Approximated Leont'ev coefficients

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Abstract

We consider Dirichlet series on convex polygons and their rate of approximation in $AC(\overline{D})$. We show that the substitution of the respective Leont'ev coefficients by appropriate interpolating sums preserves the order of approximation up to a factor $\ln n$. The estimates are given for moduli of smoothness of arbitrary order. This extends a result of Yu. I. Mel'nik in [4].

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1 Introduction

Let D be an open convex polygon with vertices at the points $a_1, \ldots, a_N$, $N \geq 3$, $\overline{D}$ its closure and $\partial D = \overline{D} \setminus D$ the boundary of D. We assume $0 \in D$.

By $AC(\overline{D})$ we denote the the space of all functions $f(z)$ holomorphic in D and continuous on $\overline{D}$ with finite norm of uniform convergence $\|f\|_{AC(\overline{D})} = \sup_{z \in \overline{D}} |f(z)| < \infty$.

Consider the quasipolynomial $L(z) = \sum_{k=1}^{N} d_k e^{a_k z}$, where $d_k \in \mathbb{C} \setminus \{0\}$, $k = 1, \ldots, N$. For the set of zeros Λ of the quasipolynomial L the following results are well known [2, Ch. 1, §2][3]:

a) The zeros $\lambda_n^{(j)}$ of L with $|\lambda_n^{(j)}| > C$ for sufficient large C have the form

$$\lambda_n^{(j)} = \widetilde{\lambda}_n^{(j)} + \delta_n^{(j)}, \tag{1}$$

where $\widetilde{\lambda}_n^{(j)} = \frac{2\pi n i}{a_{j+1} - a_j} + q_j e^{i\beta_j}$ and $|\delta_n^{(j)}| \leq e^{-an}$. Here $0 < a = \text{const.}$, $j = 1, \ldots, N$, $n > n_0$ and $a_{N+1} := a_1$. The parameters β_j and q_j are given by $e^{q_j (a_{j+1} - a_j) e^{i\beta_j}} =$

$-\frac{d_j}{d_{j+1}}$, where $d_{N+1} := d_1$. Hence these zeros are simple. The set of zeros Λ can be represented in the form

$$\Lambda = \{\lambda_n\}_{n=1,\dots,n_0} \cup \left(\bigcup_{j=1}^{N} \{\lambda_n^{(j)}\}_{n=n(j),n(j)+1,\dots} \right).$$

b) There is a constant $c_2 > 0$ such that there exists a positive constant A with

$$\left| \frac{e^{\lambda_n^{(j)} z}}{L'(\lambda_n^{(j)})} - (-1)^n B_j e^{\widetilde{\lambda}_n^{(j)} \left(z - \frac{a_{j+1}+a_j}{2} \right)} \right| \le A\, e^{-c_2 n} \quad \text{for all } n > n_0.$$

Here all $B_j \neq 0$ are constant, $j = 1, \dots, N$. This inequality is true for all $z \in \overline{D}$.

For simplicity reasons we assume that all zeros of L are simple.

We can expand functions $f \in AC(\overline{D})$ with respect to the family $\mathcal{E}(\Lambda) := \{e^{\lambda z}\}_{\lambda \in \Lambda}$ into a series of complex exponentials, the so called Dirichlet series

$$f(z) \sim \sum_{\lambda \in \Lambda} \kappa_f(\lambda) \frac{e^{\lambda z}}{L'(\lambda)}, \tag{2}$$

where

$$\begin{aligned} \kappa_f(\lambda) &= \sum_{k=1}^{N} d_k e^{a_k \lambda} \int_{a_j}^{a_k} f(\eta) e^{-\lambda \eta}\, d\eta & (3)\\ &= \frac{1}{2\pi} \sum_{k=1}^{N} d_k (a_k - a_j) \int_0^{2\pi} f\left(a_k + \frac{a_j - a_k}{2\pi}\, \theta \right) e^{-\lambda \left(\frac{a_j - a_k}{2\pi}\, \theta \right)} d\theta & (4) \end{aligned}$$

are the Leont'ev coefficients. Here, the index $j = 1, \dots, N$ is arbitrary, but fixed. Many deep results on these series are due to A. F. Leont'ev [2].

We know [1] that the partial series, weighted with the generalized Jackson kernel, approximate in the order of the modulus of continuity. The question considered in this paper is, what happens if we substitute the integration in (3) or (4) by an appropriate approximating sum. Can we choose a sum, such that the rate of approximation is preserved? This problem was first posed by Yu. I. Mel'nik in [4] and solved there for first moduli of continuity. We give positive answer to that question up to a factor $\ln n$ for moduli of arbitrary order $r \in \mathbb{N}$.

In the following section we give the rate of approximation of the series (2) weighted with the generalized Jackson kernel. Then we have a closer look on (3) and (4) and give Yu. I. Mel'nik's approach for a sum for substituting the integral, such that the order of approximation is held for first moduli. In the last section we extend this result to moduli of arbitrary order.

2 Approximation with generalized Jackson weights

To estimate the regularity of functions in $AC(\overline{D})$ we consider appropriate moduli of smoothness introduced in [6] by P. M. Tamrazov. Let $\xi \in \overline{D}$, $r \in \mathbb{N}$, $\delta > 0$ and $A > 0$. Let $U(\xi, \delta) := \{z \in \mathbb{C} : |z - \xi| \leq \delta\}$ be the closed δ-ball with center ξ. We denote by $T(\overline{D}, \xi, r, \delta, A)$ the set of all vectors $\mathbf{z} = (z_1, \ldots, z_r) \in \mathbb{C}^r$ with

(i) $z_i \in \overline{D} \cap U(\xi, \delta)$ for all $i = 1, \ldots, r$, and

(ii) $|z_i - z_j| \geq A\delta$ for all $i \neq j$, $i, j = 1, \ldots, r$.

If there is no vector satisfying these conditions we define $T(\overline{D}, \xi, r, \delta, A) := \emptyset$. Nevertheless for $A = 2^{-r}$ there is a $\delta > 0$ with $T(\overline{D}, \xi, r, \delta, A) \neq \emptyset$. Let $T_1 = T(\overline{D}, \xi, r+1, \delta, 2^{-r})$. Let $L(z, f, z_1, \ldots, z_r)$ be the polynomial in z of degree at most $r - 1$ which interpolates f at the points $z_1, \ldots, z_r$. The r-th modulus of f is defined by

$$\omega_r(f, t) = \omega_{r,\overline{D}}(f, t)_\infty := \sup_{0<\delta\leq t} \sup_{\xi\in\overline{D}} \sup_{\substack{\mathbf{z}\in T_1 \\ \mathbf{z}=(z_0,\ldots,z_r)}} |f(z_0) - L(z_0, f, z_1, \ldots, z_r)|. \tag{5}$$

Here the supremum over the empty set is defined as zero. To estimate this modulus we consider normal majorants φ: These are bounded non-decreasing functions $\varphi :]\,0, \infty\,[\to]\,0, \infty\,[$ such that for fixed $\sigma \geq 1$ and an exponent $\gamma \geq 0$ the following normality condition holds:

$$\varphi(t\delta) \leq \sigma t^\gamma \varphi(\delta)$$

for all $\delta > 0$, $t > 1$ [5, §1]. It is shown in [7] and [8, Thm. 1] that the modulus (5) is normal, i.e., $\omega_{r,\overline{D}}(f, t\delta)_\infty \leq C \cdot t^r \cdot \omega_{r,\overline{D}}(f, \delta)_\infty$, where $C > 0$ depends on r and the polygon D only. With normal majorants we thus can define classes of regularity. By $AH_r^\varphi(\overline{D})$ we denote the class of all functions $f \in AC(\overline{D})$ with $\omega_{r,\overline{D}}(f, t) \leq \text{const.} \cdot \varphi(t)$.

Let $1 \leq j \leq N$ be fixed and $r \in \mathbb{N}$. Let $f \in AC(\overline{D})$ have $r - 1$ existing derivatives at the vertices a_k, $k = 1, \ldots, N$, of the polygon. Consider for $k \neq j + 1$ the polynomial $P_{j,k}$ of degree at most r, that interpolates f at the vertices a_j and a_k and $f', \ldots, f^{(r-1)}$ at the vertex a_k. For $k = j + 1$ let $P_{j,j+1}$ denote the polynomial of degree at most $2r - 1$ that interpolated $f, f', \ldots, f^{(r-1)}$ at both points a_j and a_{j+1}. We define

$$\begin{aligned} \delta_r(f, h) \;&:=\; \max_j \sum_{\substack{k=1 \\ k\neq j}}^{N} \left\{ \int_0^h \frac{\left|f\left(a_k + \frac{a_j-a_k}{2\pi}u\right) - P_{j,k}\left(a_k + \frac{a_j-a_k}{2\pi}u\right)\right|}{u}\, du \right. \\ &+\; \left. h^r \cdot \int_h^{2\pi} \frac{\left|f\left(a_k + \frac{a_j-a_k}{2\pi}u\right) - P_{j,k}\left(a_k + \frac{a_j-a_k}{2\pi}u\right)\right|}{u^{r+1}}\, du \right\}. \end{aligned}$$

Let $\mathbf{n} = (n_1, \ldots, n_N) \in \mathbb{N}^N$ be a multi-index. Consider the corresponding quasipolynomial

$$\mathcal{P}_\mathbf{n}(f)(z) := \sum_{m=1}^{n_0} \kappa_f(\lambda_m) \frac{e^{\lambda_m z}}{L'(\lambda_m)} + \sum_{j=1}^{N} \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m}) \kappa_f(\lambda_m^{(j)}) \frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})}. \tag{6}$$

The coefficients $x_{n_j,r,m}$ are determined through the relations

$$x_{n_j,r,m} = \sum_{p=0}^{n_j} (-1)^p \binom{r}{p} J_{n_j,r,mp},$$

where $J_{n_j,r,k}$ are the Fourier coefficients of the generalized Jackson kernel

$$K_{n,r}(t) := \lambda_{n,r} \left(\frac{\sin Mt/2}{t/2} \right)^{2r} = \frac{J_{n,r,0}}{2} + \sum_{k=1}^{n} J_{n,r,k} \cos kt.$$

Here $n \in \mathbb{N}$, $r \geq 2$, $M := \lfloor \frac{n}{r} \rfloor$, and $\lambda_{n,r}$ is chosen such that

$$\frac{1}{2\pi} \int_0^{2\pi} K_{n,r}(t)\, dt = 1.$$

For the quasipolynomials (6) the following direct approximation theorem is true:

Theorem 2.1 *Let $f \in AH_r^{\omega_r}(\overline{D})$, where ω_r is a normal majorant with exponent $r \in \mathbb{N}$ satisfying the Stechkin condition*

$$\int_0^h \frac{\omega_r(f,t)}{t}\, dt + h^r \cdot \int_h^{2\pi} \frac{\omega_r(f,t)}{t^{r+1}}\, dt \quad \leq \quad c \cdot \omega_r(f,h) \tag{7}$$

for all $0 < h < \frac{2\pi}{r}$ and a positive constant c. Let f be $r-1$–times continuously differentiable at the vertices a_k, $k = 1, \ldots, N$, and

$$\sum_{k=1}^{N} d_k f^{(s)}(a_k) = 0, \quad 0 \leq s \leq r-1.$$

Let $\mathbf{n} = (n_1, \ldots, n_N) \in \mathbb{N}^N$ be a multi-index.
Then we have for approximation with the quasipolynomial $\mathcal{P}_{\mathbf{n}}(f)$ weighted with the generalized Jackson kernel

$$\|f - \mathcal{P}_{\mathbf{n}}(f)\|_{AC(\overline{D})} \leq \text{const.} \sum_{k=1}^{N} \Omega_r \left(\frac{1}{n_k} \right),$$

where Ω_r — a normal majorant with exponent r — satisfies inequality

$$\Omega_r(h) \leq \text{const.} \cdot \{\omega_r(h) + \delta_r(f,h)\}. \tag{8}$$

The proof is given in [1].

In the following section, we give Yu. I. Mel'nik's approach to the question, if this rate of approximation can be preserved, when we substitute the integral in (3) or in (4) by an appropriate sum.

3 Substitution of integrals by appropriate sums

In [4], Yu. I. Mel'nik proposed to substitute the Leont'ev coefficients in (2) by

$$\kappa_f^{(\widehat{n})}(\lambda_m^{(j)}) = \frac{1}{\lambda_m^{(j)}} \sum_{k=1}^{N} d_k \, \frac{1}{\widehat{n}} \sum_{p=0}^{\widehat{n}-1} f\left(a_k + (a_j - a_k)\,\frac{p}{\widehat{n}}\right) \left(e^{-\lambda_m^{(j)}(a_j-a_k)\frac{p+1}{\widehat{n}}} - e^{-\lambda_m^{(j)}(a_j-a_k)\frac{p}{\widehat{n}}}\right) \quad (9)$$

for all $\widehat{n} \in \mathbb{N}$. He considered functions $f \in AH_1^{\omega}(\overline{D})$ with $\sum_{n=1}^{N} d_k f(a_k) = 0$ and approximated them with partial series of the form

$$S_{n,\widehat{n}}(f)(z) = \sum_{m=1}^{n_0} \kappa_f(\lambda_m)\, \frac{e^{\lambda_m z}}{L'(\lambda_m)} + \sum_{j=1}^{N} \sum_{m=n(j)}^{n} \kappa_f^{(\widehat{n})}(\lambda_m^{(j)})\, \frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})}.$$

For the rate of approximation Mel'nik reached (see [4])

$$\|f - S_{n,\widehat{n}}(f)\|_{AC(\overline{D})} \leq \text{const.} \cdot \left\{ \omega\left(\frac{1}{\widehat{n}}\right) + \omega\left(\frac{1}{n}\right) \right\} \ln n. \quad (10)$$

The question that remains open, is, how (9) can be extended, such that an estimate for the rate of approximation can be reached for arbitrary moduli?

If we have a closer look at (9) and compare this formula with (3) and (4), we see that the integral there is decomposed in $\widehat{n}$ integrals of length $\frac{2\pi}{\widehat{n}}$:

$$\begin{aligned} \int_0^{2\pi} f\left(a_k + \frac{a_j - a_k}{2\pi}\,\theta\right) e^{-\lambda_m^{(j)}\frac{a_j-a_k}{2\pi}\theta}\, d\theta \quad &= \\ = \frac{1}{\widehat{n}} \int_p^{\widehat{n}-1} \int_{2\pi\frac{p}{\widehat{n}}}^{2\pi\frac{p+1}{\widehat{n}}} f\left(a_k + \frac{a_j - a_k}{2\pi}\,\theta\right) e^{-\lambda_m^{(j)}\frac{a_j-a_k}{2\pi}\theta}\, d\theta. & \quad (11) \end{aligned}$$

The exponential function can be integrated easily. In general, this is not the case for f: The antiderivative might not be known explicitly, or highly oscillating f may cause numerical problems. Therefore the term $f(a_k + \frac{a_j-a_k}{2\pi}\,\theta)$ is estimated by the value at the lower bound of the integral $f(a_k + (a_j - a_k)\frac{p}{\widehat{n}})$:

$$\kappa_f^{(\widehat{n})}(\lambda_m^{(j)}) = \sum_{k=1}^{N} d_k\, \frac{a_k - a_j}{2\pi}\, \frac{1}{\widehat{n}} \sum_{p=0}^{\widehat{n}-1} f\left(a_k + (a_j - a_k)\frac{p}{\widehat{n}}\right) \int_{2\pi\frac{p}{\widehat{n}}}^{2\pi\frac{p+1}{\widehat{n}}} e^{-\lambda_m^{(j)}\frac{a_j-a_k}{2\pi}\theta}\, d\theta. \quad (12)$$

Evaluating the integral explicitly gives (9).

To get a better rate of approximation with coefficients of this special form we have to find a better approximation of the function f on the staight-line interval $[\,a_j, a_k\,]$. We give a solution to this problem in the following section.

4 Higher order approximation

In this section we consider the question, if a better choice of $\kappa_f^{(\widehat{n})}(\lambda_m^{(j)})$ allows a higher rate of approximation and estimation with r-th moduli of smoothness, $r \in \mathbb{N}$. The key to this problem is the estimation of f in (11). We substitute f by the value of the modified r-th difference operator

$$\Delta^r_{\frac{2\pi}{r\widehat{n}}} f(z) - f\left(z + \frac{2\pi}{\widehat{n}}\right) = \sum_{k=0}^{r-1} (-1)^k \binom{r}{k} f\left(z + k\,\frac{2\pi}{r\widehat{n}}\right). \tag{13}$$

For $r = 1$ this expression yields $\Delta^1_{\frac{2\pi}{r\widehat{n}}} f(z) - f(z + \frac{2\pi}{\widehat{n}}) = f(z)$. If we put $z = a_k + (a_j - a_k)\,\frac{p}{\widehat{n}}$ here, we get Mel'nik's formulas (9) and (12).

Substituting f in (11) with (13) for $z = a_k + (a_j - a_k)\,\frac{p}{\widehat{n}}$ and arbitrary $r \in \mathbb{N}$ yields

$$\begin{aligned}\kappa_f^{(\widehat{n})}(\lambda_m^{(j)}) \;&=\; \frac{1}{\lambda_m^{(j)}} \sum_{k=1}^{N} d_k\, \frac{1}{\widehat{n}} \sum_{p=0}^{\widehat{n}-1} \sum_{k=0}^{r-1} (-1)^k \binom{r}{k} f\left(a_k + \frac{a_j - a_k}{\widehat{n}}\left(p + \frac{k}{r}\right)\right) \\ &\cdot\; \left(e^{-\lambda_m^{(j)}(a_j-a_k)\frac{p+1}{\widehat{n}}} - e^{-\lambda_m^{(j)}(a_j-a_k)\frac{p}{\widehat{n}}}\right).\end{aligned} \tag{14}$$

Now we can formulate the following approximation theorem:

Theorem 4.1 *Let ω_r be a normal majorant with exponent r satisfying the Stechkin condition (7). Let $f \in AH_r^{\omega_r}(\overline{D})$ and*

$$\sum_{k=1}^{N} d_k f^{(s)}(a_k) = 0 \quad \text{for all} \;\; 0 \le s \le r-1.$$

Let $\mathbf{n} = (n_1, \ldots, n_N)$ and $\widehat{\mathbf{n}} = (\widehat{n}_1, \ldots, \widehat{n}_N)$, $\mathbf{n}, \widehat{\mathbf{n}} \in \mathbb{N}^N$, be multi-indices.
Consider the partial series $\mathcal{P}_{\mathbf{n},\widehat{\mathbf{n}}}$ weighted with the generalized Jackson kernel

$$\begin{aligned}\mathcal{P}_{\mathbf{n},\widehat{\mathbf{n}}}(f)(z) \;&=\; \sum_{m=1}^{n_0} \kappa_f(\lambda_m) \frac{e^{\lambda_m z}}{L'(\lambda_m)} \\ &+\; \sum_{j=1}^{N} \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m})\, \kappa_f^{(\widehat{n}_j)}(\lambda_m^{(j)})\, \frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})},\end{aligned} \tag{15}$$

where $\kappa_f^{(\widehat{n}_j)}(\lambda_m^{(j)})$ as in (14).
Then the approximation of f with $\mathcal{P}_{\mathbf{n},\widehat{\mathbf{n}}}(f)$ yields

$$\|f - \mathcal{P}_{\mathbf{n},\widehat{\mathbf{n}}}(f)\|_{AC(\overline{D})} \le \text{const.} \cdot \left\{ \sum_{k=1}^{N} \Omega_r\left(\frac{1}{n_k}\right) + \sum_{k=1}^{N} \omega_r\left(\frac{1}{\widehat{n}_k}\right) \ln n_k \right\},$$

where Ω_r as in (8).

Proof. We show that

$$\|\mathcal{P}_{\mathbf{n},\widehat{\mathbf{n}}}(f) - \mathcal{P}_{\mathbf{n}}(f)\|_{AC(\overline{D})} \le C \sum_{k=1}^{N} \omega_r \left(\frac{1}{\widehat{n}_k}\right) \ln n_k.$$

with $C > 0$ independent of f, $\mathbf{n}$ and $\widehat{\mathbf{n}}$ and conclude with Theorem 2.1.

It is by (6) and (15) for all $z \in \overline{D}$

$$\begin{aligned} \mathcal{P}_{\mathbf{n},\widehat{\mathbf{n}}}(f)(z) - \mathcal{P}_{\mathbf{n}}(f)(z) &= \\ = \sum_{j=1}^{N} \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m}) \left(\kappa_f^{(\widehat{n}_j)}(\lambda_m^{(j)}) - \kappa_f(\lambda_m^{(j)})\right) \frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})}. \end{aligned} \tag{16}$$

We have a closer look at the difference $\kappa_f^{(\widehat{n}_j)}(\lambda_m^{(j)}) - \kappa_f(\lambda_m^{(j)})$. Using (4), (12) and (14) we get

$$\begin{aligned} \kappa_f^{(\widehat{n}_j)}(\lambda_m^{(j)}) - \kappa_f(\lambda_m^{(j)}) &= \\ = \frac{1}{\lambda_m^{(j)}} \sum_{k=1}^{N} d_k \frac{1}{\widehat{n}_j} \sum_{p=0}^{\widehat{n}_j-1} \sum_{l=0}^{r-1} (-1)^l \binom{r}{l} f\left(a_k + \frac{a_j - a_k}{\widehat{n}_j}\left(p + \frac{l}{r}\right)\right) \left(e^{-\lambda_m^{(j)}(a_j-a_k)\frac{p+1}{\widehat{n}_j}}\right. \\ \left. - e^{-\lambda_m^{(j)}(a_j-a_k)\frac{p}{\widehat{n}_j}}\right) - \frac{1}{2\pi} \sum_{k=1}^{N} d_k (a_k - a_j) \int_0^{2\pi} f\left(a_k + \frac{a_j - a_k}{2\pi}\theta\right) e^{-\lambda_m^{(j)} \frac{a_j - a_k}{2\pi}\theta} d\theta \\ = \sum_{k=1}^{N} d_k \left\{ \frac{a_k - a_j}{2\pi} \frac{1}{\widehat{n}_j} \sum_{p=0}^{\widehat{n}_j-1} \sum_{p=0}^{r-1} (-1)^l \binom{r}{l} f\left(a_k + \frac{a_j - a_k}{\widehat{n}_j}\left(p + \frac{l}{r}\right)\right) \right. \\ \cdot \int_{2\pi\frac{p}{\widehat{n}_j}}^{2\pi\frac{p+1}{\widehat{n}_j}} e^{-\lambda_m^{(j)}\frac{a_j-a_k}{2\pi}\theta} d\theta \\ \left. - \frac{1}{2\pi}(a_k - a_j) \sum_{p=0}^{\widehat{n}_j-1} \int_{2\pi\frac{p}{\widehat{n}_j}}^{2\pi\frac{p+1}{\widehat{n}_j}} f\left(a_k + \frac{a_j - a_k}{2\pi}\theta\right) e^{-\lambda_m^{(j)}\frac{a_j-a_k}{2\pi}\theta} d\theta \right\} \\ = \sum_{k=1}^{N} d_k \frac{a_k - j}{2\pi} \sum_{p=0}^{\widehat{n}_j-1} \int_{2\pi\frac{p}{\widehat{n}_j}}^{2\pi\frac{p+1}{\widehat{n}_j}} \left\{ \sum_{l=0}^{r-1} (-1)^l \binom{r}{l} f\left(a_k + \frac{a_j - a_k}{\widehat{n}_j}\left(p + \frac{l}{r}\right)\right) \right. \\ \left. - f\left(a_k + \frac{a_j - a_k}{2\pi}\theta\right) \right\} e^{-\lambda_m^{(j)}\frac{a_j-a_k}{2\pi}\theta} d\theta. \end{aligned}$$

Thus for the series (16) we can write

$$\begin{aligned} \mathcal{P}_{\mathbf{n},\widehat{\mathbf{n}}}(f)(z) - \mathcal{P}_{\mathbf{n}}(f)(z) &= \\ = \sum_{j=1}^{N} \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m}) \sum_{k=1}^{N} d_k \frac{a_k - a_j}{2\pi} \sum_{p=0}^{\widehat{n}_j-1} \int_{2\pi\frac{p}{\widehat{n}_j}}^{2\pi\frac{p+1}{\widehat{n}_j}} \left\{ \sum_{l=0}^{r-1} (-1)^l \binom{r}{l} \right. \end{aligned}$$

$$\cdot f\left(a_k + \frac{a_j - a_k}{\widehat{n}_j}\left(p + \frac{l}{r}\right)\right) - f\left(a_k + \frac{a_j - a_k}{2\pi}\,\theta\right)\Bigg\}\, e^{-\lambda_m^{(j)} \frac{a_j - a_k}{2\pi}\,\theta}\, d\theta\, \frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})}$$

$$= \sum_{j=1}^{N}\sum_{k=1}^{N} d_k \frac{a_k - a_j}{2\pi} \sum_{p=0}^{\widehat{n}_j - 1} \int_{2\pi\frac{p}{\widehat{n}_j}}^{2\pi\frac{p+1}{\widehat{n}_j}} \left\{\sum_{l=0}^{r-1}(-1)^l \binom{l}{r} f\left(a_k + \frac{a_j - a_k}{\widehat{n}_j}\left(p + \frac{l}{r}\right)\right)\right.$$

$$\left. - f\left(a_k + \frac{a_k - a_j}{2\pi}\,\theta\right)\right\} \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m}) e^{-\lambda_m^{(j)} \frac{a_j - a_k}{2\pi}\,\theta}\, \frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})}\, d\theta.$$

Hence

$$\|\mathcal{P}_{\mathbf{n},\widehat{\mathbf{n}}}(f) - \mathcal{P}_{\mathbf{n}}(f)\|_{AC(\overline{D})} =$$

$$\leq \max_{z\in\overline{D}} \sum_{j=1}^{N}\sum_{k=1}^{N} |d_k|\, \frac{|a_j - a_k|}{2\pi} \sum_{p=0}^{\widehat{n}_j - 1} \int_{2\pi\frac{p}{\widehat{n}_j}}^{2\pi\frac{p+1}{\widehat{n}_j}} \left| \sum_{l=0}^{r-1}(-1)^l \binom{r}{l} f\left(a_k + \frac{a_j - a_k}{\widehat{n}_j}\left(p + \frac{l}{r}\right)\right)\right.$$

$$\left. - f\left(a_k + \frac{a_j - a_k}{2\pi}\,\theta\right)\right| \left| \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m}) e^{-\lambda_m^{(j)} \frac{a_j - a_k}{2\pi}\,\theta}\, \frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})} \right| d\theta$$

$$\leq \max_{z\in\overline{D}} \sum_{j=1}^{N}\sum_{k=1}^{N} |d_k|\, \frac{|a_k - a_j|}{2\pi} \cdot \omega_r\left(f, \frac{|a_j - a_k|}{\widehat{n}_j}\right)$$

$$\cdot \int_0^{2\pi} \left| \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m}) e^{-\lambda_m^{(j)} \frac{a_j - a_k}{2\pi}\theta} \cdot \frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})} \right| d\theta \tag{17}$$

Now it is enough to estimate the integral in (17). First let $k \neq j+1$. Then

$$\Re\left(i\, \frac{a_j - a_k}{a_{j+1} - a_j}\right) > 0. \tag{18}$$

By (1) we infer

$$\left| e^{-\lambda_m^{(j)} \frac{a_j - a_k}{2\pi}\,\theta} \right| = \left| e^{-mi\frac{a_j - a_k}{a_{j+1} - a_j}\,\theta}\, e^{-q_j e^{i\beta_j} \frac{a_j - a_k}{2\pi}\,\theta}\, e^{-\delta_n^{(j)}\theta} \right| \leq C\, e^{-am\theta}$$

for some positive constants C, a and all $\theta \in [\,0, 2\pi\,]$. Thus we obtain for all $z \in \overline{D}$

$$\int_0^{2\pi} \left| \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m}) e^{-\lambda_m^{(j)} \frac{a_j - a_k}{2\pi}\,\theta}\, \frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})} \right| d\theta$$

$$\leq \text{const.} \int_0^{2\pi} \left| \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m}) e^{-am\theta} \right| d\theta$$

$$= \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m}) \cdot \frac{e^{-2\pi am} - 1}{-am} \quad \leq \quad \text{const.} \ln(n_j), \tag{19}$$

since the family $\left\{ \frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})} \right\}_{m \geq n(j)}$ and the generalized Jackson coefficients $1 - x_{n_j,r,m}$ are bounded.

For $k = j+1$ we have for the integral in (17) and property b)

$$\int_0^{2\pi} \left| \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m})\, e^{-\lambda_m^{(j)} \frac{a_j - a_{j+1}}{2\pi} \theta} \frac{e^{\lambda_m^{(j)} z}}{L'(\lambda_m^{(j)})} \right| d\theta$$

$$\leq \int_0^{2\pi} \left| \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m})\, e^{-\lambda_m^{(j)} \frac{a_j - a_{j+1}}{2\pi} \theta} \cdot \left| e^{-am} + A \cdot e^{\widetilde{\lambda}_m^{(j)} \left(z - \frac{a_j + a_{j+1}}{2} \right)} \right| \right| d\theta$$

$$\leq \left\{ \int_0^{2\pi} \left| \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m})\, e^{-\lambda_m^{(j)} \frac{a_j - a_{j+1}}{2\pi} \theta}\, e^{-am} \right| \right.$$

$$\left. + \left| \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m})\, e^{-\lambda_m^{(j)} \frac{a_j - a_{j+1}}{2\pi} \theta} \cdot \left| e^{\widetilde{\lambda}_m^{(j)} \left(z - \frac{a_j + a_{j+1}}{2} \right)} \right| \right| d\theta \right\}. \tag{20}$$

for constants $a, A > 0$. Hence for the first sum by (1)

$$\left| \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m})\, e^{-\lambda_m^{(j)} \frac{a_j - a_{j+1}}{2\pi} \theta}\, e^{-am} \right|$$

$$\leq \quad \text{const.} \cdot \left| \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m})\, e^{-am} e^{im\theta} \right| \quad \leq \quad C$$

independently of n_j. Now to the second sum in (20) by property a):

$$\left| \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m})\, e^{-\lambda_m^{(j)} \frac{a_j - a_{j+1}}{2\pi} \theta} \left| e^{\widetilde{\lambda}_m^{(j)} \left(z - \frac{a_j + a_{j+1}}{2} \right)} \right| \right|$$

$$\leq \quad \text{const.} \left| \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m})\, e^{im\theta} \left| e^{\widetilde{\lambda}_m^{(j)} \left(z - \frac{a_j + a_{j+1}}{2} \right)} \right| \right|. \tag{21}$$

Further we can write with property a)

$$e^{\widetilde{\lambda}_m^{(j)} \left(z - \frac{a_j + a_{j+1}}{2} \right)} = e^{\frac{2\pi m i}{a_{j+1} - a_j} \left(z - \frac{a_j + a_{j+1}}{2} \right)}\, e^{q_j e^{i\beta_j} \left(z - \frac{a_j + a_{j+1}}{2} \right)}. \tag{22}$$

The second complex exponential on the right hand side in (22) can be estimated by the constant $\max_{z \in \overline{D}} e^{q_j e^{i\beta_j} \left(z - \frac{a_j - a_{j+1}}{2} \right)}$, which is independent of m. Because of (18) we have to estimate the first exponential for $z \in [\,a_j, a_{j+1}\,]$ only. Let $z = a_j + \frac{a_{j+1}-a_j}{2\pi}(t+i\varepsilon)$, where $t \in [\,0, 2\pi\,]$ and $\varepsilon \in]\,0, \varepsilon_1\,]$, $\varepsilon_1 > 0$. Then

$$e^{\frac{2\pi mi}{a_{j+1}-a_j}\left(a_j+\frac{a_{j+1}-a_j}{2\pi}(t+i\varepsilon)-\frac{a_j+a_{j+1}}{2}\right)} = e^{mi(t+i\varepsilon)}\, e^{2\pi mi}.$$

Thus

$$\begin{aligned}
&\int_0^{2\pi} \left| \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m})\, e^{mi\theta} \left| e^{\widetilde{\lambda}_m^{(j)}\left(z-\frac{a_j+a_{j+1}}{2}\right)} \right| \right| d\theta \\
\leq\ & \text{const.} \int_0^{2\pi} \left| \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m})\, e^{mi\theta} \left| e^{mi(t+i\varepsilon)} \right| \right| d\theta \\
\leq\ & \text{const.} \int_0^{2\pi} \left| \sum_{m=n(j)}^{n_j} (1 - x_{n_j,r,m})\, e^{-\varepsilon m + im\theta} \right| d\theta \\
\leq\ & c \ln(n_j)
\end{aligned}$$

in D, where $c > 0$ can be chosen independently of ε. For $\varepsilon \to 0$ the claim follows for all $z \in \overline{D}$. □

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