# $H_2O$: REVERSIBLE HEXAGONAL-ORTHOGONAL GRID CONVERSION BY 1-D FILTERING

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## ABSTRACT

In this work, we propose a new grid conversion algorithm between the hexagonal lattice and the orthogonal (a.k.a. Cartesian) lattice. The conversion process, named $H_2O$, is easy to implement and is perfectly *reversible* using the same algorithm to return from one lattice to the other. The key observation of our approach is a decomposition of the lattice conversion as a sequence of shearing operations along three well-chosen directions. Hence, only 1-D fractional sample delay operators are required, which can be implemented by simple convolutions. The proposed algorithm combines reversibility and fast 1-D operations, together with high-quality resampled images.

***Index Terms***— 2-D lattices, resampling, shears, hexagonal sampling, interpolation

## 1. INTRODUCTION

Sampling on the hexagonal lattice has several attractive properties. For instance, assuming an isotropic power spectrum, information can be more efficiently represented [1]. Also, higher symmetry and well-defined connectivity is advantageous for various fundamental image processing tasks [2, 3]. Despite the fact that hexagonally-arranged sensor arrays have been built (e.g. [4]), images are still almost exclusively available on the orthogonal lattice. Two important reasons can be put forward: (1) direct visualization of the data on display devices; (2) wide availability of 1-D algorithms that can be applied in a separable way on the orthogonal lattice.

Traditionally, grid conversion between the orthogonal and hexagonal lattices is based on discrete/continuous models. Conceptually, a generator function is placed and weighted on every lattice site after which new samples on the target lattice are obtained. In general, consecutive conversions between the orthogonal and hexagonal lattices (of same sampling density) will increasingly degrade the quality of the data. Also, interpolation methods on the hexagonal lattice typically rely on non-separable generator functions, such as the three-directional box-splines [5, 6] or hex-splines [7]. The use of these functions requires a non-separable IIR prefiltering step, that is computationally expensive and whose implementation is not straightforward (see for example [7]).

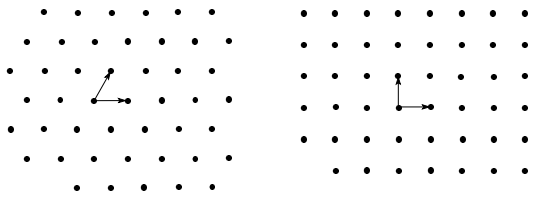

**Fig. 1**. The regular hexagonal lattice (left) and the square lattice (right), with same sampling density 1.

In this paper, we propose a new approach, named $H_2O$, to perform grid conversions between the orthogonal and hexagonal lattices of same sampling density (shown in Fig. 1), ensuring *exact reversibility*. The method is based on the observation that the hexagonal grid can be turned into the orthogonal one (or conversely) by three successive shearing operations. The principles of this method are exposed in Sect. 2. That way, converting the data can be done by 1-D filtering along each shearing direction to compensate for the respective shifts. As detailed in Sect. 3, the use of fractional delay filters enables us to obtain reversibility and treat the orthogonal-to-hexagonal case the same way as the hexagonal-to-orthogonal one. With the proposed choice of filters, we also offer a good resampling quality, as validated in Sect. 4 by experiments.

Our novel approach for resampling between these two lattices may be of particular interest in the following contexts: (1) Hexagonal to orthogonal resampling, which is often required when handling hexagonally sampled data, can be performed fast and efficiently with $H_2O$ using 1-D filtering only. More important, the reversibility ensures that no loss of information is introduced in this conversion operation; (2) Images sampled on the orthogonal lattice can be converted onto the hexagonal lattice, where many image processing tasks are advantageous, especially morphological operations [8, 9]. The reversibility ensures that the initial image can then be perfectly recovered.

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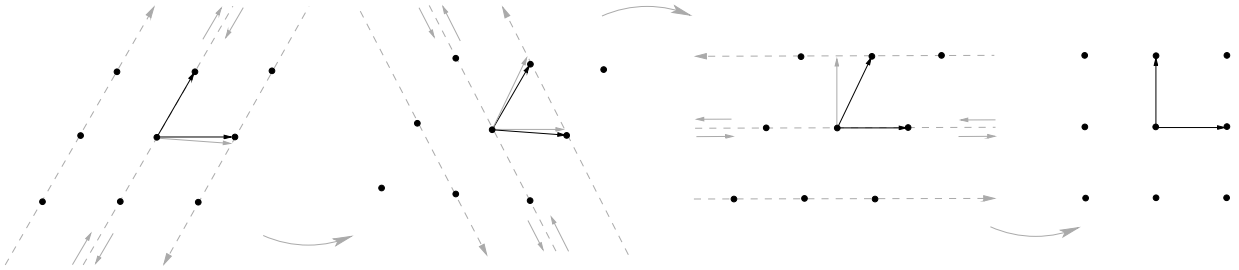


**Fig. 2.** A sequence of three shearings transforms the hexagonal lattice into the orthogonal one. These shearings are applied along the natural directions of the lattices.

### 1.1. Mathematical preliminaries

A 2-D lattice is a set of points in the plane, characterized by two linearly independant vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ grouped in a matrix $\mathbf{R} = [\mathbf{r}_1\ \mathbf{r}_2]$, such that the lattice sites are the locations $\mathbf{Rk}$ for every $\mathbf{k} = [k_1\ k_2]^{\mathrm{T}} \in \mathbb{Z}^2$. Its sampling density is $1/|\det(\mathbf{R})|$ (lattice sites per unit surface area). The orthogonal lattice $\mathbb{Z}^2$ is obtained if $\mathbf{R}$ is the identity matrix, while the regular hexagonal lattice with same sampling density 1, as in Fig. 1, corresponds to the choice

$$\mathbf{R} = \sqrt{\frac{2}{\sqrt{3}}} \begin{bmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{bmatrix}. \tag{1}$$

## 2. LATTICE CONVERSION BY SHEARINGS

A continuous shearing corresponds to a displacement of a point $\mathbf{x}$ in a direction $\mathbf{a}$, with amplitude proportional to $\langle \mathbf{x}, \mathbf{a}^{\perp} \rangle$ where $\mathbf{a}^{\perp} = [-a_2\ a_1]^{\mathrm{T}}$ is orthogonal to $\mathbf{a}$. So, we have the linear shearing operator $\mathbf{x} \mapsto \mathbf{x} + \lambda \langle \mathbf{x}, \mathbf{a}^{\perp} \rangle \mathbf{a}$, which can be characterized by the matrix $\mathbf{S} = \mathbf{I} + \lambda \mathbf{a}\mathbf{a}^{\perp \mathrm{T}}$.

Fig. 2 shows how a sequence of three shearings transforms the hexagonal lattice into the orthogonal one. Each shearing is chosen along a "natural" direction of the current lattice; thus it is performed by operating only on 1-D directions of the image. Let us detail how these shearings are determined:

- The first shearing operates on the hexagonal lattice with matrix $\mathbf{R}$. We look for a shearing operating along the second vector $\mathbf{r}_2$ of this lattice, thus with matrix $\mathbf{S}_1 = \mathbf{R}\mathrm{M}_1\mathbf{R}^{-1}$ where $\mathrm{M}_1$ has the form

$$\mathrm{M}_1 = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}. \tag{2}$$

  That is to say, with this change of basis, $\mathrm{M}_1$ is the matrix of the shearing in the basis of the lattice, while $\mathbf{S}_1$ is its matrix in the canonical basis.

- After the first shearing, the lattice has matrix $\mathbf{R}\mathrm{M}_1$. We look for a shearing operating along the "diagonal" direction of this lattice, thus with matrix $\mathbf{S}_2 = (\mathbf{R}\mathrm{M}_1)\mathrm{M}_2(\mathbf{R}\mathrm{M}_1)^{-1}$ where $\mathrm{M}_2$ has the form

$$\mathrm{M}_2 = \begin{bmatrix} 1+b & b \\ -b & 1-b \end{bmatrix}. \tag{3}$$

- The lattice has now matrix $\mathbf{R}\mathrm{M}_1\mathrm{M}_2$. We look for a shearing operating along the first vector of this lattice, thus with matrix $\mathbf{S}_3 = (\mathbf{R}\mathrm{M}_1\mathrm{M}_2)\mathrm{M}_3(\mathbf{R}\mathrm{M}_1\mathrm{M}_2)^{-1}$ where $\mathrm{M}_3$ has the form

$$\mathrm{M}_3 = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}. \tag{4}$$

At the end, the lattice has matrix $\mathbf{R}\mathrm{M}_1\mathrm{M}_2\mathrm{M}_3$. Since we want it to be the orthogonal lattice (with matrix $\mathbf{I}$), our matrices are determined by the equality

$$\mathrm{M}_1\mathrm{M}_2\mathrm{M}_3 = \mathbf{R}^{-1}. \tag{5}$$

We only have to identify the coefficients in this equality to obtain the desired values $a = 1 - \sqrt{2/\sqrt{3}}$, $b = \sqrt{\sqrt{3}/2} - 1$ and $c = \sqrt{2/\sqrt{3}} - 1 - 1/\sqrt{3}$.

We also have the factorization $\mathbf{S}_3\mathbf{S}_2\mathbf{S}_1 = \mathbf{R}^{-1}$, which is actually the decomposition, into the three designed shearings, of the geometric operation that transforms the hexagonal lattice into the orthogonal one. Note that this decomposition is not unique. It also depends on our choice of vectors characterizing the lattices, which is arbitrary. The intuition behind the decomposition presented here is to transform the vector $\mathbf{r}_1$ of the hexagonal lattice into $[1\ 0]^{\mathrm{T}}$, which requires two shearings (see Fig. 2). A third horizontal shearing then transforms $\mathbf{r}_2$ into the vertical vector $[0\ 1]^{\mathrm{T}}$ of the orthogonal lattice.

## 3. THE $\mathrm{H_2O}$ ALGORITHM

The decomposition of the lattice conversion process in three shearings suggests a new way to perform the resampling process between the hexagonal and the orthogonal lattices in three successive operations, converting the data between the successive lattices depicted on Fig. 2. The key property of this process is that it only involves 1-D operations at a time. Thus,

the practical implementation of $H_2O$ consists of the following three steps:

- On each column of the image $u$ with index $k_1$, perform a translation of amplitude $-a.k_1$ (in the direction $\uparrow$).
- On each diagonal of the image with index $k$, perform a translation of amplitude $-b.k$ (in the direction $\searrow$).
- On each row of the image with index $k_2$, perform a translation of amplitude $-c.k_2$ (in the direction $\rightarrow$).

For orthogonal to hexagonal conversion, we just have to reverse the order of the translations and make them in the opposite directions. Further on, we concentrate on the way to perform the 1-D translations.

A 1-D translation with amplitude $\tau$: $s \mapsto s'$ is achieved by means of a convolution with a *fractional delay filter* [10]: $s' = s * h_\tau$. Let us define the $\mathcal{Z}$-transform of $h_\tau$ by $H_\tau(z) = \sum_{k\in\mathbb{Z}} h_\tau[k] z^{-k}$. We now have to design a collection of filters $h_\tau$ for every $\tau \in \mathbb{R}$. To maintain the symmetry of the treatments, we first impose the condition that the delays with $\tau$ and $-\tau$ are symmetric operations, that is

$$H_{-\tau}(z) = H_\tau(z^{-1}). \tag{6}$$

In order for the hexagonal to orthogonal resampling to be reversible by mirroring the applied operations, we also impose the condition:

$$H_{-\tau}(z) = \frac{1}{H_\tau(z)}. \tag{7}$$

Eqns (6) and (7) together constrain the delay filters to be all-pass filters, that is, to have magnitude 1 in the frequency domain: $|H_\tau(e^{j\omega})| = 1$, $\forall\omega \in \mathbb{R}$. The simplest all-pass delay operator is the nearest-neighbor approximation: $H_\tau(z) = z^{-\lfloor\tau+1/2\rfloor}$. All rational all-pass fractional delay filters may be written under the form $H_\tau(z) = z^{-K} D_\tau(z)/D_\tau(z^{-1})$ for some $K \in \mathbb{Z}$ and some polynomial $D_\tau(X)$ of degree $N$ [10]. The only known class of such filters having explicit formulas of their coefficients as a function of $\tau$ is the class of Thiran's filters. We refer to [10] for their formulas as well as a review of their properties. We also consider the limit case of Thiran's filters as the order $N \to \infty$, which tends to the ideal sinc-based delay filters $H_\tau(e^{j\omega}) = e^{-j\tau\omega}$, $\forall\omega \in (-\pi, \pi)$, which can be implemented by a phase shift on the Fourier coefficients. We compare the performances of these filters in the next section.

## 4. EXPERIMENTAL VALIDATION

In order to evaluate the quality of the resampled image in a hexagonal to orthogonal setting, we need data available on both lattices. For that purpose, we generate a synthetic image by sampling on the hexagonal lattice the zoneplate pattern

| Method | hex. $\rightarrow$ Cart. | hex. $\rightarrow$ Cart. then Cart. $\rightarrow$ hex |
|---|---|---|
| $H_2O$ – nearest neighbor | 20.12 | $\infty$ |
| $H_2O$ – Thiran $N = 1$ | 36.15 | $\infty$ |
| $H_2O$ – Thiran $N = 2$ | 42.86 | $\infty$ |
| $H_2O$ – Thiran $N = 3$ | 47.28 | $\infty$ |
| $H_2O$ – sinc | 53.55 | $\infty$ |
| 2-D linear interpolation | 32.68 | 27.48 |

**Table 1**. PSNR results for the hexagonal to orthogonal resampling problem using the synthetic zoneplate image.

with analytical formula

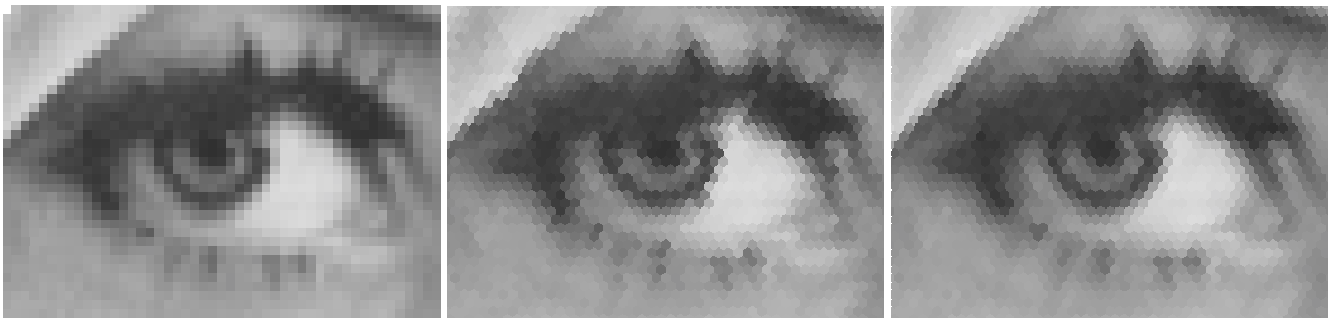
$$f(\mathbf{x}) = 127.5 + 127.5 \cos\left(\frac{1440/\pi}{1 + 512/\sqrt{8(x_1^2 + x_2^2)}}\right). \tag{8}$$

We convert this initial image onto the orthogonal lattice by means of the $H_2O$ method. The obtained image is then compared to the ground-truth image obtained by sampling the zoneplate pattern on the orthogonal lattice. The results are reported in Tab. 1 and illustrated in Fig. 4. The quality increases rapidly with the order of the filter and the simplest Thiran filter already achieves a good resampling quality. For comparison, we also give the PSNR obtained by resampling between the lattices using classical 2-D linear interpolation; that is, separable linear interpolation on the orthogonal lattice and "tri-linear" interpolation with the "tent" box-spline function [6] on the hexagonal lattice. The quality of this method is quite low and it is not reversible: going back to the orthogonal lattice does not render the initial image, but a largely distorted (mainly blurred) version of it. These preliminary results for the zoneplate image are promising and future work will investigate the performance for other types of images.

Since our filters are all-pass, they do not introduce blurring, contrary to interpolation methods. An example of resampled images with $H_2O$ is depicted in Fig. 3. The Thiran filters deliver images almost free of artifacts. Our approach is also very fast, since only three 1-D filtering are required for converting an image to the other grid. This is less expensive than with 2-D interpolation methods. Moreover, since all the treatments are 1-D, savings of memory storage are allowed and the resampling task can be performed in-place.

## 5. CONCLUSION

In this work we presented $H_2O$: a novel, fast, and high-quality method for converting data between the hexagonal and orthogonal lattices. The key property of $H_2O$ is its exact reversibility. We are currently working on the extension of the method for conversion between two arbitrary lattices of same sampling density, in 2-D and higher dimensions. For instance,



(a) (b) (c)

**Fig. 3**. The "eye of Lena" defined on the orthogonal lattice (a) is resampled on the hexagonal lattice using our $H_2O$ method with the nearest neighbor filter (b) and the order 1 Thiran filter (c). While the first approach is too rough (it simply consists in re-ordering the pixels of the image (a) on the hexagonal lattice), the Thiran filters provide images free of artifacts. Increasing the order of the Thiran filter makes no visible difference with the image (c). Note that resampling using classical interpolation methods would introduce blurring. Using $H_2O$ in reverse order exactly recovers the initial image (a) from (b) or (c).

reversible conversion between the BCC and orthogonal lattices in 3-D is of practical interest in computer graphics [11].

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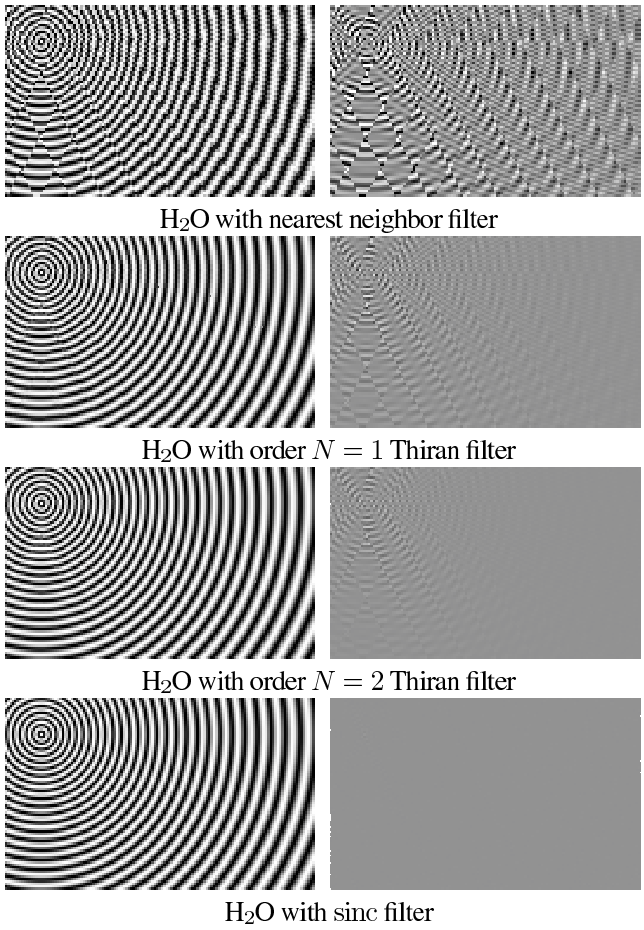


**Fig. 4.** Central part of the resampled images (left) and differences with the reference image (right) for the hexagonal to orthogonal resampling problem using the zoneplate image. The reference image is undistinguishable from the resampled image with the sinc filter.