Sparse Dictionaries for Continuous-Domain Inverse Problems

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***Abstract*—We study 1D continuous-domain inverse problems for multicomponent signals. The prior assumption on these signals is that each component is sparse in a different dictionary specified by a regularization operators. We introduce a hybrid regularization functional matched to such signals, and prove that corresponding continuous-domain inverse problems have *hybrid spline* solutions, *i.e.*, they are sums of splines matched to the regularization operators. We then propose a B-spline-based exact discretization method to solve such problems algorithmically.**

The task in an inverse problem is to recover a signal s_0 based on M measurements $\mathbf{y} \approx \boldsymbol{\nu}(s_0) \in \mathbb{R}^M$, where the measurement operator $\boldsymbol{\nu}$ models the physics of the acquisition system (forward model). This is typically achieved by minimizing the distance between the data $\mathbf{y}$ and the measurements $\boldsymbol{\nu}(s)$ of the reconstructed signal s (data fidelity). In order to inject some prior knowledge on the form of the signal, a regularization term is commonly added to the cost functional. In recent years, the advent of compressed sensing (CS) has led to an increasing popularity of sparsity-promoting regularization norms such as the ℓ_1 norm [1], [2] for discrete signals. Then, the prior assumption is that the signal s_0 is sparse in a certain dictionary basis specified by a regularization operator. However, real-worlds signals are typically composite and are thus not sparse in a single dictionary basis. We therefore focus on multicomponent signals $s = s_1 + \ldots + s_D$, where each component s_d is assumed to be sparse in a different dictionary basis. This framework is closely related to the data separation problem [2, Chapter 11], as well as the study of redundant dictionary bases [3]–[8]. This approach has been applied successfully in practice for imaging tasks such as morphological component analysis [9], [10] or image restauration [11].

These works dealing with multicomponent signals focus on purely discrete models. Yet many real-world signals are continuously defined, and this mismatch leads to discretization errors. This observation has lead to an abundance of research on continuous-domain problems with sparsity-promoting norms [12]–[15]. However, to the best of our knowledge, until our submitted work [16] that we present here, no such attempts have been made for multicomponent signals.

We focus on generalized total-variation regularization (gTV)

$$\|f\|_{\mathrm{TV}^{(N_0)}} := \|\mathrm{D}^{N_0}\{f\}\|_{\mathcal{M}} = \sup_{\substack{\varphi \in \mathcal{S}(\mathbb{R}) \\ \|\varphi\|_\infty = 1}} \langle f, \mathrm{D}^{N_0}\{\varphi\}\rangle, \quad (1)$$

where D is the derivative operator and $\mathcal{S}(\mathbb{R})$ is the Schwartz space on $\mathbb{R}$. It is known that gTV promotes sparsity in the sense that it leads to reconstructed signals that are sparse polynomial splines of order N_0 [15]. A polynomial spline of order N_0 can be expressed as $s(x) = \frac{1}{(N_0-1)!}\sum_k a_k(x - x_k)_+^{N_0-1} + p(x)$ where $a_k, x_k \in \mathbb{R}$ and p is a polynomial of degree no greater than $N_0 - 1$. The sparsity of a spline refers here to the number of knots x_k.

In order to deal with multicomponent signals, we introduce the hybrid regularization functional

$$\mathcal{R}_{\mathrm{hyb}}(f) = \min_{\substack{f_1,\ldots f_D \\ f_1+\ldots+f_D=f}} \sum_{d=1}^{D} \alpha_d \|f_d\|_{\mathrm{TV}^{(N_{0,d})}} \quad (2)$$

where the α_d control the weight between each regularization term with $\alpha_1 + \ldots + \alpha_D = 1$. This regularization function is well suited for multicomponent signals whose components s_d are sparse in the dictionaries consisting of sparse polynomial splines of degree $N_{0,d}$. We now state our main theoretical result.

Theorem 1. *Let $0 < N_{0,1} < \ldots < N_{0,D}$ and let $\boldsymbol{\nu} : f \mapsto \boldsymbol{\nu}(f) \in \mathbb{R}^M$ be a weak*-continuous operator*[1]*. Assume that $\boldsymbol{\nu}(p) \neq 0$ for all polynomials of degree less than $N_{0,D}$ (well-posedness assumption). Then, for any $\lambda > 0$, the optimization problem*

$$\mathcal{S} = \arg\min_f \left(\|\boldsymbol{\nu}(f) - \mathbf{y}\|_2^2 + \lambda \mathcal{R}_{\mathrm{hyb}}(f) \right) \quad (3)$$

has a solution s of the form $s = s_1 + \ldots + s_D + p$, where p is a polynomial of degree no greater that $N_{0,D} - 1$, and the s_d are polynomial splines of the form

$$s_d(x) = \frac{1}{(N_{0,d}-1)!} \sum_{k=1}^{K_d} a_{k,d}(x - x_{k,d})_+^{N_{0,d}-1}, \quad (4)$$

where $a_{k,d}, x_{k,d} \in \mathbb{R}$. Moreover, the sparsity indices K_d verify $K_1 + \cdots + K_D \leq M$.

Theorem 1 was proved in [16] and extends the main result of [15]. It states that Problem (3) has a sparse *hybrid spline* solution, *i.e.*, a sum of different splines. A remarkable feature of Theorem 1 is that the number of components D does not affect the sparsity of the solution, which is bounded by the number of measurements M.

In order to discretize Problem (3), we restrict its search space to the sum of spaces of splines with knots on a grid, *i.e.*, $\{(\cdot - x_n)_+^{N_{0,1}-1}\}_{n=1}^N + \ldots + \{(\cdot - x_n)_+^{N_{0,D}-1}\}_{n=1}^N + \{(\cdot)^n\}_{n=0}^{N_{0,D}-1}$ where the x_n lie on a uniform grid. This approach has many appealing properties: firstly, Theorem 1 guarantees that the search space is matched to the form of the solution (4). Next, critically, it leads to an exact discretization in the continuous domain: in the chosen search space, there is no discretization error. Finally, it allows for the use of B-spline as basis functions, which have compact support and thus lead to well-conditioned problems. These problems are then solved with a multiresolution algorithm introduced in [17] that uses a combination of ADMM [18] and the simplex algorithm [19].

We show some examples of our algorithm that demonstrate its pertinence for $D = 2$ components. Figure 1 is a curve fitting example, where the measurements are samples of the signal, *i.e.*, $\boldsymbol{\nu}(f) = (f(x_1), \ldots, f(x_M))$ where the x_m are the sample locations. We can see that he reconstructed signal satisfactorily interpolates the data points. In Figure 2, the measurements are samples of the Fourier transform of the signal. We notice that the reconstructed signal is very close to the ground truth signal s_0.

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[1] This is a mild technical assumption. We refer to [15] for the definition.

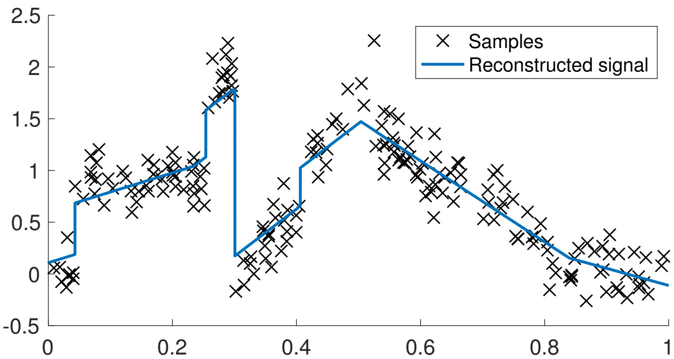

Fig. 1: Curve fitting for $\mathrm{L}_1 = \mathrm{D}$, $\mathrm{L}_2 = \mathrm{D}^2$, $M = 200$, $\lambda = 1.3$, $\alpha_1 = 0.95$, $\alpha_2 = 0.05$. This figure is taken from [16].

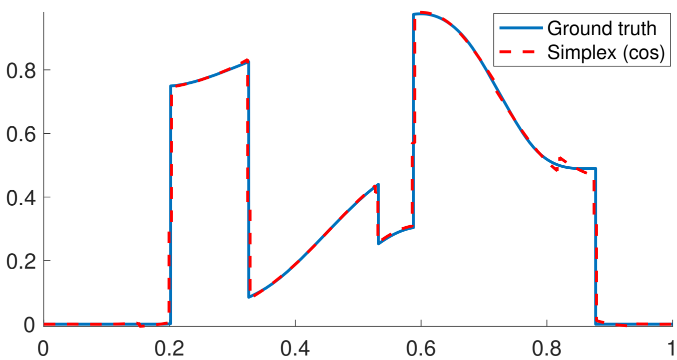

Fig. 2: Reconstruction result with noiseless Fourier measurements for $\mathrm{L}_1 = \mathrm{D}$, $\mathrm{L}_2 = \mathrm{D}^4$, $M = 30$, $\lambda = 10^{-15}$, $\alpha_1 = 1 - 5 \times 10^{-5}$, $\alpha_2 = 5 \times 10^{-5}$. This figure is taken from [16].

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