LEARNING STEERABLE WAVELET FRAMES

Nicolas Chenouard[⋆†] *and Michael Unser*[⋆]

[⋆]Biomedical Imaging Group (BIG), École polytechnique fédérale de Lausanne (EPFL)
[†]Center for Biomedical Imaging (CIBM), Université de Lausanne (UNIL)

ABSTRACT

We present a functional framework for the adaptive design of dictionaries where the invariance to translation, dilation, and rotation is built upfront into the primary representation space. Our key idea is to build an invariant signal representation *prior* to the learning stage. By doing so, we focus our effort on adapting the dictionary to the distinctive features of the signal, rather than to the cumbersome encoding of the desired invariance properties of the representation. We thus avoid the pitfall of traditional dictionary-learning techniques that need to allocate considerable computational power to laboriously obtain some degree of invariance of the representation space. Moreover, we avoid the redundancy of representation which is typical of early works on dictionary learning for image coding, where several translated, dilated, and rotated copies of the same two-dimensional function are necessary [4, 2, 1, 3], whereas we need just one.

The backbone of our construction is a primal isotropic wavelet frame that provides the multiresolution decomposition of the signal and the invariance to arbitrary translations [5]. The invariance to rotation, often considered to be challenging, is obtained by applying an N-th order Riesz operator, which is designed to be steerable and self-reversible [7]. The method amounts to applying a one-to-many mapping to the wavelets and to steering the resulting coefficients with respect to the local orientation of the signal [6]. In practice, the steered coefficients of a translated and rotated pattern depend on the pattern only, and depend neither on its translation nor on its rotation. The transformed pattern can thus be learned in the wavelet frame without regard for its initial orientation. Faster dictionary-learning techniques and lower-dimensional spaces should be feasible thanks to the reduced complexity of the learning problem which is yielded by using the primary invariant signal representation.

A crucial benefit of the Riesz-wavelet transform is its self-reversibility, which leads to fast analysis and synthesis algorithms. Moreover, our implementation relies on a primary wavelet decomposition which is subsampled, hence yielding a moderate over-completness of the representation space which can be further controlled by the degree of the Riesz operator. We thus advocate the proposed framework as a mean to gain invariance properties along with efficient algorithms and a moderate redundancy of the representation space.

To demonstrate the validity of the proposed framework for dictionary learning, we propose a simple but illustrative learning technique which consists in adaptively shaping each scale of the Riesz-wavelet transform with an $M \times P$ matrix $\mathbf{U}$ where M is the number of Riesz channels and P is greater or equal to M. We investigate further the properties of $\mathbf{U}$ such that the equalization of the channels and the self-reversibility of the full transform are preserved. It is worth noting that the learned dictionary, as a linear combination of the Riesz-wavelet vectors, naturally inherits the invariance property of the primary representation space. This approach can thus be viewed as a recipe for the design of steerable wavelet frames.

As a first application, we focus on learning optimized steerable frames for image denoising. We constrain $\mathbf{U}$ to be square and unitary for several practical reasons: the resulting transform is naturally self-reversible and equalized, it gives rise to fast decomposition/reconstruction algorithms, and no increase of the overcompletness is implied. We use a PCA learning technique which amounts to diagonalizing the covariance matrix of the multichannel Riesz-wavelet coefficients in each scale. As a result, we obtain a generalized Riesz-wavelet transform in which the SNR is maximized in the first channel. We show that applying a soft-thresholding operator to the learned image representation outperforms the non-adaptive counterpart of these transforms, which demonstrates the ability of the representation space to adapt to the signal characteristics.

***Keywords*—** Wavelet Frames, Steerability, Dictionary learning, Riesz transform.

1. REFERENCES

[1] M. Aharon, M. Elad, and A. Bruckstein. K-svd: An algorithm for designing overcomplete dictionaries for sparse representations. *IEEE Transactions on Signal Processing*, 54(11):4311–4322, 2006.

[2] M. S. Lewicki and B. A. Olshausen. Probabilistic framework for the adaptation and comparison of image codes. *Journal of the Optical Society of America A*, 16(7):1587–1601, Jul 1999.

[3] J.. Mairal, G. Sapiro, and M. Elad. Learning multiscale sparse representations for image and video restoration. *Multiscale Modeling & Simulation*, 7(1):214–241, 2008.

[4] B. A. Olshausen and D. J. Field. Sparse coding with an overcomplete basis set: A strategy employed by v1? *Vision Research*, 37(23):3311 – 3325, 1997.

[5] E. P. Simoncelli and W. T. Freeman. The steerable pyramid: a flexible architecture for multi-scale derivative computation. In *Proceedings of the 1995 International Conference on Image Processing (Vol. 3)-Volume 3 - Volume 3*, ICIP '95, pages 3444–, Washington, DC, USA, 1995. IEEE Computer Society.

[6] M. Unser, D. Sage, and D. Van De Ville. Multiresolution monogenic signal analysis using the Riesz-Laplace wavelet transform. *IEEE Transactions on Image Processing*, 18(11):2402–2418, Nov. 2009.

[7] M. Unser and D. Van De Ville. Wavelet steerability and the higher-order Riesz transform. *IEEE Transactions on Image Processing*, 19(3):636–652, Mar. 2010.