Fractional Laplacians, splines, wavelets, and fractal processes

Ricard Delgado-Gonzalo[1], Pouya Tafti[1], Christophe Rabut[2], Michael Unser[1]

[1]: Ecole Polytechnique Fédérale de Lausanne, CH-1015 Lausanne, Switzerland
[2]: CR INSA, département GMM, F-31077 Toulouse, France

Abstract

Our aim is to propose a multi-dimensional operator framework that provides a bridge between approximation theory (in particular, the construction of polyharmonic splines and wavelets) and the investigation of self-similar stochastic processes.

Our investigation starts with the identification of the linear differential operators that are translation-, scale- and rotation-invariant; these are the fractional Laplacians $(-\Delta)^{\frac{\gamma}{2}}$ with $\gamma \in \mathbb{R}^+$, which are best specified as Fourier-domain multipliers (Duchon, 1977). The corresponding family is endowed with a semi-group property. To make this statement precise, we must characterize the action of the fractional operators (boundedness, decay properties) on appropriate function spaces and also unambiguously specify the inverse operators. The mathematical difficulty is that: (1) the null space of the fractional Laplacians is non-empty (but finite-dimensional), and, (2) the inversion involves singular integral operators that need to be regularized.

Having mastered the underlying operator algebra, we define our signals of interest s as the solutions of a fractional differential equation with an appropriate excitation r: $(-\Delta)^{\frac{\gamma}{2}} s = r$. When r is a deterministic sequence of Dirac impulses, the formulation yields Duchon-type splines (Duchon, 1977), which are the fractional extensions of the classical thin-plate splines—the spline knots are simply specified by the locations of the Dirac impulses. If, on the other hand, the differential system is driven by white Gaussian noise, then it generates fractional Brownian fields which are fractal-like processes (Mandelbrot, 1968).

We take advantage of the proposed framework to construct multiresolution signal decompositions and Laplacian-like wavelets. First, we consider the extension of the existing polyharmonic wavelet constructions (Micchelli, 1991; Bacchelli, 2005; Van De Ville, 2005) for fractional orders of approximation. The crucial ingredient there is the interpolator in a cardinal spline space of twice the order γ (Madych 1990; Rabut, 1992) (which need not be integer). We show that the γth order fractional Laplacian of this interpolator yields an operator-like wavelet that generates a Riesz basis of $L_2(\mathbb{R}^d)$ in any number of dimensions, with mild conditions on the dilation matrix. Interestingly, the scheme also generalizes to the case where the knot spacing is non-uniform.

As an alternative to the above operator-like wavelets whose conditioning deteriorates as the order increases, we propose a pyramid-like decomposition of signals in terms of polyharmonic splines. This leads to the definition of the *fractional Laplacian pyramid* which

is a slightly redundant, reversible multiresolution signal representation. This pyramid constitutes a tight frame of $L_2(\mathbb{R}^d)$ and is implemented efficiently using the fast Fourier transform (Barbotin, 2008). We prove that the underlying wavelets are smoothed versions of the fractional Laplacian. Finally, we argue that the Laplacian-like behavior of the wavelets (in either construction) is beneficial for the analysis of signal with fractal-like characterics (Tafti, 2009). Indeed, we can invoke a simple duality argument to show that it will essentially whiten (resp., sparsify) fractal processes (resp., non-uniform-spline-like signals).

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