Generalized Poisson Summation Formula for Tempered Distributions

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***Abstract*—The Poisson summation formula (PSF), which relates the sampling of an analog signal with the periodization of its Fourier transform, plays a key role in the classical sampling theory. In its current forms, the formula is only applicable to a limited class of signals in L_1. However, this assumption on the signals is too strict for many applications in signal processing that require sampling of non-decaying signals. In this paper we generalize the PSF for functions living in weighted Sobolev spaces that do not impose any decay on the functions. The only requirement is that the signal to be sampled and its weak derivatives up to order $1/2+\varepsilon$ for arbitrarily small $\varepsilon > 0$, grow slower than a polynomial in the L_2 sense. The generalized PSF will be interpreted in the language of distributions.**

I. Introduction

A widely used result in engineering is that sampling in time corresponds to periodization in frequency and vice versa. This beautiful relation is captured in the celebrated Poisson summation formula

$$\sum_{k\in\mathbb{Z}} f(k)e^{-2\pi ik\xi} = \sum_{k\in\mathbb{Z}} \hat{f}(\xi+k). \tag{PSF}$$

In other words, the *discrete-time* Fourier transform of the sampled sequence is equal to the periodization of the *continuous-time* Fourier transform of the analog signal. This identity is the key behind many results in sampling theory including the classical Shannon's sampling theorem [1] for bandlimited signals. What appears to be missing, however, is a proof of (PSF) for a general function f beyond the classical hypotheses, which will be detailed later. In engineering texts, see for example [2]–[5], the common "proof" of (PSF) for an *arbitrary* function f is based on the following observations: (1) the sampled sequence is obtained by multiplying the analog signal with a Dirac comb; (2) multiplication maps to convolution in Fourier-domain; and (3) the Fourier transform of a Dirac comb is also a Dirac comb. The problem with this argument is that the multiplication of a Dirac comb with a general function is not necessarily a tempered distribution, so that the convolution theorem may not be applicable. Take, for example, the popular ramp signal $f(x) = \frac{x+|x|}{2}$. Since $f(x)$ is not differentiable at 0, the multiplication between f and the Dirac comb $\sum_{k\in\mathbb{Z}} \delta(\cdot - k)$ is prohibited. It means that the above argument for (PSF) does not even work for this seemingly well-behaved signal.

In the mathematics literature, the PSF is often stated in a dual form with the sampling occurring in the Fourier domain, and under strict conditions on the function f. Various versions of (PSF) have been proven when both f and $\hat{f}$ are in appropriate subspaces of $L_1(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$. In its most general form, when $f, \hat{f} \in L_1(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$, the RHS of (PSF) is a well-defined periodic function in $L_1([0,1])$ whose (possibly divergent) Fourier series is the LHS. If f and $\hat{f}$ additionally satisfy $|f(x)|+|\hat{f}(x)| \le C(1+|x|)^{-1-\varepsilon}, \forall x \in \mathbb{R}$, for some $C, \varepsilon > 0$, then it can be shown [6], [7] that (PSF) holds pointwise with both sides converging absolutely. It is also known [8], [9] that (PSF) holds pointwise when $f, \hat{f} \in L_1(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$ and $\hat{f}$ has bounded total variation. Other versions of the PSF on $L_1(\mathbb{R})$ can also be found in [9]–[11]. The main limitation of the above mathematical forms of the PSF is that they do not apply to functions that are non-decreasing, such as our above example of a ramp signal or any realization of a stationary stochastic process, not to mention the non-stationary ones like the Brownian motion which may even grow at infinity [12]. Intuitively, however, there does not seem to be any compelling reason why a continuous signal cannot be sampled even if it is not decaying.

In this paper we attempt to bridge the gap between the engineering and mathematics statements of the PSF. We will show that (PSF) holds in the sense of distribution theory [13] for a broad class of continuous signals included in a *weighted* Sobolev space $L^s_{2,1/w}(\mathbb{R})$, whose *weak* derivatives up to order s are in the weighted-L_2 space $L_{2,1/w}(\mathbb{R})$. Here, the decaying weight $1/w$ allows growing signals. In particular, when the weight $1/w$ decays polynomially and the order of smoothness s is above the threshold $1/2$, both sides of (PSF) are well-defined *periodic distributions* and they are equal when acting on periodic *test functions*. In light of our new result, it is easy to see that (PSF) holds for the ramp function, and more generally, for any piecewise polynomials that are continuous. For brevity, most of the proofs in this paper are omitted or just sketched. Detailed proofs can be found in our extensive papers [14], [15] that are currently in preparation.[1]

The rest of the paper is organized as follows. Section II introduces some notations and definitions that will be used throughout the paper. Section III discusses the sampling in weighted Sobolev spaces. The generalized PSF is stated in

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[1]The manuscripts are available from the authors upon request.

Section IV with a sketched proof. Finally, some concluding remarks are made in Section V.

II. Notation and Definition

Throughout the paper, we deal with *complex-valued* functions and sequences. Following the convention in signal processing, we use parentheses for functions and square brackets for sequences to specify their values at a particular point. Especially, for a function f, $f[\cdot]$ denotes the sequence $\{f(k)\}_{k\in\mathbb{Z}}$. For $p \geq 1$, the spaces $L_p(\mathbb{R})$ and $\ell_p(\mathbb{Z})$ include functions and sequences, respectively, whose p-norms are finite. The scalar product between two functions in $L_2(\mathbb{R})$ is defined as

$$\langle f, g\rangle := \int_{\mathbb{R}} f(x)g(x)\mathrm{d}x, \quad \text{for } f, g \in L_2(\mathbb{R})$$

As usual, the notation $\langle\cdot,\cdot\rangle$ is also used for the action of a distribution on a test function. Sometimes, it is useful to write the complex sinusoid $e^{-2\pi i k\xi}$ as $e^{-2\pi i\langle\cdot,k\rangle}$ to imply that it is a function in variable ξ. We also adopt standard notations used in Schwartz's distribution theory [13]. We use the symbols $\wedge$ and $\vee$ to denote the forward and inverse Fourier transforms, respectively, of a tempered distribution $f \in \mathcal{S}'(\mathbb{R})$, i.e.

$$\left\langle \hat{f}, \varphi\right\rangle = \langle (f)^{\wedge}, \varphi\rangle := \langle f, \hat{\varphi}\rangle, \forall\varphi \in \mathcal{S}(\mathbb{R}),$$
$$\left\langle \check{f}, \varphi\right\rangle = \langle (f)^{\vee}, \varphi\rangle := \langle f, \check{\varphi}\rangle, \forall\varphi \in \mathcal{S}(\mathbb{R})$$

where

$$\hat{\varphi}(\xi) = (\varphi)^{\wedge}(\xi) := \int_{\mathbb{R}} \varphi(x)e^{-2\pi i\xi x}\mathrm{d}x;$$
$$\check{\varphi}(x) = (\varphi)^{\vee}(x) := \int_{\mathbb{R}} \varphi(\xi)e^{2\pi i\xi x}\mathrm{d}\xi.$$

The set of continuous functions is denoted by $\mathcal{C}(\mathbb{R})$, whereas $\mathcal{C}_c^{\infty}(\mathbb{R}) = \mathcal{D}(\mathbb{R})$ denotes the space of *bump functions* that are infinitely differentiable and compactly supported. For an interval $\mathbb{T}$ of $\mathbb{R}$, $\mathcal{C}^{\infty}(\mathbb{T})$ denotes the space of $\mathbb{T}$-periodic test functions that are infinitely differentiable. Here, a function φ is $\mathbb{T}$-periodic if $\varphi = \varphi(\cdot + |\mathbb{T}|k), \forall k \in \mathbb{Z}$. The constants throughout the paper are denoted by C with some subscripts denoting the dependence of the constants on those parameters. For example, $C_{x,y,z}$ is a constant that depends only on x, y and z.

The non-decaying signals in this paper will be modeled as elements of weighted L_2 spaces w.r.t. some decaying weighting function that controls the growth of the signals. We assume implicitly throughout the paper that any weighting function is real-valued, continuous, positive, and symmetric.

Definition 1 (Submultiplicative weights)**.** *A weighting function w is called submultiplicative if there exists a constant C_w such that*

$$w(x+y) \leq C_w\, w(x)w(y), \quad \forall x, y \in \mathbb{R}. \tag{1}$$

Note that the submultiplicativity of w implies $w(x) \leq C_w\, w(x)w(0)$, or $w(0) \geq 1/C_w$. On the other hand, $w(0) = w(x-x) \leq C_w\, w(x)w(-x) = C_w\,(w(x))^2$, and thus $w(x) \geq \sqrt{w(0)/C_w} \geq 1/C_w$, for all x. This means that every submultiplicative weighting function is lower-bounded. A prototypical example of submultiplicative weights is $w(x) = (1+|x|^2)^{\alpha/2}$, for some $\alpha \geq 0$. For this weight, we can choose C_w to be $C_\alpha = 2^{\alpha/2}$.

Definition 2 (Weighted L_p and ℓ_p spaces)**.** *For $p \geq 1$ and a weighting function w, a function f is in $L_{p,w}(\mathbb{R})$ if fw is in $L_p(\mathbb{R})$; a sequence c is in $\ell_{p,w}(\mathbb{Z})$ if $\{c[k]w(k)\}_{k\in\mathbb{Z}}$ is in $\ell_p(\mathbb{Z})$. The corresponding weighted norms are defined as*

$$\|f\|_{L_{p,w}(\mathbb{R})} := \|fw\|_{L_p(\mathbb{R})}; \qquad \|c\|_{\ell_{p,w}(\mathbb{Z})} := \|c\,w[\cdot]\|_{\ell_p(\mathbb{Z})}.$$

Note that if w is submultiplicative then $L_{p,w}(\mathbb{R}) \subset L_p(\mathbb{R})$, and $\ell_{p,w}(\mathbb{Z}) \subset \ell_p(\mathbb{Z})$ because w is lower-bounded.

Definition 3 (Weighted hybrid norm spaces)**.** *For $p, q \geq 1$, the hybrid norm space $W_{p,q}(\mathbb{R})$ includes all functions $f : \mathbb{R} \to \mathbb{C}$ whose following norm is finite*

$$\|f\|_{W_{p,q}(\mathbb{R})} := \left(\int_0^1 \left(\sum_{k\in\mathbb{Z}} |f(x+k)|^p\right)^{q/p} \mathrm{d}x\right)^{1/q},$$

with usual adjustments when p or q is infinity. The weighted hybrid norm space $W_{p,q,w}(\mathbb{R})$ w.r.t. a weighting function w is defined according to the following weighted norm

$$\|f\|_{W_{p,q,w}(\mathbb{R})} := \|fw\|_{W_{p,q}(\mathbb{R})}.$$

These hybrid (mixed) norm spaces were mentioned in [9]. They are similar to Wiener amalgam spaces $\widetilde{W}_{p,q}(\mathbb{R})$ [16]–[18], where the discrete and continuous norms are mixed in reverse order, i.e.

$$\|f\|_{\widetilde{W}_{p,q}(\mathbb{R})} := \left(\sum_{k\in\mathbb{Z}} \left(\int_0^1 |f(x+k)|^q\, \mathrm{d}x\right)^{p/q}\right)^{1/p}.$$

These amalgam spaces play a central role in many important results of the non-uniform sampling theory in shift-invariant (spline-like) spaces [19]–[23]. Deeper results regarding Wiener amalgam spaces and their generalizations can be found in [24]–[29]. We note some obvious inclusions of hybrid norm spaces: $W_{p,q_2,w}(\mathbb{R}) \subset W_{p,q_1,w}(\mathbb{R})$ when $1 \leq q_1 \leq q_2 \leq \infty$, and $W_{p_1,q,w}(\mathbb{R}) \subset W_{p_2,q,w}(\mathbb{R})$ when $1 \leq p_1 \leq p_2 \leq \infty$. Also, it is easy to see that $W_{p,p,w}(\mathbb{R}) = L_{p,w}(\mathbb{R})$.

Definition 4 (Weighted Sobolev spaces)**.** *For $s \in \mathbb{R}$, and a weighting function w, the weighted Sobolev space $L^s_{2,w}(\mathbb{R})$ is defined as*

$$L^s_{2,w}(\mathbb{R}) := \left\{ f \in \mathcal{S}'(\mathbb{R}) : \left((1+|\cdot|^2)^{\frac{s}{2}}\hat{f}\right)^{\vee} \in L_{2,w}(\mathbb{R})\right\}.$$

In the above definition, we note that $(1+|\xi|^2)^{\frac{s}{2}}$ is an infinitely differentiable and slowly increasing function, and so the multiplication $(1+|\cdot|^2)^{\frac{s}{2}}\hat{f}$ is a well-defined tempered distribution.

III. Sampling in Weighted Sobolev Spaces

For a slowly increasing weight w, the sampling of a slowly increasing signal in $L_{2,1/w}(\mathbb{R})$ might be troublesome because the sampled sequence might be rapidly increasing even when the signal is continuous, and so its discrete-time Fourier transform as in the LHS of (PSF) might not be well-defined. It is therefore desirable to impose some order of smoothness on the signal to be sampled by switching $L_{2,1/w}(\mathbb{R})$ to $L^s_{2,1/w}(\mathbb{R})$. In what follows, we show that if the signal is continuous and the order s of the Sobolev space is above the threshold $1/2$ then the sampling is *stable* in the sense that the discrete $\ell_{2,1/w}$-norm is bounded by the continuous $L_{2,1/w}$-norm.

We first want to make an important observation: any distribution $f \in L^s_{2,w}(\mathbb{R})$ can be written as a convolution $f = f_s * \varphi_s$, where $f_s := \left((1+|\cdot|^2)^{\frac{s}{2}}\hat{f}\right)^\vee \in L_{2,w}(\mathbb{R})$, and $\varphi_s := \left((1+|\cdot|^2)^{-\frac{s}{2}}\right)^\vee$. In the literature, the function φ_s when $s>0$ is often referred to as *Bessel potential kernel*. The following properties of this kernel will be useful later. First, it is easy to see that $\varphi_s(x)$ is a real and symmetric function. Second, it is well-known from Sobolev space theory (see, for example, [30, Prop. 6.1.5]) that $\varphi_s(x)$ is a positive function that decays exponentially outside a neighborhood of the origin. More precisely, there exists a constant C_s such that

$$\varphi_s(x) \le C_s \cdot e^{-\pi|x|}, \quad \forall |x| > \frac{1}{\pi}. \tag{2}$$

Third, it is also known that $\varphi_s \in L_2(\mathbb{R})$ whenever $s > 1/2$. Based on these properties we can show the following.

Proposition 1. *Suppose $s > 1/2$, and $w(x) = (1+|x|^2)^{\alpha/2}$ for some $\alpha \ge 0$. The Bessel potential kernel φ_s is then an element of the weighted hybrid norm space $W_{1,2,w}(\mathbb{R})$.*

Proof. See [14]. □

Note that $L^s_{2,1/w}(\mathbb{R})$ is identical to $L_{2,1/w}(\mathbb{R})$ when $s = 0$. Is it true that every element of $L^s_{2,1/w}(\mathbb{R})$ is an $L_{2,1/w}$ function? The following result gives an affirmative answer to that question when $s \ge 0$ and the weighting function $1/w$ is decaying polynomially.

Proposition 2. *If $s > 0$, and $w(x) = (1+|x|^2)^{\alpha/2}$ for some $\alpha \ge 0$, then $L^s_{2,1/w}(\mathbb{R}) \subset L_{2,1/w}(\mathbb{R})$.*

Proof. As a special case of Proposition 1, we have $\varphi_s \in W_{1,1,w}(\mathbb{R}) = L_{1,w}(\mathbb{R})$, for all $s > 0$. Let f be an element of $L^s_{2,1/w}(\mathbb{R})$. By duality we can write

$$\|f\|_{L_{2,1/w}(\mathbb{R})} = \sup_{\|g\|_{L_{2,w}(\mathbb{R})}=1} \langle f, g\rangle .$$

Using the submultiplicativity of w, Cauchy-Schwarz and Young inequalities we obtain

$$\begin{aligned}
\|f\|_{L_{2,1/w}(\mathbb{R})} &= \sup_{\|g\|_{L_{2,w}(\mathbb{R})}=1} \langle f_s * \varphi_s, g\rangle \\
&= \sup_{\|g\|_{L_{2,w}(\mathbb{R})}=1} \left\langle \frac{f_s}{w}, w\cdot(g*\varphi_s)\right\rangle \\
&\le \sup_{\|g\|_{L_{2,w}(\mathbb{R})}=1} \|f_s\|_{L_{2,1/w}(\mathbb{R})}\, C_\alpha\, \|g\|_{L_{2,w}(\mathbb{R})}\, \|\varphi_s\|_{L_{1,w}(\mathbb{R})} \\
&= C_\alpha\, \|f_s\|_{L_{2,1/w}(\mathbb{R})}\, \|\varphi_s\|_{L_{1,w}(\mathbb{R})} < \infty,
\end{aligned}$$

which implies that $f \in L_{2,1/w}(\mathbb{R})$, completing the proof. □

Before stating the central result of this section we need the following key lemma whose proof can be found in [14].

Lemma 1. *Let w be a submultiplicative weighting function. If $f \in L_{2,1/w}(\mathbb{R})$ and $\varphi \in W_{1,2,w}(\mathbb{R})$ then $f*\varphi$ is a well-defined continuous function and its sampled sequence $\{c[k] := (f*\varphi)(k)\}_{k\in\mathbb{Z}}$ belongs to $\ell_{2,1/w}(\mathbb{Z})$, and the following bound holds*

$$\|c\|_{\ell_{2,1/w}(\mathbb{R})} \le C_w\ \|f\|_{L_{2,1/w}(\mathbb{R})}\, \|\varphi\|_{W_{1,2,w}(\mathbb{R})},$$

where C_w is the constant given in (1).

Theorem 1. *Let $s > 1/2$, and $w(x) = (1+|x|^2)^{\alpha/2}$ for some $\alpha \ge 0$. The sampling operator $f \mapsto f|_{\mathbb{Z}}$ is then bounded from $L^s_{2,1/w}(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$ to $\ell_{2,1/w}(\mathbb{Z})$, i.e., there exists a constant $C_{\alpha,s}$ such that*

$$\|f[\cdot]\|_{\ell_{2,1/w}(\mathbb{Z})} \le C_{\alpha,s}\, \|f\|_{L^s_{2,1/w}(\mathbb{R})}, \quad \forall f \in L^s_{2,1/w}(\mathbb{R}).$$

Proof. From Proposition 1, we know that $f = f_s * \varphi_s$ *almost everywhere*, where $f_s \in L_{2,1/w}$ and $\varphi_s \in W_{1,2,w}(\mathbb{R})$. It then follows from Lemma 1 that $f_s * \varphi_s$ is continuous. Combining this with the fact that f is continuous, we deduce that $f = f_s * \varphi_s$ everywhere.

Now we can safely write $f(k) = (f_s*\varphi_s)(k)$, for all $k \in \mathbb{Z}$, and again invoke Lemma 1 to get

$$\begin{aligned}
\|f[\cdot]\|_{\ell_{2,1/w}(\mathbb{Z})} &\le C_\alpha\ \|\varphi_s\|_{W_{1,2,w}(\mathbb{R})}\, \|f_s\|_{L_{2,1/w}(\mathbb{R})} \\
&= \underbrace{C_\alpha\ \|\varphi_s\|_{W_{1,2,w}(\mathbb{R})}}_{C_{\alpha,s}}\, \|f\|_{L^s_{2,1/w}(\mathbb{R})},
\end{aligned}$$

which is the desired bound. □

IV. Generalized Poisson Summation Formula

Now we are ready to generalize the Poisson summation formula for functions in weighted Sobolev spaces and show that both the discrete-domain Fourier transform of the sampled sequence and the periodization of the continuous-domain Fourier transform are well-defined periodic distributions and they are equal in $\mathcal{S}'(\mathbb{T})$, for $\mathbb{T} := [0,1]$. First we need to define the periodization $\sum_{k\in\mathbb{Z}} \hat{f}(\cdot+k)$ by means of a *unitary function* $\psi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R})$ such that $\sum_{k\in\mathbb{Z}} \psi(\cdot+k) = 1$. In particular, the action of $\sum_{k\in\mathbb{Z}} \hat{f}(\cdot+k)$ on a periodic test function $\varphi_{\mathrm{perio}} \in \mathcal{C}^\infty(\mathbb{T})$ is given by

$$\left\langle \sum_{k\in\mathbb{Z}} \hat{f}(\cdot+k), \varphi_{\mathrm{perio}} \right\rangle_{\mathbb{T}} := \sum_{k\in\mathbb{Z}} \left\langle \hat{f}(\cdot+k), \varphi_{\mathrm{perio}}\psi \right\rangle. \tag{3}$$

The next result shows that the RHS of (3) is finite for appropriate weighted Sobolev spaces, whereas Theorem 2 guarantees that this definition is independent of the choice of ψ.

Proposition 3. *For $s > 1/2$, and $w(x) = (1+|x|^2)^n$, for some $n \in \mathbb{Z}^+$, if $f \in L^s_{2,1/w}(\mathbb{R})$ then the series $\sum_{k\in\mathbb{Z}} \left\langle \hat{f}(\cdot + k), \varphi \right\rangle$ is absolutely convergent, for every bump function $\varphi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R})$.*

Proof. See [15]. □

Theorem 2 (Generalized PSF)**.** *Suppose $s > 1/2$, and $w(x) = (1 + |x|^2)^n$ for some $n \in \mathbb{Z}^+$. For all $f \in L^s_{2,1/w}(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$, the following equality holds*

$$\sum_{k\in\mathbb{Z}} f(k) e^{-2\pi i \langle \cdot, k\rangle} = \sum_{k\in\mathbb{Z}} \hat{f}(\cdot + k),$$

in the distributional sense, i.e., for all $\varphi_{\mathrm{perio}} \in \mathcal{C}^\infty(\mathbb{T})$ and every unitary function ψ, we have

$$\sum_{k\in\mathbb{Z}} \left\langle f(k) e^{-2\pi i\langle\cdot,k\rangle}, \varphi_{\mathrm{perio}} \right\rangle_{\mathbb{T}} = \sum_{k\in\mathbb{Z}} \left\langle \hat{f}(\cdot + k), \varphi_{\mathrm{perio}}\psi \right\rangle, \tag{4}$$

with both sides converging absolutely.

Sketch of Proof. Fix a periodic test function φ_{perio}, a unitary function ψ, and put $\varphi := \varphi_{\mathrm{perio}}\, \psi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R})$, and $g := f\hat{\varphi}$. As $\hat{\varphi} \in \mathcal{S}(\mathbb{R})$, and $f \in L_{2,1/w}(\mathbb{R})$ (according to Proposition 2) we can argue that $g \in L_1(\mathbb{R})$. Hence, the Fourier transform $\hat{g}$ of g is a continuous function. After some algebraic manipulations, we can rewrite the LHS of (4) as

$$\mathrm{LHS} = \sum_{k\in\mathbb{Z}} f(k)\hat{\varphi}(k) = \sum_{k\in\mathbb{Z}} g(k). \tag{5}$$

We note that the series $\sum_{k\in\mathbb{Z}} g(k)$ is absolutely convergent because $f[\cdot] \in \ell_{2,1/w}(\mathbb{Z})$ (according to Theorem 1) and that

$$\|f[\cdot]\hat{\varphi}[\cdot]\|_{\ell_1(\mathbb{Z})} \leq \|f[\cdot]\|_{\ell_{2,1/w}(\mathbb{Z})} \|\hat{\varphi}[\cdot]\|_{\ell_{2,w}(\mathbb{Z})} < +\infty.$$

Similarly, we can show that the function $h(x) := \sum_{k\in\mathbb{Z}} g(x+k)$ is well-defined everywhere. This function $h(x)$ is obviously periodic and continuous. Now we can invoke the PSF for $g(x) \in L_1(\mathbb{R})$ to obtain that the Fourier coefficients of $h(x)$ are given by $\hat{h}[n] = \hat{g}(n)$, for $n \in \mathbb{Z}$. On the other hand, let us define the function $u(x) := \left\langle \hat{f}, \varphi(\cdot - x) \right\rangle$. Using the convolution theorem for the product $f\hat{\varphi}$ (which is of course applicable because $\hat{\varphi} \in \mathcal{S}(\mathbb{R})$), and the properties of distributions we can show that $u(x) = \hat{g}(x)$, for *almost all* $x \in \mathbb{R}$, and that $u(x)$ is continuous. As $\hat{g}$ is also continuous, it must be that $u = \hat{g}$ everywhere. Therefore, for every $k \in \mathbb{Z}$,

$$\hat{h}[k] = \hat{g}(k) = u(k) = \left\langle \hat{f}, \varphi(\cdot - k) \right\rangle = \left\langle \hat{f}(\cdot + k), \varphi \right\rangle. \tag{6}$$

From this equality and Proposition 3, we know that the Fourier coefficients of $h(x)$ is absolutely summable. This together with the continuity of $h(x)$ establishes that its Fourier series converges uniformly to $h(x)$. Especially, at $x = 0$, we have $h(0) = \sum_{k\in\mathbb{Z}} \hat{h}[k]$, which gives us the desired equality (4) thanks to (5), (6), and the definition of $h(x)$. □

V. Conclusion

We have provided a distributional proof of the Poisson summation formula for non-decaying signals whose weak derivatives up to order $1/2 + \varepsilon$ are slowly growing. These signals are modeled as tempered distributions in weighted Sobolev spaces. Although our results are stated only for the sampling on $\mathbb{Z}$, they can be easily extended to $T\mathbb{Z}$ for an arbitrary sampling period T by the scaling property of the Fourier transform. This generalized PSF may be used as a foundational result for the sampling theory of non-decaying signals. It also confirms the general validity of the statement: "A periodization in the frequency domain maps into a sampling in the time domain and vice versa."

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