The Sliding Frank-Wolfe Algorithm for the BLASSO

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***Abstract*—This paper showcases the Sliding Frank-Wolfe (SFW), which is a novel optimization algorithm to solve the BLASSO sparse spikes super-resolution problem. The BLASSO is the continuous (i.e. off-the-grid or grid-less) counterpart of the well-known ℓ^1 sparse regularisation method (also known as LASSO or Basis Pursuit). Our algorithm is a variation on the classical Frank-Wolfe (also known as conditional gradient) which follows a recent trend of interleaving convex optimization updates (corresponding to adding new spikes) with non-convex optimization steps (corresponding to moving the spikes). We prove theoretically that this algorithm terminates in a finite number of steps under a mild non-degeneracy hypothesis.**

I. Introduction

A. Sparse Spikes Super-Resolution

The sparse spikes super-resolution problem aims at recovering an approximation of an unknown input discrete measure $m_{a_0,x_0} \stackrel{\text{def.}}{=} \sum_{i=1}^N a_{0,i}\delta_{x_{0,i}}$ from noisy measurements $y \stackrel{\text{def.}}{=} y_0 + w \in \mathbb{R}^M$ where $y_0 \stackrel{\text{def.}}{=} \Phi(m_{a_0,x_0})$ and w models the acquisition noise. The linear operator Φ is defined over the space of Radon measures $\mathcal{M}(\mathbb{R}^d)$ by $\Phi(m) \stackrel{\text{def.}}{=} \int_{\mathbb{R}^d} \varphi(x)\mathrm{d}m(x)$ (where φ belongs to a general class of kernels detailed in Definition 1) and models the acquisition process. Here $a_{0,i} \in \mathbb{R}$ are the amplitudes of the Dirac masses at positions $x_{0,i} \in \mathbb{R}^d$. This is an ill-posed inverse problem and the BLASSO

$$\min_{m\in\mathcal{M}(\mathbb{R}^d)} T_\lambda(m) \stackrel{\text{def.}}{=} \frac{1}{2}\left\|\Phi(m)-y\right\|^2_{\mathbb{R}^M} + \lambda|m|(\mathbb{R}^d), \qquad (\mathcal{P}_\lambda(y))$$

is a way to solve it in a stable way by introducing a sparsity-enforcing convex regularization. The problem $\mathcal{P}_\lambda(y)$ is a convex optimization problem that generalizes the LASSO over the non-reflexive Banach space $\mathcal{M}(\mathbb{R}^d)$ where the ℓ^1 norm becomes its continuous counterpart represented by the total variation norm $|\cdot|(\mathbb{R}^d)$.

B. Previous Works

The performance of the BLASSO has been theoretically studied in many papers, see for example [1], [2], [3], [4], [5], [6]. On the numerical standpoint, it is possible to solve the BLASSO by considering its dual and recasting it, for Fourier measurements and in a one dimensional setting, as a finite dimensional SDP [1]. In dimension greater than 2, one need to use the Lasserre's hierarchy [7], [8], [9]. In order to solve directly $\mathcal{P}_\lambda(y)$, algorithms that do not use any Hilbertian structure and can instead deal with measures are required. The authors of [2] proposed a modified version of the Frank-Wolfe (FW) where the amplitudes and positions are updated separately to further decrease the objective function. This idea was numerically studied in [10]. We follow a similar approach.

II. The Algorithm

From now on, we suppose for simplicity that $d = 1$. Our algorithm is presented in Algorithm 1 (see [11] for more details). It consists in recursively adding a new Dirac mass to the estimated measure (Step 3), computing the new amplitudes by solving the LASSO (Step 7), and moving continuously *both* the amplitudes and positions by finding a critical point of a non-convex problem (Step 8). See Figures 1 and 2 for illustrations of the algorithm.

Algorithm 1 Sliding Frank-Wolfe (SFW) Algorithm

1: Initialize with $m^{[0]} = 0$ and $n = 0$.
2: **for** $k = 0, \ldots, n$ **do**
3: $\quad m^{[k]} = \sum_{i=1}^{N^{[k]}} a_i^{[k]}\delta_{x_i^{[k]}}$, $a_i^{[k]} \in \mathbb{R}$, $x_i^{[k]}$ pairwise distincts, find $x_*^{[k]} \in \mathbb{R}$ s.t.:
$$x_*^{[k]} \in \arg\max_{x\in\mathbb{R}} |\eta^{[k]}(x)| \quad \text{where} \quad \eta^{[k]} \stackrel{\text{def.}}{=} \frac{1}{\lambda}\Phi^*(y-\Phi(m^{[k]})),$$
4: $\quad$ **if** $|\eta^{[k]}(x_*^{[k]})| \leqslant 1$ **then**
5: $\quad\quad m^{[k]}$ is a solution of $\mathcal{P}_\lambda(y)$. Stop.
6: $\quad$ **else**
7: $\quad\quad$ Get $m^{[k+1/2]} = \sum_{i=1}^{N^{[k]}} a_i^{[k+1/2]}\delta_{x_i^{[k]}} + a_{N^{[k]}+1}^{[k+1/2]}\delta_{x_*^{[k]}}$, s.t.
$$a^{[k+1/2]} \in \arg\min_{a\in\mathbb{R}^{N^{[k]}+1}} \frac{1}{2}\left\|\Phi(m_{a,x^{[k+1/2]}})-y\right\|^2_{\mathbb{R}^M} + \lambda\left\|a\right\|_1$$
$$\text{where} \quad x^{[k+1/2]} = (x_1^{[k]}, \ldots, x_{N^{[k]}}^{[k]}, x_*^{[k]})$$
8: $\quad\quad$ Find $m^{[k+1]} = \sum_{i=1}^{N^{[k]}+1} a_i^{[k+1]}\delta_{x_i^{[k+1]}}$ by initializing with $(a^{[k+1/2]}, x^{[k+1/2]})$ and obtaining a critical point of
$$(a,x) \in \mathbb{R}^{N^{[k]}+1}\times\mathbb{R}^{N^{[k]}+1} \mapsto \frac{1}{2}\left\|\Phi(m_{a,x})-y\right\|^2_{\mathbb{R}^M} + \lambda\left\|a\right\|_1.$$
9: $\quad$ **end if**
10: **end for**

III. Convergence Result

Definition 1 (Admissible kernels φ)**.** *We denote by* $\mathrm{KER}^{(k)}$*, the set of admissible kernels of order* k*. A function* $\varphi : X \to \mathbb{R}^M$ *belongs to* $\mathrm{KER}^{(k)}$ *if:*

- $\varphi \in \mathscr{C}^k(\mathbb{R}, \mathbb{R}^M)$,
- *For all* $p \in \mathbb{R}^M$, $x \in X \mapsto \langle\varphi(x), p\rangle_{\mathbb{R}^M}$ *vanishes at infinity,*
- *for all* $0 \leqslant i \leqslant k$, $\sup_{x\in\mathbb{R}} \left\|\varphi^{(i)}(x)\right\|_{\mathbb{R}^M} < +\infty$.

The next theorem gives a finite time convergence guarantee under mild assumptions that significantly improves the previously known convergence for this kind of algorithm (weak-* convergence).

Theorem 1. *Suppose that* $\varphi \in \mathrm{KER}^{(2)}$*, that* $m_{a,x} = \sum_{i=1}^N a_i\delta_{x_i}$ *is the unique solution of* $\mathcal{P}_\lambda(y)$*, and that* $\eta_\lambda = \frac{1}{\lambda}\Phi^*(y-\Phi(m_{a,x}))$ *is nondegenerate, i.e. for all* $x \in \mathbb{R}\setminus\bigcup_{i=1}^N\{x_i\}$

$$|\eta_\lambda(x)| < 1 \quad \text{and} \quad \forall i \in \{1,\ldots,N\}, \quad \eta_\lambda''(x_i) \neq 0. \qquad (1)$$

Then Algorithm 1 recovers $m_{a,x}$ *after a finite number of steps (i.e. there exists* $k \in \mathbb{N}$ *such that* $m^{[k]} = m_{a,x}$*).*

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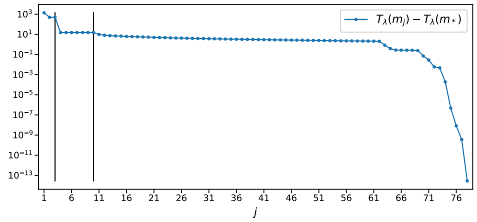
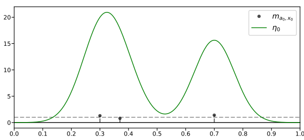
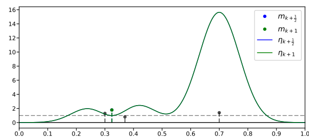


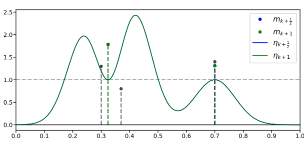
Fig. 1: Values of the objective function throughout the SFW algorithm (cumulative iterations of the BFGS used to find a critical point of the non-convex problem of Step 8). The vertical black lines separate the main outer iterations of the algorithm. The measure m_* represents the unique solution of the BLASSO. It is the same example as in Figure 2.



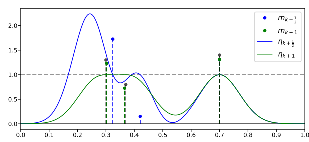
$k = 0$. Start of the loop.



$k = 0$. End of the loop.



$k = 1$. End of the loop.



$k = 2$. End of the loop.

Fig. 2: Main steps of the SFW algorithm.

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