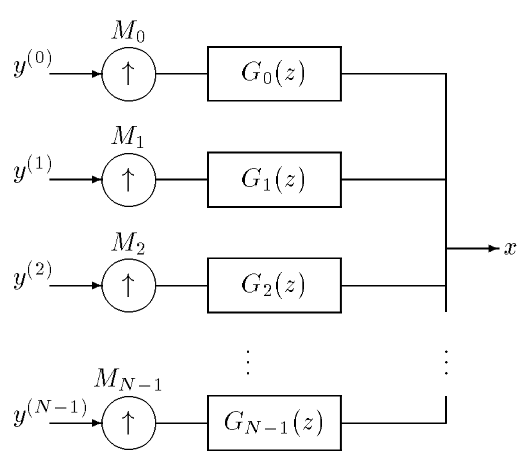


Figure 2. Synthesis filterbank

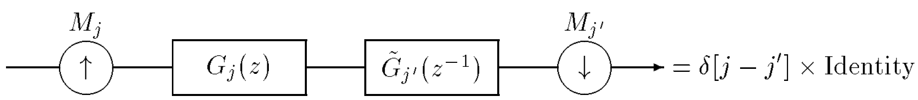
2.1. Perfect reconstruction filterbanks

When the input and output rates are equal (i.e., $1 = \sum_{j=0}^{N-1} M_j^{-1}$), then the filterbank is said to be critically sampled. This condition is frequently required in practice, even though it may be advisable for some applications to allow oversampling.[7]

The implementation of a dyadic discrete wavelet transform takes precisely the form of an analysis filterbank where $M_j = 2^{j+1}$ for $j = 0 \ldots N-2$ and $M_{N-1} = 2^{N-1}$, and where the G_j take a special form.

Another example is the implementation of a Short Time Fourier Transform (STFT) which takes the same form, but where the analysis filterbank is now uniform; i.e., $M_j = M$ for all $j = 0 \ldots N-1$.

In those two cases, the inverse operator of an analysis filterbank takes the form of a synthesis filterbank, but this property is not so general.[8,9] We will from now on concentrate on such filterbanks. The perfect reconstruction condition is equivalent to the N^2 graphical equations



for all $j, j' = 0 \ldots N-1$. These well-known conditions in wavelet theory are equivalent to the biorthonormality conditions. Their mathematical formulation is

$$\sum_k g_j[nM+k]\, \tilde{g}_{j'}[k] = \delta[j-j']\delta[n] \tag{1}$$

for all $j, j' = 0 \ldots N-1$ and all $n \in \mathbb{Z}$.

We will see that it is possible to cast the perfect reconstruction problem into an approximation problem for which we have developped efficient tools.[10–12]

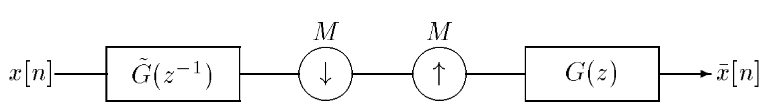
2.2. The basic ℓ^2-approximation problem

In most cases, the interest of transforming a signal into downsampled subbands arises from the need to find a more compact representation of the input signal. At first sight, increasing the number of signals—although downsampled—does not compress the information; however, if most of the subband signals have such a negligible energy that they will contribute little to the reconstruction, then a substantial coding gain can be achieved by discarding these subbands. The natural question that one is thus led to formulate is: is there an optimal way to choose the filters G_j and $\tilde{G}_j$ so that most subbands contribute for nothing to the reconstruction? The extreme case is obviously when *all* subbands, except for the first, can be neglected.

We have thus to answer the dual question: how much of the signal do we loose if we reconstruct with the first branch only? This is truly an approximation problem, as we are going to see.

We first need to define a measure for estimating the approximation error. Our choice is the ℓ^2 norm for discrete sequences $\|x\|_{\ell^2} = \left(\sum_n x[n]^2\right)^{\frac{1}{2}}$.

The reconstruction of a signal with only one subband can be depicted graphically as



(2)

and we are interested in evaluating $\|x - \bar{x}\|_{\ell^2}$. The mathematical expression of the sequence $\bar{x}$ is

$$\bar{x}[n] = \sum_{k,k'} g[n-k'M]\, \tilde{g}[k-k'M]\, x[k] \tag{3}$$

This transformation is a projector, (i.e., $\bar{\bar{x}} = \bar{x}$) if and only if G and $\tilde{G}$ are biorthonormal. However, it is not integer-shift invariant: if we shift x by one sample we do not obtain a shifted version of $\bar{x}$. In fact, we have

$$\bar{x}_m = \overline{x_m} \text{ if } M \text{ divides } m$$

2.2.1. Continuous formulation of the problem

Notice that if we build the function $x(t)$

$$x(t) = \sum_n x[n] \operatorname{sinc}(t-n) \tag{4}$$

then we have $\|x\|_{\mathbf{L}^2} = \|x\|_{\ell^2}$. That is, if we are to evaluate the ℓ^2 difference of the discrete sequences, we might as well evaluate the $\mathbf{L}^2$ difference of the functions $x(t)$ and $\bar{x}(t)$. Let us emphasize that by using this discrete-to-continuous reformulation, we do not mean that the signal from which we got the samples $x[n]$ would be best represented within a sinc space. Instead, the present formulation only aims at defining an equivalent—continuous—form of the problem.

Using the same "discrete to continuous" mapping, we define the two functions

$$\varphi(t) = \sqrt{M} \sum_k g[k] \operatorname{sinc}(Mt-k) \tag{5}$$

$$\tilde{\varphi}(t) = \sqrt{M} \sum_k \tilde{g}[k] \operatorname{sinc}(Mt-k) \tag{6}$$

A consequence of the *orthonormality* of the sinc basis and of the perfect reconstruction conditions (1) is that φ and $\tilde{\varphi}$ are *biorthonormal*, i.e., $\int \tilde{\varphi}(\tau - n)\varphi(\tau)\, d\tau = \delta[n]$ for all $n \in \mathbb{Z}$.

Then, it is not too difficult to see that

$$\bar{x}(t) = \sum_k \left\{ \int x(\tau)\tilde{\varphi}\left(\frac{\tau}{M} - k\right) d\frac{\tau}{M} \right\} \varphi\left(\frac{t}{M} - k\right) \tag{7}$$

This expression takes exactly the same form as the usual approximation of a function onto an integer shift-invariant space $V_T = \operatorname{span}_{k\in\mathbb{Z}}\{\varphi(\frac{t}{T} - k\}$ with $T = M$; $\tilde{\varphi}(t)$ is the so-called sampling distribution.[10] We can thus use the powerful results obtained in the field of approximation theory to evaluate $\|x - \bar{x}\|_{\ell^2}$.

3. CONTINUOUS APPROXIMATION RESULTS

We apply here one of the main theorems we have obtained in the evaluation of the approximation error of a function $x(t) \in \mathbf{W}_2^r$, the Sobolev space of order r (i.e., $x(t)$ has r derivatives in $\mathbf{L}^2$) where $r > 0.5$.[10,11] We assume that $|G(e^{i\omega})|$ and $|\tilde{G}(e^{i\omega})|$ are upper bounded over $[-\pi, \pi]$.

Theorem 3.1. *Let $E(\omega)$ be defined by*

$$E(\omega) = \left|1 - \hat{\tilde{\varphi}}(\omega)^* \hat{\varphi}(\omega)\right|^2 + |\hat{\tilde{\varphi}}(\omega)|^2 \sum_{n\neq 0} |\hat{\varphi}(\omega + 2n\pi)|^2. \tag{8}$$

Then, for any integer N greater than 1, we have

$$\frac{1}{M}\int_0^M \|x_\mu - \overline{x_\mu}\|_{\mathbf{L}^2}^2\, d\mu = \int |\hat{x}(\omega)|^2 E(\omega M)\, \frac{d\omega}{2\pi}. \tag{9}$$

Moreover, if $\hat{x}(\omega)\hat{x}(\omega + \frac{2k\pi}{M}) = 0$ for all $k \in \mathbb{Z}_*$ and $\omega \in \mathbb{R}$, then $\|x_\mu - \overline{x_\mu}\|_{\ell^2} = \|x - \bar{x}\|_{\ell^2}$ for all real μ, i.e., $\|x - \bar{x}\|_{\ell^2}^2$ can be expressed exactly as the right-hand side of (9).

Proof. It suffices to verify that the conditions required in[10] are met. Obviously, $x(t)$ belongs to $\mathbf{W}_2^{\frac{1}{2}+\varepsilon}$ since $\{x[n]\}_{n\in\mathbb{Z}}$ is in ℓ^2; it is even in $\mathbf{W}_2^a$ for any $a > 0$, because of our choice of the sinc basis bunction.

The other requirements for these results to hold are that $\hat{\tilde{\varphi}}$ be upper bounded, and that φ satisfy the upper Riesz condition $\sum_n |\hat{\varphi}(\omega + 2n\pi)|^2 \leq \text{Const} < \infty$ for almost every $\omega \in [-\pi, \pi]$. This is clearly the case if $|G(e^{i\omega})|$ and $|\tilde{G}(e^{i\omega})|$ are bounded. Thus we can apply the main approximation theorem.[10] $\square$

From this "continuous" average result we can deduce the following "discrete" average theorem.

THEOREM 3.2. *Let $F_M(\omega)$ be defined by*

$$F_M(\omega) = \left|1 - \frac{\tilde{G}(e^{i\omega})^* G(e^{i\omega})}{M}\right|^2 + \frac{|\tilde{G}(e^{i\omega})|^2}{M^2} \sum_{k=1}^{M-1} |G(e^{j(\omega + \frac{2k\pi}{M})})|^2 \tag{10}$$

Then we have

$$\frac{1}{M} \sum_{m=0}^{M-1} \|x_m - \overline{x_m}\|_{\ell^2}^2 = \int_{-\pi}^{\pi} |X(e^{i\omega})|^2 F_M(\omega) \frac{d\omega}{2\pi} \tag{11}$$

This theorem states that the quality of the approximation is directly given by the rhs of (11). In particular, if the signal's main energy is concentrated in a frequency interval I, then the quality of the approximation is given by the values of $F_M(\omega)$ over this interval.

As an example, if $I = [-\frac{\pi}{M}, \frac{\pi}{M}]$ and if x is white within this frequency band, then

$$J(G, \tilde{G}) = \int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} F_M(\omega)\, d\omega \tag{12}$$

is the expression to minimize in a corresponding design problem. Note that, when G is orthonormal and $\tilde{G} = G$, we recover the classical $\mathbf{L}^2$ design measure $J(G) = \frac{2}{M} \int_{\frac{\pi}{M}}^{\pi} |G(e^{i\omega})|^2\, d\omega$, i.e., the energy of the attenuated band.

3.1. Least-squares error

We transform the expression (10) into

$$F_M(\omega) = \underbrace{1 - \frac{|G(e^{i\omega})|^2}{A(e^{Mi\omega})}}_{F_{\text{opt}}(\omega)} + \frac{A(e^{Mi\omega})}{M^2} \left|\tilde{G}(e^{i\omega}) - \tilde{G}_{\text{opt}}(e^{i\omega})\right|^2 \tag{13}$$

where $A(e^{i\omega}) = \sum_{k=0}^{M-1} |G(e^{i\frac{\omega+2k\pi}{M}})|^2$ and $\tilde{G}_{\text{opt}}(e^{i\omega}) = M \dfrac{G(e^{i\omega})}{A(e^{Mi\omega})}$.

It is clear from (13) that the approximation kernel $F_M(\omega)$ is always larger than the expression $F_{\text{opt}}(\omega)$, which is attained only when $\tilde{G} = \tilde{G}_{\text{opt}}$, a special choice of the analysis filter. This optimum corresponds to the orthogonal projection of the sequence x onto the vector space generated by the sequences $\{g_{\cdot - kM}\}$.

The optimal kernels corresponding to Daubechies' filter of order 4 (i.e., length 8) and to the cubic spline filter (i.e., $G(z) = \sqrt{2}(\frac{1+z^{-1}}{2})^4$) are plotted in Fig. 3.

3.2. Approximation order

It is particularly interesting to characterize the approximating behavior of the projector depicted in (2) at low frequencies, since many natural signals are essentially low-pass (e.g., images). For this, we extend the classical notion of approximation order[4] to these discrete schemes.

We can continuously change the scale of the signal x by building $x^f(t) = \sqrt{f}.x(ft)$; the amplitude factor $\sqrt{f}$ ensures that $\|x^f\|_{\mathbf{L}^2} = \|x\|_{\mathbf{L}^2}$. When $f < 1$, this transformation amounts to shrinking the digital spectrum of the sequence x by a ratio f and to setting to 0 the values that are between $f\pi$ and π. Thus, when $f < 1$, f is the measure of the bandwidth of the scaled signal x^f.

We are thus interested in evaluating $\|x^f - \overline{x^f}\|_{\ell^2}$ when f tends to 0. We say that the filters $(G, \tilde{G})$ are of approximation order L if $\|x^f - \overline{x^f}\|_{\ell^2} \propto f^L$ as $f \to 0$.

THEOREM 3.3. *Assume that the biorthonormal filters $G(e^{i\omega})$ and $\tilde{G}(e^{i\omega})$ are continuous at $\omega = 0$. Then, they define an order L approximation scheme iff $\left(\frac{1-z^{-M}}{1-z^{-1}}\right)^L$ divides $G(z)$.*

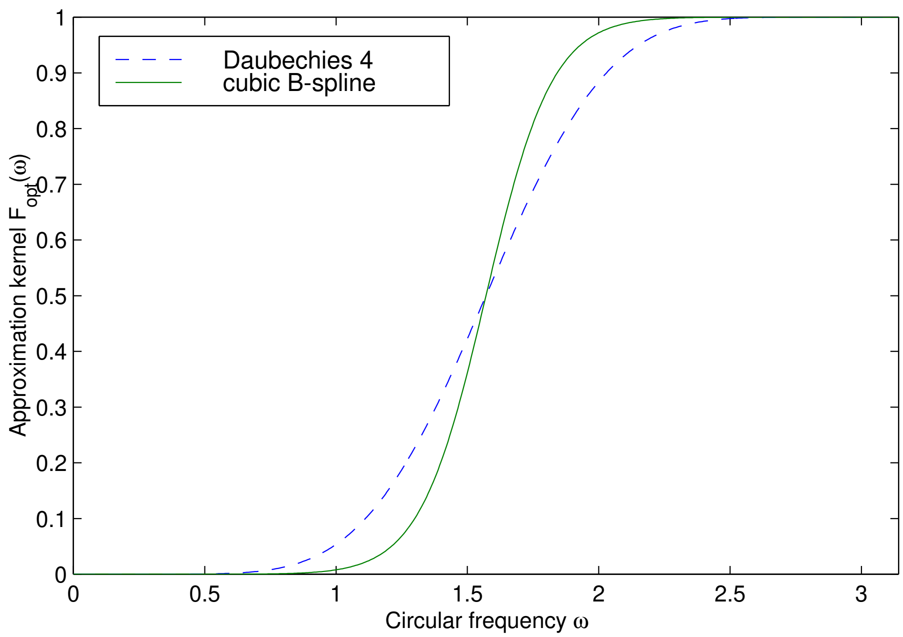


Figure 3. Plot of the least-squares approximation kernels of Daubechies' order 4 filter and of the B-spline filter $\sqrt{2}(\frac{1+z^{-1}}{2})^4$; Although both filters have the same order, the B-spline outperforms Daubechies' in the first halfband

Moreover, we have $\|x^f - \overline{x^f}\|_{\ell^2} = C_{G,\tilde{G}}\, f^L\, \|x^{(L)}\|_{\mathbf{L}^2} + o(f^L)$ *with*

$$C^2_{G,\tilde{G}} = \frac{1}{M^2}\left|\sum_{k=1}^{M-1}\frac{\tilde{G}(e^{\frac{2ik\pi}{M}})^* Q(e^{\frac{2ik\pi}{M}})}{(1-e^{\frac{-2ik\pi}{M}})^L}\right|^2 + \frac{1}{M}\sum_{k=1}^{M-1}\frac{|Q(e^{\frac{2ik\pi}{M}})|^2}{|1-e^{\frac{-2ik\pi}{M}}|^{2L}} \tag{14}$$

where we have defined $Q(z)$ *by* $Q(z) = \left(\frac{1}{M}\frac{1-z^{-M}}{1-z^{-1}}\right)^{-L} G(z)$.

Proof. When f is smaller than $\frac{1}{M}$ we know from Theorem 3.1 that $\|x^f - \overline{x^f}\|_{\ell^2}$ is exactly given by the right-hand side of (11). Thus, the condition that $(G,\tilde{G})$ be of approximation order L is equivalent to the condition $\int_{-\pi}^{\pi}|X(e^{i\omega})|^2 F_M(f\omega)\,d\omega \propto f^{2L}$ as $f\to\infty$. Then, using a similar technique as in,[10] we find that this condition is equivalent to requiring that $F_M(\omega)$ cancels $2L$ times at $\omega = 0$. Since the filters are biorthonormal, this is finally equivalent to require that $\left(\frac{1-z^{-M}}{1-z^{-1}}\right)^L$ divides $G(z)$. □

The first part of this theorem is the discrete analog of the well-known Strang-Fix equivalence.[4] When $\tilde{G}(e^{\frac{2ik\pi}{M}}) = 0$ for $k = 1 \ldots M-1$, the second part of the theorem is also the discrete analog of Unser's expression of the asymptotic constant[13]; under this mild condition, always met in practice, we get

$$C_{G,\tilde{G}} = C_G = \sqrt{\frac{1}{M}\sum_{k=1}^{M-1}\frac{|Q(e^{\frac{2ik\pi}{M}})|^2}{|1-e^{\frac{-2ik\pi}{M}}|^{2L}}}. \tag{15}$$

This expression is in fact the least-squares asymptotic constant; it might prove useful to minimize it when the input signal x has a very low-pass behavior.

3.3. Approximation results for wavelets

The expression C_G can be computed for filters that are generated by a multiresolution analysis; i.e., for filters G_j of the form $G_j(z) = G(z)G(z^2)\ldots G(z^{2^{j-1}})$ and for $M = 2^j$.

THEOREM 3.4. *Let* $A_j(z)$ *be defined by induction by* $A_0 = 1$ *and by*

$$A_j(z^2) = \frac{1}{2}\left[G(z)G(z^{-1})A_{j-1}(z) + G(-z)G(-z^{-1})A_{j-1}(-z)\right]$$

for $j \geq 1$*. Then, if* $G(z) = \left(\frac{1+z^{-1}}{2}\right)^L Q(z)$*, we have*

$$C^2_{G_j} = C^2_{G_{j-1}} + 2^{2L(j-2)-1} Q(-1)^2 A_{j-1}(-1), \tag{16}$$

which allows to compute $C^2_{G_j}$ *inductively from* $C^2_{G_0} = 0$ *.*

As a special case, when G *is orthonormal, then we have* $A_j(z) = 1$ *for all* j*, and thus*

$$C^2_{G_j} = \frac{|Q(-1)|^2}{2^{2L+1}} \frac{2^{2Lj} - 1}{2^{2L} - 1} \tag{17}$$

Proof. The demonstration uses the same trick as for the asymptotic "continuous" constant.[12] □

Of course, when j tends to infinity, we recover the asymptotic constant for the limit functions generated by the filter G, and its expression is given in.[10,12]

4. EXAMPLES: SPLINES AND DAUBECHIES

It turns out that it is possible to express the constant for Daubechies and spline filters. In the first case, taking into account the formula $|Q(-1)|^2 = \binom{2L}{L}$ that was given in,[12] we get

$$D_j = \underbrace{4^{-L}\sqrt{\frac{\binom{2L}{L}}{2(1-4^{-L})}}}_{\text{“continuous” asymptotic constant}^{12}} \sqrt{2^{2Lj} - 1} \tag{18}$$

for the Daubechies discrete asymptotic constant.

On the other side, we use the direct expression of the B-spline filter, i.e., $G(z) = \sqrt{M}\left(\frac{1}{M}\frac{1-z^{-M}}{1-z^{-1}}\right)^L$ in (15) for computing the B-spline constant, and we get

$$S_M = \sqrt{\sum_{k=1}^{M-1} \frac{1}{|1 - e^{\frac{-2ik\pi}{M}}|^{2L}}} \tag{19}$$

We can compare the constants, when the approximation order, L, tends to infinity. We see that

$$D_j \approx \frac{2^{L(j-1)}}{\sqrt[4]{4\pi L}} \text{ as } L \to \infty$$

using Stirling formula, as in.[12] On the other side, for the spline constant, we get

$$S_M \approx \frac{\sqrt{2}}{|1 - e^{-\frac{2i\pi}{M}}|^L} \text{ as } L \to \infty$$

We thus see that asymptotically as $L \to \infty$, a downsampling by

$$M = \frac{\pi}{\arcsin 2^{-j}} \tag{20}$$

for the spline scheme yields the same low-pass approximation error as the Daubechies scheme with a downsampling by 2^j. In other words, this means that, by using the spline filter, we are able to compress the data $\frac{\pi}{2^j \arcsin 2^{-j}}$ times more than with an iterated Daubechies filter. When $j \to \infty$, we observe that we get the result stated in[12] that splines are asymptotically π-times better than Daubechies. In fact, even for $j = 1$ we obtain a compression gain of 3, a result that is already close to π.

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